Margaret Bilu
Office 604


bilu@cims.nyu.edu

# Algebra homework 8
# Permutations

Due November 13th, 2019

Please hand in your homework stapled, with your name written on it. All answers have to be justified.
For every $n \geq 1$ we denote by $\mathfrak{S}_n$ the $n$-th group of permutations.

**Exercise 1.** Compute the signs of the following permutations:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 4 & 6 & 2 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 1 & 3 & 8 & 6 & 2 & 7 \end{pmatrix}, \quad \sigma_3 = (1,2,3,4)^{1001}$$

$$\sigma_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 6 & 7 & 1 & 5 & 2 \end{pmatrix}, \quad \sigma_5 = (1,2,4)(5,3), \quad \sigma_6 = (1,7)(1,6)(7,3)(5,2).$$

**Exercise 2.** Let $\sigma \in \mathfrak{S}_n$. Prove that

1. $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$.
2. for all $\alpha \in \mathfrak{S}_n$, $\operatorname{sgn}(\alpha\sigma\alpha^{-1}) = \operatorname{sgn}(\sigma)$.

**Exercise 3.** Let $n \geq 1$ and let $e_1, \ldots, e_n$ be the usual basis vectors of $\mathbf{R}^n$, that is, for every $i \in \{1, \ldots, n\}$, we have

$$e_i = (0, \ldots, 0, 1, 0, \ldots 0)$$

where the 1 is in the $i$-th coordinate. For all $\mathfrak{S}_n$ we define the matrix $M_\sigma \in M_n(\mathbf{R})$ to be the matrix such that for all $i \in \{1, \ldots, n\}$ its coefficient at column $i$ and row $\sigma(i)$ is 1, all other coefficients being equal to zero. For example, when $n = 2$, for the transposition $(12)$ in $\mathfrak{S}_2$, we have $M_{(12)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

1. In this question, we study the case $n = 3$. Compute $M_\sigma$ for all $\sigma \in \mathfrak{S}_3$.
2. Now we go back to general $n$. Compute $M_{\text{id}}$ where $\text{id} \in \mathfrak{S}_n$ is the identity permutation.
3. Explain why for all $\sigma \in \mathfrak{S}_n$, there is exactly one coefficient equal to 1 in each row of $M_\sigma$, as well as in each column.
4. What is the image $M_\sigma e_i$ of the basis vector $e_i$ by $M_\sigma$?
5. Show that for all permutations $\sigma, \tau \in \mathfrak{S}_n$, we have $M_{\sigma\tau} = M_\sigma M_\tau$.
6. Show that for every $\sigma \in \mathfrak{S}_n$, $M_\sigma$ is an invertible matrix, by computing its inverse.
7. Show that the map $\phi : \mathfrak{S}_n \to (GL_n(\mathbf{R}), \cdot)$ given by $\sigma \mapsto M_\sigma$ is well-defined and is an injective group homomorphism.

**Exercise 4.** Recall that the center of the group $\mathfrak{S}_n$ is defined by

$$Z(\mathfrak{S}_n) = \{\sigma \in \mathfrak{S}_n | \text{ for all } \alpha \in \mathfrak{S}_n,\ \alpha\sigma = \sigma\alpha\}.$$

1. Show that $\text{id} \in Z(\mathfrak{S}_n)$.
2. Determine $Z(\mathfrak{S}_n)$ for $n = 1, 2, 3$.
3. We now assume $n \geq 3$ and pick $\sigma \in \mathfrak{S}_n$ different from the identity.
   (a) Show that there exists $i \in \{1, \ldots, n\}$ such that $\sigma(i) \neq i$. We denote $j = \sigma(i)$.
   (b) Construct a transposition $\alpha$ such that $\alpha\sigma\alpha^{-1}(i) \neq j$.
   (c) Conclude that $Z(\mathfrak{S}_n) = \{\text{id}\}$.