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Algebra homework 8
Permutations

Due November 13th, 2019

Please hand in your homework stapled, with your name written on it. All answers have to be justified.
For every $n \geq 1$ we denote by $\mathfrak{S}_n$ the n-th symmetric group.

Exercise 1. Compute the signs of the following permutations:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 4 & 6 & 2 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 1 & 3 & 8 & 6 & 2 & 7 \end{pmatrix}, \quad \sigma_3 = (1,2,3,4)^{1001}$$

$$\sigma_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 6 & 7 & 1 & 5 & 2 \end{pmatrix}, \quad \sigma_5 = (1,2,4)(5,3), \quad \sigma_6 = (1,7)(1,6)(7,3)(5,2).$$

Solution.
We can use the definition of sgn as $(-1)^r$, where r is the number of transpositions in a decomposition of σ as a product of transpositions, together with the fact that sgn is a group homomorphism. Recall also that the sign of a k-cycle is $(-1)^{k-1}$. We have:

1. $\text{sgn}(\sigma_1) = \text{sgn}((1,5,2,3,4,6)) = (-1)^5$, so $\text{sgn}(\sigma_1) = -1$.
2. $\text{sgn}(\sigma_2) = \text{sgn}((1,4,3)(2,5,8,7)) = \text{sgn}((1,4,3))\text{sgn}((2,5,8,7)) = (-1)^2(-1)^3 = -1$
3. $\text{sgn}(\sigma_3) = \text{sgn}((1,2,3,4))^{1001} = (-1)^{1001} = -1$
4. $\text{sgn}(\sigma_4) = \text{sgn}((1,4,7,2,3,6,5)) = (-1)^6 = 1$.
5. $\text{sgn}(\sigma_5) = \text{sgn}((1,2,4))\text{sgn}((5,2)) = (-1)^3 = -1$.
6. $\text{sgn}(\sigma_6) = (-1)^4 = 1$.

Exercise 2. Let $\sigma \in \mathfrak{S}_n$. Prove that

1. $\text{sgn}(\sigma) = \text{sgn}(\sigma^{-1})$.

   Solution.

   By definition, $\sigma\sigma^{-1} = \text{id}$.

   Since $\text{sgn} : \mathfrak{S}_n \mapsto \{-1, 1\}$ is a group homomorphism,

   $$1 = \text{sgn}(\text{id}) = \text{sgn}(\sigma\sigma^{-1}) = \text{sgn}(\sigma)\text{sgn}(\sigma^{-1})$$

   So $\text{sgn}(\sigma)$ is the inverse of $\text{sgn}(\sigma^{-1})$ in $\{-1, 1\}$. This implies $\text{sgn}(\sigma) = \text{sgn}(\sigma^{-1})$.

2. for all $\alpha \in \mathfrak{S}_n$, $\text{sgn}(\alpha\sigma\alpha^{-1}) = \text{sgn}(\sigma)$.

   Solution.

   Since sgn is a group homomorphism,

   $$\text{sgn}(\alpha\sigma\alpha^{-1}) = \text{sgn}(\alpha)\text{sgn}(\sigma)\text{sgn}(\alpha^{-1}) = \text{sgn}(\sigma)$$

   since $\text{sgn}(\alpha)\text{sgn}(\alpha^{-1}) = 1$ using the first question, and since the group $\{1, -1\}$ is commutative.

Exercise 3. Let $n \geq 1$ and let $e_1, \dots, e_n$ be the usual basis vectors of $\mathbf{R}^n$, that is, for every $i \in \{1, \dots, n\}$, we have

$$e_i = (0, \dots, 0, 1, 0, \dots 0)$$

where the 1 is in the i-th coordinate. For all $\mathfrak{S}_n$ we define the matrix $M_\sigma \in M_n(\mathbf{R})$ to be the matrix such that for all $i \in \{1, \dots, n\}$ its coefficient at column i and row $\sigma(i)$ is 1, all other coefficients being equal to zero. For example, when $n = 2$, for the transposition (12) in $\mathfrak{S}_2$, we have $M_{(12)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

1. In this question, we study the case $n = 3$. Compute M_σ for all $\sigma \in \mathfrak{S}_3$.

   Solution.

   Using the definition, we have:

$$M_{\text{id}} = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M_{(1,2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M_{(2,3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$M_{(1,3)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, M_{(1,2,3)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, M_{(1,3,2)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

2. Now we go back to general n. Compute M_{id} where $\text{id} \in \mathfrak{S}_n$ is the identity permutation.

   Solution.

   By definition, $\text{id}(i) = i$ for any i. So:

$$M_{\text{id}} = I_n = \begin{bmatrix} 1 & 0 & \dots & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & \dots & 0 & 1 \end{bmatrix}$$

3. Explain why for all $\sigma \in \mathfrak{S}_n$, there is exactly one coefficient equal to 1 in each row of M_σ, as well as in each column.

   Solution. Let $j \in \{1, \dots, n\}$. Then there is a 1 in the i-th cell of the j-th row of M_σ if and only if $\sigma(i) = j$. Since σ is bijective, this happens exactly for one value of i, namely $i = \sigma^{-1}(j)$.

   Let $i \in \{1, \dots, n\}$. Then there is a 1 in the j-th cell of the i-th column of M_σ if and only if $j = \sigma(i)$, so there is exactly one value of j for which this happens.

4. What is the image $M_\sigma e_i$ of the basis vector e_i by M_σ?

   Solution.

   By definition, we have: $M_\sigma e_i = e_{\sigma(i)}$.

5. Show that for all permutations $\sigma, \tau \in \mathfrak{S}_n$, we have $M_{\sigma\tau} = M_\sigma M_\tau$.

   Solution.

   Using the previous question, we have for all basis vector e_i, $M_{\sigma\tau} e_i = e_{\sigma\tau(i)} = M_\sigma e_{\tau(i)} = M_\sigma M_\tau e_i$.

   Therefore, $M_{\sigma\tau} = M_\sigma M_\tau$, since they coincide on all vectors of a basis.

6. Show that for every $\sigma \in \mathfrak{S}_n$, M_σ is an invertible matrix, by computing its inverse.

   Solution. The result of the previous question combined with question 2 gives us the result since $M_\sigma M_{\sigma^{-1}} = M_{\text{id}} = I_n$. Therefore, M_σ is invertible with inverse $M_{\sigma^{-1}}$.

7. Show that the map $\phi : \mathfrak{S}_n \to (GL_n(\mathbf{R}), \cdot)$ defined by $\sigma \mapsto M_\sigma$ is an injective group homomorphism.

   Solution.

   With the result of question 6, this map is well defined. With the result of question 5, it is a group homomorphism. To check that it is injective, it suffices to see that its kernel is trivial. But $\phi(\sigma) = M_\sigma = I_n$ implies $\sigma(e_i) = e_i$ for all i, so $\sigma = \text{id}$.

Exercise 4. Recall that the center of the group $\mathfrak{S}_n$ is defined by

$$Z(\mathfrak{S}_n) = \{\sigma \in \mathfrak{S}_n | \text{ for all } \alpha \in \mathfrak{S}_n,\ \alpha\sigma = \sigma\alpha\}.$$

1. Show that $\text{id} \in Z(\mathfrak{S}_n)$.

   Solution.

   By definition of the identity, $\alpha \circ \text{id} = \text{id} \circ \alpha = \alpha$ for any $\alpha \in \mathfrak{S}_n$.

   So $\text{id} \in Z(\mathfrak{S}_n)$.

2. Compute $Z(\mathfrak{S}_n)$ for $n = 1, 2, 3$.

   Solution.

   For $n = 1$, we clearly have $Z(\mathfrak{S}_1) = \{\text{id}\} = \mathfrak{S}_1$.

   For $n = 2$, $Z(\mathfrak{S}_2) = \{\text{id},\ (1,2)\} = \mathfrak{S}_2$.

   For $n = 3$, we have $Z(\mathfrak{S}_3) = \{\text{id}\}$. You can check separately for each element of $\mathfrak{S}_3$ other than the identity, that it does not belong to the center by finding an element α such that $\sigma\alpha \neq \alpha\sigma$. For example, we have

   $$(1,2,3)(1,2) = (13)$$

   whereas

   $$(1,2)(1,2,3) = (23)$$

   This shows that both $(1,2,3)$ and $(1,2)$ are not in the center. In the same way, we have

   $$(1,3,2)(1,3) = (1,2)$$

   whereas

   $$(1,3)(1,3,2) = (2,3)$$

   This shows that both $(1,3,2)$ and $(1,3)$ are not in the center. Finally, we have

   $$(1,2)(2,3) = (1,2,3) \neq (1,3,2) = (2,3)(1,2),$$

   which shows that $(2,3)$ is not in the center either.

   Another way of seeing this is by looking at the Cayley table of $\mathfrak{S}_3$: an element σ of the center has $\alpha\sigma = \sigma\alpha$ for all α. Thus, the elements in the column corresponding to σ and in the row corresponding to σ must be in exactly the same order.

$\circ$	id	(123)	(132)	(12)	(23)	(13)
id	id	(123)	(132)	(12)	(23)	(13)
(123)	(123)	(132)	id	(13)	(12)	(23)
(132)	(132)	id	(123)	(23)	(13)	(12)
(12)	(12)	(23)	(13)	id	(123)	(132)
(23)	(23)	(13)	(12)	(132)	id	(123)
(13)	(13)	(12)	(23)	(123)	(132)	id

id is the only element for which this is satisfied.

3. We now assume $n \geq 3$ and pick $\sigma \in \mathfrak{S}_n$ different from the identity.

   (a) Show that there exists $i \in \{1, \ldots, n\}$ such that $\sigma(i) \neq i$. We denote $j = \sigma(i)$.

   Solution.

   By contradiction, if for all $i \in \{1, \ldots, n\}$ we had $\sigma(i) = i$, we would have $\sigma = \text{id}$, which is not the case. So there exists i such that $\sigma(i) = j \neq i$.

   (b) Construct a transposition α such that $\alpha\sigma\alpha^{-1}(i) \neq j$.

   Solution.

   You can take any transposition which moves j but not i.

   Indeed:

   For $\alpha = (j, k)$, with $k \neq i$ (which exists since $n \geq 3$), we have: $\alpha\sigma\alpha^{-1}(i) = \alpha\sigma(i) = \alpha(j) = k$.

   So $\alpha\sigma\alpha^{-1}(i) \neq j$.

   (c) Conclude that $Z(\mathfrak{S}_n) = \{\text{id}\}$.

   Solution.

   With the last result, for any $\sigma \in \mathfrak{S}_n$ different from id, we can find an α such that

   $$\alpha\sigma\alpha^{-1}(i) = k \neq j, \; j = \sigma(i)$$

   If α and σ were to commute, we would have $\alpha\sigma\alpha^{-1}(i) = \sigma(i) = j$, which we see is not the case.

   Therefore, $Z(\mathfrak{S}_n) = \{\text{id}\}$.