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Algebra homework 9
Permutations, Cosets, Lagrange's theorem

Due November 20th, 2019

Please hand in your homework stapled, with your name written on it. All answers have to be justified.

Exercise 1. Describe the left and the right cosets of the following subgroups H_i of the following groups G_i, and compute $[G_i : H_i]$ for every i.

1. $H_1 = \langle 4 \rangle$ in $G_1 = \mathbf{Z}/12\mathbf{Z}$.
2. $H_2 = 3\mathbf{Z}$ in $G_2 = \mathbf{Z}$.
3. $H_3 = \{\mathrm{id}, (1,2,3), (1,3,2)\}$ in $G_3 = \mathfrak{S}_3$.
4. $H_4 = \{\mathrm{id}, (1,3)\}$ in $G_4 = \mathfrak{S}_3$.
5. $H_5 = \mathfrak{A}_n$ in $G_5 = \mathfrak{S}_n$ for $n \geq 2$ (Hint: show that it cannot have more than two different cosets).

Exercise 2. Let H be a subgroup of a group G, and let $a, b \in G$. Show that $b \in aH$ if and only if $aH = bH$.

Exercise 3. Let G be a group and H a subgroup of G. We assume that H is such that for all $h \in H$ and all $g \in G$, the product ghg^{-1} is an element of H (we say that H is a *normal* subgroup of G).

1. Show that all subgroups of an abelian group are normal.
2. Show that for all $g \in G$, $gH = Hg$, that is, the right and the left cosets of H are the same.
3. Find an example of a group G and of a subgroup H of G which is not normal.

Exercise 4. Let $n \geq 2$ and let $\phi : \mathfrak{S}_n \to \{1, -1\}$ be a non-trivial group homomorphism. The aim of this exercise is to prove that ϕ is equal to the sgn homomorphism. In other words, the sgn homomorphism is the only non-trivial homomorphism from $\mathfrak{S}_n$ to $\{1, -1\}$.

1. What does it mean for ϕ to be non-trivial? Deduce that there exists at least one transposition $\tau \in \mathfrak{S}_n$ such that $\phi(\tau) = -1$.
2. Show that for all permutations $\alpha, \sigma \in \mathfrak{S}_n$, we have $\phi(\sigma\alpha\sigma^{-1}) = \phi(\alpha)$.
3. Let α, α' be two transpositions.
   (a) Put $\alpha = (a, b)$. Show that if $\alpha' \neq \alpha$, we may assume that either $\alpha' = (c, d)$ with c, d distinct from a, b, or $\alpha' = (a, c)$ with c distinct from a, b.
   (b) Show that there exists a permutation σ such that $\sigma\alpha\sigma^{-1} = \alpha'$.
4. Show that for any transposition $\tau' \in \mathfrak{S}_n$, we have $\phi(\tau') = -1$.
5. Deduce that $\phi = \mathrm{sgn}$, that is, that $\phi(\sigma) = \mathrm{sgn}(\sigma)$ for all $\sigma \in \mathfrak{S}_n$.

Exercise 5. Let G be a group of order 25.

1. Prove that G has at least one element of order 5.
2. Deduce that G has at least one subgroup of order 5.
3. If G is cyclic, show that it contains exactly one subgroup of order 5.
4. Show that if G is not cyclic, it must have more than one subgroup of order 5.