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Midterm

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The use of notes and electronic devices is forbidden. All answers have to be justified. You can use all results proved in lectures without proof, unless you are explicitly asked for a proof. If you use a result seen in a homework or in recitation you need to prove it. You may use the back of the last sheet if you need extra space.

Name: **NetId:**

1	/12
2	/20
3	/10
4	/9
5	/9
6	/13
7	/7
Total	/80

Exercise 1. (12 points) Prove or disprove: for any of the following statements, say if it is true or false, by either proving it or providing a counterexample.

1. Let a, b be integers. If there exist integers u and v such that $ua + vb = 2$, then $\gcd(a, b) = 2$.

2. Let G be a group and let $x \in G$ be of order 7. Then $x^{43} = x$.

3. Let G and H be cyclic groups. Then $G \times H$ is cyclic.

4. The groups $(\mathbf{R}, +)$ and $(\mathbf{R}_{>0}, \cdot)$ are isomorphic.

Exercise 2. Let G be the group $\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

1. (2 points) Determine the subgroup $\langle (2,0) \rangle$. Is it a proper subgroup of G?

2. (3 points) Determine the subgroup $\langle (1,1) \rangle$. Is it a proper subgroup of G?

3. (4 points) Describe all of the elements of finite order in G, and give their orders.

4. (5 points) Is G cyclic?

5. We consider the map $p : G \to \mathbf{Z}/3\mathbf{Z}$ given by $p(x, y) = [x]_3$ (that is, (x, y) is sent to the congruence class of x modulo 3).

   (a) (2 points) Show that it is a homomorphism.

   (b) (4 points) Determine its kernel and image.

Exercise 3. 1. (2 points) What is the order of a group G?

2. (2 points) What does it mean for a group G to be cyclic?

3. (2 points) How many cyclic groups of order 20 are there up to isomorphism?

4. (2 points) How many infinite cyclic groups are there up to isomorphism?

5. (2 points) Give an example of a group which is commutative, but not cyclic.

Exercise 4. Let $*$ be the law of composition on $\mathbf{R}$ given by $x * y = (xy)^2$.

1. (4 points) Is it associative?

2. (2 points) Is it commutative?

3. (3 points) Does it have an identity element?

Exercise 5. 1. (2 points) State Bézout's theorem.

2. (3 points) Prove that for all integers n, the integers $4n + 3$ and $3n + 2$ are relatively prime.

3. (4 points) Prove that for all integers n, $\gcd(2n+4, 5n+9)$ divides 2. Determine the value of this gcd when n is odd and when n is even.

Exercise 6. Let G be a group with identity element e.

1. (2 points) Let a be an element of G. Define what it means for a to be of finite order.

2. (4 points) Assume that $a \in G$ is of finite order. Show that any integer n such that $a^n = e$ is divisible by the order of a.

3. Let $a, b \in G$ be two elements such that $a \neq e$ and $bab^{-1} = a^2$.

   (a) (4 points) Show that $b^2ab^{-2} = a^4$.

   (b) (3 points) Assume that b is of order 2. Determine the order of a.

Exercise 7.

1. (3 points) Give a list of the generators of $\mathbf{Z}/10\mathbf{Z}$.

2. (4 points) Is $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/5\mathbf{Z}$ isomorphic to $\mathbf{Z}/10\mathbf{Z}$?