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Midterm

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The use of notes and electronic devices is forbidden. All answers have to be justified. You can use all results proved in lectures without proof. If you use a result seen in a homework or in recitation you need to prove it. You may use the back of the last sheet if you need extra space.

Name: **NetId:**

1	/12
2	/18
3	/10
4	/9
5	/7
6	/18
7	/6
Total	/80

Exercise 1. (12 points) Prove or disprove: for any of the following statements, say if it is true or false, by either proving it or providing a counterexample.

1. Let a, b, c be integers. If a divides bc then a divides b or c.

2. Let G and H be commutative groups. Then $G \times H$ is commutative.

3. Let G be a group and H a subgroup of G. If G is commutative, then so is H.

4. Let G be a group with identity element e and $x \in G$ an element of order 4. Then $x^{20} = e$.

Exercise 2. Let G be the group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$.

1. (5 points) Write out the list of the elements of G, and for each element, give its order.

2. (2 points) What is the order of G?

3. (2 points) Is G cyclic?

4. (2 points) Is G isomorphic to $\mathbf{Z}/8\mathbf{Z}$?

5. (3 points) Give an example of a proper subgroup of G.

6. (4 points) We consider the map $p : G \to \mathbf{Z}/2\mathbf{Z}$ given by $p(x, y) = x$. Determine its kernel and image.

Exercise 3. Let G be a group.

1. (2 points) What does it mean for G to be commutative?

2. (2 points) What does it mean for G to be cyclic?

3. (3 points) Show that if G is cyclic, then G is commutative.

4. (3 points) Give an example of a group which is commutative, but not cyclic.

Exercise 4. Let $*$ be the law of composition of $\mathbf{Z}$ given by $x * y = 2x + 2y$.

1. (4 points) Is it associative?

2. (2 points) Is it commutative?

3. (3 points) Does it have an identity element?

Exercise 5. 1. (2 points) State Bézout's theorem.

2. (5 points) Let a, b, c be integers such that $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$. Show that $\gcd(a, bc) = 1$.

Exercise 6. Let G be the set given by

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in M_2(\mathbf{R}),\ a \in \{1, -1\}, b \in \mathbf{Z} \right\}.$$

1. (2 points) Show that G is a subset of $GL_2(\mathbf{R})$, the set of invertible 2×2 matrices with real coefficients.

2. (8 points) Show that G is a subgroup of $(GL_2(\mathbf{R}), \cdot)$.

3. (3 points) Is G commutative?

4. (5 points) Determine all elements of order 2 of G.

Exercise 7. (6 points) If $a, b \in \mathbf{Z}$ are such that $a \equiv b \pmod{n}$, show that $\gcd(a, n) = \gcd(b, n)$.