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Final

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The use of notes and electronic devices is forbidden. All answers have to be justified. You can use all results stated in lectures without proof, unless explicitly asked to produce a proof. If you use a result seen in a homework or in recitation you need to prove it.
Within each of the exercises, you can use the result of a question in subsequent questions even if you haven't solved that particular question. Thus, if you cannot solve a question, do move on to the next one after some time, assuming what had to be proved.
You may use the back of the last sheet if you need extra space.

Name: **NetId:**

1	/12
2	/10
3	/12
4	/10
5	/10
6	/25
7	/11
Total	/90

Exercise 1. (12 points) Prove or disprove: for any of the following statements, say if it is true or false, by either proving it or providing a counterexample.

1. For any integer a, one has $a^7 \equiv a \pmod 7$.

2. $H = \{\mathrm{id}, (12)\}$ is a normal subgroup of $\mathfrak{S}_3$.

3. $\mathbf{Z}/100\mathbf{Z}$ has a subgroup of order 20.

4. Let G be a group and H a normal subgroup of G. If H and G/H are both cyclic, then G is cyclic.

Exercise 2. 1. Let $f : \mathbf{Z}/10\mathbf{Z} \to \mathbf{Z}/3\mathbf{Z}$ be a group homomorphism.

(a) (2 points) Show that $\mathrm{Im} f$ must have order dividing 10.

(b) (2 points) Recall that a homomorphism $\phi : G \to G'$ between two groups is said to be trivial if for every $g \in G$, $\phi(g) = e'$ where e' is the identity element of G'. Conclude that f is trivial.

2. Let $f : \mathbf{Z}/15\mathbf{Z} \to \mathbf{Z}/3\mathbf{Z}$ be a non-trivial group homomorphism.

(a) (4 points) Show that f is surjective and determine the order of $\mathrm{Ker} f$.

(b) (2 points) Deduce $\mathrm{Ker} f$.

Exercise 3. Let $G = \mathbf{Z} \times \mathbf{Z}$ and let H be the subset of G given by

$$H = \{(x, y),\ x \text{ and } y \text{ are even integers}\}.$$

1. (5 points) Show that H is a subgroup of G and that it is normal.

2. (4 points) Describe the cosets of H and deduce the index of H.

3. (3 points) Which familiar group is the quotient G/H isomorphic to?

Exercise 4. Let p be a prime number.

1. (2 points) How many multiples of p are there in the set $\{1, 2, \ldots, p^2\}$?

2. (2 points) Compute $\phi(p^2)$ where ϕ is the Euler function.

3. (2 points) Show that for any integer x not divisible by p, we have $x^{p(p-1)} \equiv 1 \pmod{p^2}$.

4. (4 points) Determine the remainder of 54^{86} in the Euclidean division by 49.

Exercise 5. 1. (2 points) How many abelian groups of order 210 are there up to isomorphism?

2. (3 points) Let p and q be two distinct prime numbers. How many abelian groups of order p^2q^2 are there up to isomorphism? Describe all the possibilities.

3. Let G be an abelian group of order 100.

   (a) (2 points) Could it be that G does not contain an element of order 5?

(b) (3 points) Assume that G does not contain an element of order 4. What can you conclude about G?

Exercise 6. The aim of this exercise is to classify all groups of order 6, by proving that a group of order 6 is either cyclic isomorphic to $\mathbf{Z}/6\mathbf{Z}$, or isomorphic to $\mathfrak{S}_3$.

1. (4 points) *Preliminary question.* Show that if x, y are elements of a group G such that x is of order 3, y is of order 2 and $xy = yx$, then xy is of order 6.

From now on, let G be a group of order 6. **We assume that G has an element a of order 3 and an element b of order 2.**

2. (2 points) Consider the subgroup $H = \langle a \rangle$ generated by a. What is its index? By looking at the coset bH, show that $G = \{e, a, a^2, b, ba, ba^2\}$.

3. (4 points) Explain why we cannot have ab equal to any of the following: e, a, a^2, b.

Thus, it remains to consider the cases $ab = ba$ and $ab = ba^2$.

4. (2 points) Show that if $ab = ba$, then G is cyclic.

5. (2 points) **From now on we assume that** $ab = ba^2$**.** Show that $bab = a^2$.

6. (2 points) Deduce that ba is of order 2.

7. (3 points) Show that ba^2 is of order 2.

8. (6 points) Complete the following Cayley table for G. Use the space provided to give some details on how you do it, and write down a isomorphism between G and $\mathfrak{S}_3$. You do not need to prove that it is an isomorphism. *Do try this question even if you haven't solved some of the previous ones!*

	e	a	a^2	b	ba	ba^2
e						
a						
a^2						
b						
ba						
ba^2						

Exercise 7. 1. (5 points) Give a list of all of the subgroups of $\mathfrak{S}_3$, and say which of them are normal. It should be apparent from your answer why your list is exhaustive.

2. Let $f : \mathfrak{S}_3 \to G$ be a non-trivial group homomorphism to some group G. We assume moreover that f is not injective.

   (a) (3 points) What is $\mathrm{Ker} f$ equal to?

   (b) (2 points) Show that G must have a subgroup of order 2.

   (c) (1 point) Give an example of such a homomorphism.