Margaret Bilu
Office 604 Fall 2017
bilu@cims.nyu.edu

Algebra homework 1
Reminders on set theory

1 Sets

Exercise 1. In the set $X = \mathbf{R}$, consider the intervals $A = [2, \infty)$ and $B = [0, 5)$. Compute the following sets:

(a) A^c,

(b) $A \cup B$,

(c) $A \cap B$,

(d) $A \setminus B$,

(e) $B \setminus A$.

Exercise 2. Let A and B be two subsets of a set X. Prove the De Morgan laws:

(a) $(A \cap B)^c = A^c \cup B^c$

(b) $(A \cup B)^c = A^c \cap B^c$.

2 Mappings

Exercise 3. Let $f : \mathbf{R} \to \mathbf{R}$ be the map $f : x \mapsto x^2$. Compute the inverse image sets $f^{-1}(A)$ of the following sets A:

(a) $[0, 1]$,

(b) $\{-2\}$,

(c) $\{-1, 0, 4\}$,

(d) $[0, +\infty)$.

Exercise 4. Let $f : X \to Y$ be a map between sets.

1. For any two subsets A, B of X, show that

$$f(A) \cup f(B) = f(A \cup B).$$

2. (a) Show that in general

$$f(A) \cap f(B) \neq f(A \cap B) \tag{1}$$

by giving a counterexample. (Hint: draw a picture)

(b) Show that we do get equality in (1) if we furthermore assume that f is injective.

Exercise 5. Let $f : X \to Y$ be a map between sets.

1. For any two subsets A, B of Y, show that

$$f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B).$$

2. For any two subsets A, B of Y, show that

$$f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B).$$

Exercise 6. Let $f : X \to Y$ and $g : Y \to Z$ be maps between sets.

1. Show that if f and g are surjective, then so is $g \circ f$.
2. Show that if f and g are injective, then so if $g \circ f$.

3 Equivalence relations

Exercise 7. For an element $x = (x_1, x_2)$ of the plane $\mathbf{R}^2$, we denote by $||x|| = \sqrt{x_1^2 + x_2^2}$ its Euclidean norm. Let R be the relation on the plane $\mathbf{R}^2$ given by

$$x \sim_R y \quad \text{if} \quad ||x|| = ||y||.$$

Show that R is an equivalence relation and describe its equivalence classes.

Exercise 8. We define a relation R on the subsets of a nonempty set X by

$$A \sim_R B \quad \text{if} \quad A \cap B = \varnothing.$$

1. Is R reflexive?
2. Is it symmetric?
3. Is it transitive?

You must justify your answers.