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Algebra homework 10
Permutations, cosets

All answers have to be justified.

Exercise 1. Compute the left and the right cosets of the following subgroups H_i of the following groups G_i.

1. $H_1 = \langle 4 \rangle$ (subgroup generated by 4) in $G_1 = \mathbf{Z}/12\mathbf{Z}$.
2. $H_2 = 5\mathbf{Z}$ in $G_2 = \mathbf{Z}$.
3. $H_3 = \{\mathrm{id}, (1,2,3), (1,3,2)\}$ in $G_3 = \mathfrak{S}_3$.
4. $H_4 = \{\mathrm{id}, (1,3)\}$ in $G_4 = \mathfrak{S}_3$.
5. $H_5 = \mathfrak{A}_n$ in $G_5 = \mathfrak{S}_n$ (Hint: show that it cannot have more than two different cosets).

Exercise 2. Let G be a group and H a subgroup of G. We assume that H is such that for all $h \in H$ and all $g \in G$, the product ghg^{-1} is an element of H (we say that H is a *normal* subgroup of G).

1. Show that all subgroups of an abelian group are normal.
2. Show that for all $g \in G$, $gH = Hg$, that is, the right and the left cosets of H are the same.
3. Find an example of a group G and of a subgroup H of G which is not normal.

Exercise 3. Let H and K be subgroups of a group G.

1. Show that $H \cap K$ is a subgroup of G.
2. Show that for every $g \in G$, $gH \cap gK$ is a coset of $H \cap K$ in G.

Exercise 4. Let $n \geq 2$ and let $\phi : \mathfrak{S}_n \to \{1, -1\}$ be a non-trivial group homomorphism. The aim of this exercise is to prove that ϕ is equal to the sgn homomorphism.

1. Show that there exists at least one transposition $\tau \in \mathfrak{S}_n$ such that $\phi(\tau) = -1$.
2. Show that for all permutations $\alpha, \sigma \in \mathfrak{S}_n$, we have $\phi(\sigma\alpha\sigma^{-1}) = \phi(\alpha)$.
3. Show that for any transposition $\tau' \in \mathfrak{S}_n$, we have $\phi(\tau') = -1$.
4. Deduce that $\phi = \mathrm{sgn}$.

Exercise 5. Let $n \geq 3$ be an integer. The aim of this exercise is to show that $\mathfrak{A}_n$ is generated by cycles of length 3 of the form $(1, 2, i)$, $i \in \{3, \ldots, n\}$.

1. Check that this is true for $n = 3$.
2. Show that for all distinct $a, b, c \in \{2, \ldots, n\}$, we have the equality $(a, b, c) = (1, a, b)(1, b, c)$.
3. Deduce that $\mathfrak{A}_n$ is generated by cycles of length 3 of the form $(1, i, j)$, $i, j \in \{2, \ldots, n\}$.
4. Let $\sigma \in \mathfrak{A}_n$ be a cycle of length 3 containing 1 and 2. Show that either σ or σ^{-1} is of the form $(1, 2, i)$.
5. Let $\sigma \in \mathfrak{A}_n$ be a cycle of the form $(1, i, j)$. Show that we have
$$(1, i, j) = (1, 2, j)^{-1}(1, 2, i)(1, 2, j).$$
6. Conclude that $\mathfrak{A}_n$ is generated by cycles of length 3 of the form $(1, 2, i)$.