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Algebra homework 11
Index, Lagrange's theorem

All answers have to be justified.

Exercise 1. Compute the indexes of the following subgroups H_i of the following groups G_i.

1. $H_1 = \langle 3 \rangle$ (subgroup generated by 3) in $G_1 = \mathbf{Z}/81\mathbf{Z}$.
2. $H_2 = 23\mathbf{Z}$ in $G_2 = \mathbf{Z}$.
3. $H_3 = \{\mathrm{id}, (1,2,3), (1,3,2)\}$ in $G_3 = \mathfrak{S}_3$.
4. $H_4 = \{\mathrm{id}, (1,3)\}$ in $G_4 = \mathfrak{S}_3$.

Exercise 2. Let $f : \mathbf{Z}/9\mathbf{Z} \to \mathbf{Z}/9\mathbf{Z}$ given by $f(x) = 3x$.

1. Prove that f is a group homomorphism.
2. Compute $\mathrm{Ker}\, f$ and $\mathrm{Im}\, f$.
3. Check that $[\mathbf{Z}/9\mathbf{Z} : \mathrm{Ker}\, f] = |\mathrm{Im}\, f|$.

Exercise 3.
1. Give a list of all the subgroups of $\mathbf{Z}/14\mathbf{Z}$ together with their orders.
2. Check that
$$14 = \sum_{d|14} \phi(d)$$
where ϕ is Euler's function.

Exercise 4. Let G be a group of order 25.

1. Prove that G has at least one subgroup of order 5.
2. Prove that if G contains only one subgroup of order 5, then it is cyclic.

Exercise 5. Let $\phi : G \to G'$ be a group homomorphism. Assume that G is of order 18, G' is of order 15 and that ϕ is not the trivial homomorphism. What is the order of $\mathrm{Ker}\, \phi$?

Exercise 6. Does a group of order 16 necessarily contain an element of order 4? Justify your answer by either proving the existence of such an element or providing an example where it doesn't exist.

Exercise 7. Let $n \geq 1$ be an integer and let G be a group of order $2n$.

1. Prove that the number of elements of order 2 in G is odd. (Hint: recall that elements of order 2 are their own inverses.)
2. Show that G must contain a subgroup of order 2.

Exercise 8.
1. Find an integer x such that $x^2 \equiv -1 \pmod 5$.
2. Find an integer x such that $x^2 \equiv -1 \pmod{13}$.
3. Let p be a prime congruent to 3 modulo 4. Show that there is no solution to the equation $x^2 \equiv -1 \pmod p$.