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Algebra homework 5
Product groups, cyclic groups

Exercise 1. Show that every cyclic subgroup of a group G is abelian.

Exercise 2. Describe the following groups:

(a) The subgroup of $(\mathbf{Z}, +)$ generated by 5.

(b) The subgroup of $(M_n(\mathbf{R}), +)$ generated by the $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) The subgroup of $\mathbf{Z}/12\mathbf{Z}$ generated by 4.

(d) The subgroup of $\mathbf{Z}/14\mathbf{Z}$ generated by 3.

(e) The subgroup of $(\mathbf{C}^*, \cdot)$ generated by i.

(f) The subgroup of $(GL_2(\mathbf{R}), \cdot)$ generated by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Exercise 3.

1. Compute the orders of the elements of the group $((\mathbf{Z}/8\mathbf{Z})^\times, \cdot)$. Is it cyclic?
2. Show that the group $((\mathbf{Z}/7\mathbf{Z})^\times, \cdot)$ is cyclic by finding a generator.

Exercise 4. Show that the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is an element of infinite order of the group $(GL_2(\mathbf{R}), \cdot)$ of invertible 2×2 matrices with real coefficients. Give a formula for A^n in terms of $n \in \mathbf{Z}$.

Exercise 5. Let a and b be elements of a group G.

1. Show that a and a^{-1} have the same order.
2. Show that ab and ba have the same order.

Exercise 6. Let a and b be elements of a group G. Assume that each of the elements a, b and ab has order 2. Show that the set $H = \{1, a, b, ab\}$ is a subgroup of G, of order exactly 4.

Exercise 7. Let r, s be positive integers. Let x be an element of order r in a group G and y an element of order s in a group H. Show that the order of the element (x, y) in the group $G \times H$ is the least common multiple $\mathrm{lcm}(r, s)$ of r and s, that is, the smallest positive integer divisible both by r and by s.

Exercise 8. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

be elements of $GL_2(\mathbf{R})$. Show that A and B have finite orders but that their product AB has infinite order.