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Algebra homework 6
Homomorphisms, isomorphisms

Exercise 1. Show that the following maps are group homomorphisms and compute their kernels.

(a) $f : (\mathbf{R}^\times, \cdot) \to (GL_2(\mathbf{R}), \cdot)$ given by

$$f(x) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) $g : (\mathbf{R}, +) \to (GL_2(\mathbf{R}), \cdot)$ given by

$$g(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

(c) $h : (\mathbf{R}^2, +) \to (\mathbf{R}, +)$ given by $h(x, y) = x$.

(d) The complex conjugation map $j : (\mathbf{C}, +) \to (\mathbf{C}, +)$, given by $j(x + iy) = x - iy$.

(e) $k : G \to G$ given by $k(x) = x^n$ if G is an abelian group (written in multiplicative notation). What if G is not abelian?

Exercise 2. Let $\phi : G \to H$ be a group homomorphism.

1. Show that if G is abelian, then $\mathrm{Im}(\phi)$ is also abelian.
2. Show that if G is cyclic, then $\mathrm{Im}(\phi)$ is also cyclic.

Exercise 3. Let T denote the group of invertible upper triangular 2×2 matrices

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad a, b, d \in \mathbf{R}, \ ad \neq 0.$$

1. Show that T is a subgroup of $GL_2(\mathbf{R})$.
2. Let $\phi : T \to \mathbf{R}^\times$ be the map given by sending a matrix A as above to a^2. Show that ϕ is a homomorphism, and give its kernel and image.

Exercise 4. Show that all the non-trivial subgroups of $\mathbf{Z}$ are isomorphic to $\mathbf{Z}$.

Exercise 5. Recall that the group $(\mathbf{Z}/8\mathbf{Z})^\times$ is of order 4. Is it isomorphic to $\mathbf{Z}/4\mathbf{Z}$? If not, find another group of order 4 it is isomorphic to.

Exercise 6.
1. Show that for any $a \in \mathbf{Z}$, the map $\phi : \mathbf{Z} \to \mathbf{Z}$ defined by $\phi(n) = an$ is a group homomorphism. Give its kernel and image.
2. Conversely, show that a homomorphism $\phi : \mathbf{Z} \to \mathbf{Z}$ is of the form $\phi(n) = an$ for some $a \in \mathbf{Z}$. Thus, the homomorphisms $\mathbf{Z} \to \mathbf{Z}$ are exactly the maps $n \mapsto an$.
3. Determine all the automorphisms of $\mathbf{Z}$, that is, the isomorphisms $\mathbf{Z} \to \mathbf{Z}$.

Exercise 7. Let G and H be two groups. Show that $G \times H$ is isomorphic to $H \times G$.

Exercise 8. Are the groups $\mathbf{Z}/6\mathbf{Z}$ and $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$ isomorphic? Justify your answer by either constructing an isomorphism or explaining why it does not exist.