Margaret Bilu
Office 604

bilu@cims.nyu.edu

Algebra homework 7
Groups

Exercise 1. Prove that if a group G has no proper subgroups, then G is a cyclic group.

Exercise 2. Show that of p is prime, $\mathbf{Z}/p\mathbf{Z}$ has no proper subgroups.

Exercise 3. An *automorphism* of a group G is an isomorphism from G to itself.

1. Let $n \geq 1$ be an integer. Prove that for all $B \in GL_n(\mathbf{R})$, the map $A \mapsto BAB^{-1}$ is an automorphism of $SL_n(\mathbf{R})$.
2. We denote by $\mathrm{Aut}(G)$ the set of automorphisms of the group G. Show that $(\mathrm{Aut}(G), \circ)$ is a group.
3. Find $\mathrm{Aut}(G)$ when
   (a) $G = \mathbf{Z}/2\mathbf{Z}$,
   (b) $G = \mathbf{Z}$,
   (c) $G = \mathbf{Z}/6\mathbf{Z}$.
4. Find two non-isomorphic groups G and H such that $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$ are isomorphic.
5. Let G be a group and $g \in G$. Define a map $i_g : G \to G$ by
   $$i_g(x) = gxg^{-1}.$$
   Prove that i_g defines an automorphism of G. Such an automorphism is called an *inner automorphism*. The set of all inner automorphisms is denoted by $\mathrm{Inn}(G)$.
6. Prove that $\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G)$.
7. What is $\mathrm{Inn}(G)$ if G is abelian?
8. Find an example of an automorphism of some group G which is not an inner automorphism.
9. For a group G, define a map $i : G \to \mathrm{Aut}(G)$ by sending $g \in G$ to the automorphism i_g defined above. Show that i is a group homomorphism. Show that its kernel is the center of G (see homework 4). What is its image?

Exercise 4. In this exercise, we want to describe the homomorphisms $f : (\mathbf{Q}, +) \to (\mathbf{Q}, +)$, where $\mathbf{Q}$ is the set of rational numbers.

1. Show that for all $x \in \mathbf{Q}$ and for all $n \in \mathbf{Z}$, we have $f(nx) = nf(x)$.
2. Deduce from this that for any rational number $\frac{p}{q}$, we have
   $$f\left(\frac{p}{q}\right) = \frac{f(p)}{q}.$$
3. Conclude that f is of the form $x \mapsto ax$ for some rational number a.
4. Check that any map of this form is indeed a homomorphism.
5. Deduce $\mathrm{Aut}(\mathbf{Q})$.