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Algebra homework 8
Permutations

All answers have to be justified. For every $n \geq 1$ we denote by $\mathfrak{S}_n$ the n-th symmetric group.

Exercise 1. Consider the following permutations:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 4 & 2 & 1 & 5 & 7 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 3 & 7 & 6 & 1 & 8 & 4 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 6 & 7 & 1 & 5 & 2 \end{pmatrix}.$$

1. Compute $\sigma_1\sigma_3$.
2. Recall that both σ_1 and σ_3 can be seen as elements of $\mathfrak{S}_8$ by putting $\sigma_1(8) = \sigma_3(8) = 8$. Compute the products $\sigma_1\sigma_2$ and $\sigma_2\sigma_3$ in $\mathfrak{S}_8$.
3. Decompose σ_1, σ_2 and σ_3 into products of disjoint cycles, and then write each of them as a product of transpositions.

Exercise 2. Write the following permutations as products of disjoint cycles:

1. $(1, 2, 7, 5)(2, 6, 1)$
2. $(1, 5, 2)(2, 3)(5, 7)(1, 2, 3)$
3. $(1, 3, 4)^{100}$
4. $(1, 3, 5, 6)^{-1}$

Exercise 3. Let $a_1, \ldots, a_k$ be distinct elements of $\{1, \ldots, n\}$. Compute the inverse in $\mathfrak{S}_n$ of the cycle $(a_1, \ldots, a_k)$.

Exercise 4. Compute the sets

$$E = \{\sigma \in \mathfrak{S}_4,\ \sigma(1) = 3\}$$

and

$$F = \{\sigma \in \mathfrak{S}_4,\ \sigma(2) = 2\}.$$

Are they subgroups of $\mathfrak{S}_4$?

Exercise 5. Prove that if σ is a cycle of odd length, then σ^2 is also a cycle. Show that this is not true for cycles of even length by giving a counterexample.

Exercise 6. Let $\sigma \in \mathfrak{S}_n$.

1. Show that σ can be written as a product of at most $n - 1$ transpositions.
2. Show that if σ is not a cycle, then σ can be written as a product of at most $n-2$ transpositions.

Exercise 7. Let $\sigma = \sigma_1 \ldots \sigma_k \in \mathfrak{S}_n$, where $\sigma_1, \ldots, \sigma_k$ are disjoint cycles. Prove that the order of σ is the least common multiple of the lengths of the cycles $\sigma_1, \ldots, \sigma_k$.

Exercise 8.

1. Give a list of all possible orders of an element of $\mathfrak{S}_4$, then of $\mathfrak{S}_5$.
2. Show that an element of $\mathfrak{S}_7$ can't have order 8, 9 or 11.
3. Give a list of all possible orders of an element of $\mathfrak{S}_7$.