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Algebra homework 9
Permutations

All answers have to be justified. For every $n \geq 1$ we denote by $\mathfrak{S}_n$ the n-th symmetric group.

Exercise 1. Compute the signs of the following permutations:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 4 & 6 & 2 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 1 & 3 & 8 & 6 & 2 & 7 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 6 & 7 & 1 & 5 & 2 \end{pmatrix}, \quad \sigma_4 = (1,3,4)(5,2), \quad \sigma_5 = (1,2)(1,6)(2,3)(5,3).$$

Exercise 2. Let $\sigma \in \mathfrak{S}_n$. Prove that

1. $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$.
2. for all $\alpha \in \mathfrak{S}_n$, $\operatorname{sgn}(\alpha\sigma\alpha^{-1}) = \operatorname{sgn}(\sigma)$.

Exercise 3. Let $n \geq 1$ and let $e_1, \dots, e_n$ be the usual basis vectors of $\mathbf{R}^n$. For all $\mathfrak{S}_n$ we define the matrix $M_\sigma \in M_n(\mathbf{R})$ to be the matrix such that for all $i \in \{1, \dots, n\}$ its coefficient at column i and row $\sigma(i)$ is 1, all other coefficients being equal to zero. For example, for the transposition (12) in $\mathfrak{S}_2$, we have $M_{(12)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

1. Compute M_{id} where $\text{id} \in \mathfrak{S}_n$ is the identity permutation.
2. Compute M_σ for all $\sigma \in \mathfrak{S}_3$.
3. Explain why for all $\sigma \in \mathfrak{S}_n$, there is exactly one coefficient equal to 1 in each row of M_σ, as well as in each column.
4. What is the image $M_\sigma e_i$ of the basis vector e_i by M_σ?
5. Show that for all permutations $\sigma, \tau \in \mathfrak{S}_n$, we have $M_{\sigma\tau} = M_\sigma M_\tau$.
6. Show that for every $\sigma \in \mathfrak{S}_n$, M_σ is an invertible matrix.
7. Show that the map $\phi : \mathfrak{S}_n \to (GL_n(\mathbf{R}), \cdot)$ defined by $\sigma \mapsto M_\sigma$ is an injective group homomorphism.

Exercise 4. Recall that the center of the group $\mathfrak{S}_n$ is defined by

$$Z(\mathfrak{S}_n) = \{\sigma \in \mathfrak{S}_n | \text{ for all } \alpha \in \mathfrak{S}_n,\ \alpha\sigma = \sigma\alpha\}.$$

1. Show that $\text{id} \in Z(\mathfrak{S}_n)$.
2. Compute $Z(\mathfrak{S}_n)$ for $n = 1, 2, 3$.
3. We now assume $n \geq 3$ and pick $\sigma \in \mathfrak{S}_n$ different from the identity.
   (a) Show that there exist $i, j \in \{1, \dots, n\}$ distinct such that $\sigma(i) = j$.
   (b) Construct a transposition α such that $\alpha\sigma\alpha^{-1}(i) \neq j$.
   (c) Conclude that $Z(\mathfrak{S}_n) = \{\text{id}\}$.