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Midterm
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The use of notes and electronic devices is forbidden. All answers have to be justified. You can use all results proved in lectures without proof. If you use a result seen in a homework or in recitation you need to prove it.

Exercise 1. 1. Let $f : G \to H$ be a group homomorphism. Define Ker f and show that it is a subgroup of G.

2. Let A be the subset of $M_2(\mathbf{R})$ given by

$$A = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{R}), \ a + d = 0 \right\}.$$

Is it a subgroup of $(M_2(\mathbf{R}), +)$?

Exercise 2. We denote the class of an integer n in $\mathbf{Z}/10\mathbf{Z}$ by $[n]$.

1. (a) Give the order of $[2]$ in the group $(\mathbf{Z}/10\mathbf{Z}, +)$. Is it a generator?

   (b) Show that the group $\mathbf{Z}/10\mathbf{Z}$ is cyclic and give the list of all its generators.

   (c) Give an element of order 2 in $\mathbf{Z}/10\mathbf{Z}$.

   (d) Give a subgroup of order 5 of $\mathbf{Z}/10\mathbf{Z}$.

2. Let $\phi : \mathbf{Z}/10\mathbf{Z} \to \mathbf{Z}/10\mathbf{Z}$ be a group homomorphism.

   (a) Can we have $\phi([0]) = [3]$?

   (b) If $\phi([2]) = [4]$, what is $\phi([8])$?

   (c) If $\phi([1]) = [2]$, can ϕ be an isomorphism?

   (d) Show that ϕ is of the form $[n] \mapsto a[n]$ for some $a \in \mathbf{Z}/10\mathbf{Z}$.

   (e) Check that any map of this type is indeed a homomorphism.

   (f) For which values of a is the homomorphism $[n] \mapsto a[n]$ an isomorphism?

3. (a) What is the order of the group $((\mathbf{Z}/10\mathbf{Z})^\times, \cdot)$? Give all its elements and compute their inverses.

   (b) Describe the group $(\mathbf{Z}/10\mathbf{Z})^\times$ by giving its Cayley table.

   (c) Is $(\mathbf{Z}/10\mathbf{Z})^\times$ cyclic? If yes, give a generator.

   (d) Is there an element of order 3 in $(\mathbf{Z}/10\mathbf{Z})^\times$?

   (e) Give an example of a proper subgroup of $(\mathbf{Z}/10\mathbf{Z})^\times$.

Exercise 3. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be the map defined by $(x, y) \mapsto (x + y, y)$.

1. Show that f is a group homomorphism.

2. Determine its kernel and its image.

3. Is f an isomorphism?

Exercise 4. 1. Give a list of all groups of order at most 4 up to isomorphism.

2. Let G be a group of order 4 having at least two distinct elements of order 2. Determine which of the groups in the list from the previous question it is isomorphic to.

3. Give a list of all the subgroups of this group.