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Algebra practice problems

Exercise 1. Let G be a group and let H_1, H_2 be normal subgroups of G. Show that $H_1 \cap H_2$ is a normal subgroup of G.

Exercise 2. Let H be a subgroup of a group G. The *centralizer* of H in G is defined to be the set

$$C_H(G) = \{x \in G,\ xh = hx \text{ for all } h \in H\}.$$

1. Show that $C_H(G)$ is a subgroup of G.
2. Show that if H is normal, then $C_H(G)$ is normal.

Exercise 3.
1. Find a permutation $\sigma \in \mathfrak{S}_9$ such that $\sigma(1,2)(3,4)\sigma^{-1} = (5,6)(3,1)$.
2. Does there exist $\sigma \in \mathfrak{S}_9$ such that $\sigma(1,2,3)\sigma^{-1} = (2,3)(1,6,7)$?
3. Does there exist $\sigma \in \mathfrak{S}_9$ such that $\sigma(1,2,4)\sigma^{-1} = (2,5)(1,3)$?

Exercise 4. The orthogonal group $O_n(\mathbf{R})$ is the subset of $M_n(\mathbf{R})$ given by

$$O_n(\mathbf{R}) = \{M \in M_n(\mathbf{R}),\ M^tM = MM^t = I_n\}$$

where M^t denotes the transpose of a matrix M. We recall that for any matrix M, M and M^t have the same determinant.

1. Show that $O_n(\mathbf{R})$ is a subgroup of $(GL_n(\mathbf{R}), \cdot)$.
2. We define the special orthogonal group $SO_n(\mathbf{R})$ to be the subset of $O_n(\mathbf{R})$ of matrices with determinant 1:
$$SO_n(\mathbf{R}) = \{M \in O_n(\mathbf{R}),\ \det(M) = 1.\}$$
Show that $SO_n(\mathbf{R})$ is a normal subgroup of $O_n(\mathbf{R})$.
3. Show that $SO_n(\mathbf{R})$ has index 2 in $O_n(\mathbf{R})$ and that $O_n(\mathbf{R})/SO_n(\mathbf{R})$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z}, +)$.
4. Check that for any real number θ, the matrix
$$M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
is an element of $SO_2(\mathbf{R})$.
5. Check that the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is an element of $O_2(\mathbf{R})$. Is it an element of $SO_2(\mathbf{R})$?

Exercise 5. let G be a group and let H be the *commutator subgroup* of G, that is, the set of all finite products of elements of the form $aba^{-1}b^{-1}$ for $a, b \in G$.

1. Show that H is a normal subgroup of G.
2. Show that the quotient G/H is abelian.
3. More generally, for any normal subgroup N of G, show that G/N is abelian if and only if N contains H.

Exercise 6. Let σ be the element of $\mathfrak{S}_9$ given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 4 & 7 & 9 & 6 & 1 & 3 & 5 & 2 \end{pmatrix}.$$

1. Give a decomposition of σ into disjoint cycles.
2. Determine the sign of σ.
3. What is the order of σ in $\mathfrak{S}_9$?

Exercise 7. In $\mathfrak{S}_4$, consider the subset

$$H = \left\{ \mathrm{id}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \right\}.$$

1. Compute the inverses of the elements of H in $\mathfrak{S}_4$.
2. Is H a subgroup of $\mathfrak{S}_4$?

Exercise 8. Let $n \geq 1$ be an integer and let $H = \{\sigma \in \mathfrak{S}_n,\ \sigma(1) = 1.\}$.

1. Show that H is a subgroup of $\mathfrak{S}_n$.
2. Write down all the elements of H when $n = 1$, $n = 2$ and $n = 3$.
3. When $n \geq 3$, show that H is not a normal subgroup of $\mathfrak{S}_n$.

Exercise 9. Let G be a group. Recall that the *center* of G is the subgroup of G given by

$$Z(G) = \{x \in G,\ xg = gx \text{ for all } g \in G\}.$$

1. Show that $Z(G)$ is a normal subgroup of G.
2. We assume that the quotient group $G/Z(G)$ is cyclic.
   (a) Show that this implies the existence of some element $t \in G$ such that for all $a \in G$, the coset $aZ(G)$ is equal to $t^n Z(G)$ for some $n \in \mathbf{Z}$.
   (b) Show that if $aZ(G) = t^n Z(G)$, then there exists $x \in Z(G)$ such that $a = t^n x$.
   (c) Deduce from this that G is abelian.

Exercise 10. Let G be a group and let H be a subgroup of G. Recall that for all $g \in G$, gHg^{-1} is a subgroup of G. We define N to be the intersection of all these subgroups.

1. Show that it is a normal subgroup of G.
2. Show that if H is normal, then $H = N$.
3. Compute N when $G = \mathfrak{S}_3$ and $H = \{\mathrm{id}, (12)\}$.