Margaret Bilu
Office 604 Fall 2017
bilu@cims.nyu.edu

Algebra practice problems
Hints and solutions

Note: we do not give solutions to the questions where one needs to prove that something is a (normal) subgroup. The procedure is always the same, one should check the three axioms SG1, SG2, SG3 (closure, identity, inverses). To check that it is normal, use the definition.

Exercise 1. Let G be a group and let H_1, H_2 be normal subgroups of G. Show that $H_1 \cap H_2$ is a normal subgroup of G.

Exercise 2. Let H be a subgroup of a group G. The *centralizer* of H in G is defined to be the set

$$C_H(G) = \{x \in G,\ xh = hx \text{ for all } h \in H\}.$$

1. Show that $C_H(G)$ is a subgroup of G.

2. Show that if H is normal, then $C_H(G)$ is normal.

   Solution. Assume that H is normal. We need to prove that for every $x \in C)H(G)$ and for every $g \in G$, $gxg^{-1} \in C_H(G)$. For this, by definition of $C_H(G)$, we need to prove that for every $h \in H$,
   $$gxg^{-1}h = hgxg^{-1}. \tag{1}$$
   Now, since h is normal, we have $g^{-1}hg \in H$, and therefore, since $x \in C_H(G)$, by definition of $C_H(G)$, we have that
   $$x(g^{-1}hg) = (g^{-1}hg)x.$$
   Multiplying by g on the left and by g^{-1} on the right, we get equality (1).

Exercise 3.

1. Find a permutation $\sigma \in \mathfrak{S}_9$ such that $\sigma(1,2)(3,4)\sigma^{-1} = (5,6)(3,1)$.

   Solution. We have
   $$\sigma(1,2)(3,4)\sigma^{-1} = \sigma(1,2)\sigma^{-1}\sigma(3,4)\sigma^{-1},$$
   so it suffices to find σ such that simultaneously,
   $$\sigma(1,2)\sigma^{-1} = (5,6)$$
   and
   $$\sigma(3,4)\sigma^{-1} = (3,1).$$
   By a formula seen in class, it suffices to find σ such that $\sigma(1) = 2$, $\sigma(2) = 6$, $\sigma(3) = 3$ and $\sigma(4) = 1$. Take
   $$\sigma = (4,1,5)(2,6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 3 & 1 & 4 & 2 & 7 & 8 & 9 \end{pmatrix}.$$

2. Does there exist $\sigma \in \mathfrak{S}_9$ such that $\sigma(1,2,3)\sigma^{-1} = (2,3)(1,6,7)$?

   Solution. Look at signs: the left-hand side is even, the right-hand side is odd, so this is impossible.

3. Does there exist $\sigma \in \mathfrak{S}_9$ such that $\sigma(1,2,4)\sigma^{-1} = (2,5)(1,3)$?

   Solution. Here both sides are even, so the sign argument does not work. However, by a formula seen in class, we must have

   $$\sigma(1,2,4)\sigma^{-1} = (\sigma(1), \sigma(2), \sigma(3)),$$

   that is, the left-hand side is a cycle of length 3, whereas the right-hand side is not, so this is impossible.

Exercise 4. The orthogonal group $O_n(\mathbf{R})$ is the subset of $M_n(\mathbf{R})$ given by

$$O_n(\mathbf{R}) = \{M \in M_n(\mathbf{R}),\ M^t M = M M^t = I_n\}$$

where M^t denotes the transpose of a matrix M. We recall that for any matrix M, M and M^t have the same determinant.

1. Show that $O_n(\mathbf{R})$ is a subgroup of $(GL_n(\mathbf{R}), \cdot)$.

2. We define the special orthogonal group $SO_n(\mathbf{R})$ to be the subset of $O_n(\mathbf{R})$ of matrices with determinant 1:

   $$SO_n(\mathbf{R}) = \{M \in O_n(\mathbf{R}),\ \det(M) = 1.\}$$

   Show that $SO_n(\mathbf{R})$ is a normal subgroup of $O_n(\mathbf{R})$.

3. Show that $SO_n(\mathbf{R})$ has index 2 in $O_n(\mathbf{R})$ and that $O_n(\mathbf{R})/SO_n(\mathbf{R})$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z}, +)$.

   Hint: Show that the determinant of an element of $O_n(\mathbf{R})$ is either 1 or -1.

4. Check that for any real number θ, the matrix

   $$M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

   is an element of $SO_2(\mathbf{R})$.

5. Check that the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is an element of $O_2(\mathbf{R})$. Is it an element of $SO_2(\mathbf{R})$?

Exercise 5. Let G be a group and let H be the *commutator subgroup* of G, that is, the set of all finite products of elements of the form $aba^{-1}b^{-1}$ for $a, b \in G$.

1. Show that H is a normal subgroup of G.

   Solution. To check that it is a subgroup, check all the subgroup axioms. To show that it is normal, write for all $g \in G$

   $$gaba^{-1}b^{-1}g^{-1} = gag^{-1}gbg^{-1}ga^{-1}g^{-1}gb^{-1}g^{-1} = (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1}.$$

2. Show that the quotient G/H is abelian.

   Solution. Let aH and bH be two cosets. We want to show that

   $$(aH)(bH) = (bH)(aH).$$

   By definition, $(aH)(bH) = abH$ and $(bH)(aH) = baH$. Since H is normal, these left cosets are equal to On the other hand, since $ab(ba)^{-1} = aba^{-1}b^{-1} \in H$, we have that $ab \in Hba = baH$ (right cosets and left cosets are the same), so the cosets abH and baH are the same, whence the result.

3. More generally, for any normal subgroup N of G, show that G/N is abelian if and only if N contains H.

   Solution. By the same method as above, if N contains H, then G/N is abelian. Conversely, if G/N is abelian, then this means that for all $a, b \in G$, $(aN)(bN) = (bN)(aN)$, that is, $abN = baN$. Since N is normal, this implies $Nab = Nba$ (right cosets are same as left cosets), so $ab(ba)^{-1} \in N$, i.e. $aba^{-1}b^{-1} \in N$. Thus, N contains all of the elements of the form $aba^{-1}b^{-1}$ for $a, b \in G$. By closure, it contains all the finite products of such elements, and therefore it contains H.

Exercise 6. Let σ be the element of $\mathfrak{S}_9$ given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 4 & 7 & 9 & 6 & 1 & 3 & 5 & 2 \end{pmatrix}.$$

1. Give a decomposition of σ into disjoint cycles.

   Solution. We have $\sigma = (1, 8, 5, 6)(2, 4, 9)(3, 7)$.

2. Determine the sign of σ.

   Solution. Using multiplicativity of the sign and the fact that a cycle of length k has sign $(-1)^{k-1}$, we see that is even.

3. What is the order of σ in $\mathfrak{S}_9$?

   Solution. Observe that $\sigma^{12} = \mathrm{id}$ (to compute more quickly, use that disjoint cycles commute, and that a cycle of length k is of order k), so that the order of σ divides 12. It is therefore equal to $1, 2, 3, 4, 6$ or 12. It cannot be 1 because $\sigma \neq \mathrm{id}$. We compute

$$\sigma^2 = (1, 8, 5, 6)^2(2, 4, 9)^2(3, 7)^2 = (1, 5)(8, 6)(2, 9, 4) \neq \mathrm{id}$$

$$\sigma^3 = (1, 8, 5, 6)^3(2, 4, 9)^3(3, 7)^3 = (1, 6, 5, 8)(3, 7) \neq \mathrm{id}.$$

$$\sigma^4 = (1, 8, 5, 6)^4(2, 4, 9)^4(3, 7)^4 = (2, 4, 9) \neq \mathrm{id}.$$

$$\sigma^6 = (1, 8, 5, 6)^6(2, 4, 9)^6(3, 7)^6 = (1, 8, 5, 6)^2 = (1, 5)(8, 6) \neq \mathrm{id}.$$

   Therefore, the order of σ is 12.

Exercise 7. In $\mathfrak{S}_4$, consider the subset

$$H = \left\{ \mathrm{id}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \right\}.$$

1. Compute the inverses of the elements of H in $\mathfrak{S}_4$.

   Solution. You should find that $\mathrm{id}^{-1} = \mathrm{id}$, the first two non-trivial elements are inverse to each other, and the last element is its own inverse.

2. Is H a subgroup of $\mathfrak{S}_4$?

   Solution. No, it does not satisfy closure, looking e.g. at the product of the two last elements.

Exercise 8. Let $n \geq 1$ be an integer and let $H = \{\sigma \in \mathfrak{S}_n,\ \sigma(1) = 1.\}$.

1. Show that H is a subgroup of $\mathfrak{S}_n$.

2. Write down all the elements of H when $n = 1$, $n = 2$ and $n = 3$.

3. When $n \geq 3$, show that H is not a normal subgroup of $\mathfrak{S}_n$.

   Solution. If $n \geq 3$, then H is not the trivial subgroup $\{\mathrm{id}\}$, and therefore it contains some $\sigma \neq \mathrm{id}$. Then there exists $i > 1$ such that $\sigma(i) \neq i$. By injectivity of σ, $\sigma(i) \neq \sigma(1) = 1$. Look at the permutation $\alpha = (1,i)\sigma(1,i)^{-1}$: we have

   $$\alpha(1) = (1,i)\sigma(i) = \sigma(i) \neq 1,$$

   since $\sigma(i) \notin \{1, i\}$. Therefore, $\alpha \notin H$, and so H is not normal.

Exercise 9. Let G be a group. Recall that the *center* of G is the subgroup of G given by

$$Z(G) = \{x \in G,\ xg = gx \text{ for all } g \in G\}.$$

1. Show that $Z(G)$ is a normal subgroup of G.

2. We assume that the quotient group $G/Z(G)$ is cyclic.

   (a) Show that this implies the existence of some element $t \in G$ such that for all $a \in G$, the coset $aZ(G)$ is equal to $t^n Z(G)$ for some $n \in \mathbf{Z}$.

   Solution. Since $G/Z(G)$ is cyclic, it is generated by some coset $tZ(G)$ for some $t \in G$. This means that for all $aZ(G)$, there is $n \in \mathbf{Z}$ such that $aZ(G) = (tZ(G))^n = t^n Z(G)$, where the last equality comes from the definition of the group law in $G/Z(G)$.

   (b) Show that if $aZ(G) = t^n Z(G)$, then there exists $x \in Z(G)$ such that $a = t^n x$. *Solution.* Two elements a and b define the same coset if and only if $b^{-1}a \in Z(G)$, so if and only if $a = bx$ for some $x \in Z(G)$. Apply this to $b = t^n$.

   (c) Deduce from this that G is abelian.

   Solution. Let $a, b \in G$. We want to prove that $ab = ba$. Using the previous question, we may write $a = t^n x$ and $b = t^m y$ for $m, n \in \mathbf{Z}$ and $x, y \in Z(G)$. Then we have

   $$\begin{aligned} ab &= t^n x t^m y \\ &= t^n t^m x y \quad \text{because } x \in Z(G) \\ &= t^{n+m} x y \\ &= t^m t^n x y \\ &= t^m t^n y x \quad \text{because } x \in Z(G) \\ &= t^m y t^n x \quad \text{because } y \in Z(G) \\ &= ba \end{aligned}$$

Exercise 10. Let G be a group and let H be a subgroup of G. Recall that for all $g \in G$, gHg^{-1} is a subgroup of G. We define N to be the intersection of all these subgroups.

1. Show that it is a normal subgroup of G.

2. Show that if H is normal, then $H = N$.

   Solution. If H is normal, then for all $g \in G$, $gHg^{-1} = H$, so the intersection of all of these subgroups is H.

3. Compute N when $G = \mathfrak{S}_3$ and $H = \{\mathrm{id}, (12)\}$.

   Solution. Compute e.g. $(12)H(12)^{-1} = H$ and $(13)H(13)^{-1} = \{\mathrm{id}, (23)\}$. Already the intersection of these two subgroups is trivial, so the total intersection will be trivial as well.