Motivic Euler products

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1 What do we mean by "motivic"?

1.1 The Grothendieck ring of varieties

The Grothendieck group of varieties was introduced by Grothendieck in the same 1964 letter to Serre in which he also introduced motives.

Definition 1.1. The Grothendieck group of varieties over a field k is the quotient of the free abelian group generated by isomorphism classes of varieties over k, by the relations

$$[X] - [Z] - [X \setminus Z]$$

for any variety X and any closed subscheme Z of X.

This group may be endowed with a commutative ring structure, given by $[X][X'] = [X \times_k X']$, with unit element the class $[\mathrm{Spec}\, k]$.

We denote by $\mathbf{L}$ the class of the affine line $[\mathbf{A}^1_k]$. We then have, for every $n \geq 1$, $[\mathbf{A}^n_k] = \mathbf{L}^n$, and $[\mathbf{P}^n_k] = \mathbf{L}^n + \ldots + \mathbf{L} + 1$.

One also often considers the localisation

$$\mathscr{M}_k = \mathrm{KVar}_k[\mathbf{L}^{-1}]$$

which is a non-trivial operation, since $\mathbf{L}$ is a zero-divisor in KVar_k (this is a result by Borisov, [Bor]).

1.2 The counting measure

Assume $k = \mathbf{F}_q$. Then there is a well-defined ring morphism

$$\mathrm{KVar}_k \to \mathbf{Z}$$

sending the class of a variety X to $|X(\mathbf{F}_q)|$.

Assume you are given some result involving point counts of a variety over $\mathbf{F}_q$. Then one may ask if this result is already true on the level of classes in KVar_k, and if yes, if it is also true when k is no longer necessarily a finite field, and the counting measure doesn't exist any more.

1.3 Kapranov's motivic zeta function

For a variety X over a finite field $k = \mathbf{F}_q$, we may define its zeta function:

$$\zeta_X(t) = \exp\left(\sum_{m\geq 1} \frac{|X(\mathbf{F}_{q^m})|}{m} t^m\right) \in \mathbf{Q}[[t]],$$

It has the following Euler product decomposition:

$$\zeta_X(t) = \prod_{x\in X} \frac{1}{1-t^{\deg x}} = \prod_{x\in X}(1+t^{\deg x}+t^{2\deg x}+\ldots).$$

Expanding this Euler product, one gets another expression for $\zeta_X(t)$, which shows in particular that it has integral coefficients:

$$\zeta_X(t) = \sum_{n\geq 0} |\{\text{effective zero-cycles of degree } n \text{ on } X\}| t^n \in \mathbf{Z}[[t]].$$

For a quasi-projective variety X, define $S^nX = X^n/\mathfrak{S}_n$ the n-th symmetric power of X. Then the rational points of S^nX are exactly the effective zero-cycles of degree n on X, and therefore we may define the following motivic analogue of $\zeta_X(t)$:

$$Z_X(t) = \sum_{n\geq 0}[S^nX]t^n \ \in \ \mathrm{KVar}_k[[t]].$$

This makes sense for any field k, and for quasi-projective X. For $k = \mathbf{F}_q$, this lifts the above zeta function $\zeta_X(t)$.

One may therefore ask which properties of $\zeta_X(t)$ are already true at the motivic level. For example, since it is known that $\zeta_X(t)$ is always rational, one could ask about the rationality of $Z_X(t)$.

1. When X is a curve with a rational point, it has been proved by Kapranov himself in [Kapr] that $Z_X(t)$ is rational.

2. However, Larsen and Lunts showed in [LL] that this fails in general for surfaces: more precisely, they proved that for a surface X over an algebraically closed field k, $Z_X(t)$ is rational in $\mathrm{KVar}_k[[t]]$ if and only if X is of negative Kodaira dimension. They also very recently showed in [LL18] that there exist surfaces X (namely, some K3's), such that $Z_X(t)$ is not rational in $\mathscr{M}_k[[t]]$.

In what follows we are going to address the existence of an Euler product decomposition for $Z_X(t)$.

2 Motivic Euler products

We want to have a definition of Euler products such that in particular, $Z_X(t)$ would have decomposition

$$\prod_{x\in X}\left(1+t^{\deg x}+t^{2\deg x}+\ldots\right)$$

To achieve this, we are in fact going to endow a more general family of power series with an Euler product decomposition.

From now on, we assume k to be algebraically closed, so that we don't have to bother ourselves with degrees of points.

Let X be a variety over k, and $\mathscr{X} = (X_i)_{i\geq 1}$ a family of varieties over X. Our aim is to make sense of an infinite product of the form

$$\prod_{x\in X}\left(1+X_{1,x}t+X_{2,x}t^2+\ldots\right),$$

where each coefficient $X_{i,x}$ has to be thought of as the class of the fibre on $X_{i,x}$ in KVar_k.

Expanding this product naively gives us

$$1 + \sum_{x \in X} X_{1,x} t$$

for the two first terms. For the term of degree two, we notice that there are two different contributions: either we choose the term $X_{1,x}t$ in two distinct factors, and 1 in all the others, or we choose the term $X_{2,x}t^2$ in one factor and 1 in all the others. In other words, the coefficient of order two decomposes into pieces corresponding to the partitions $[1,1]$ and $[2]$ of the integer 2. More generally, the coefficient of degree n will decompose into pieces corresponding to different partitions of the integer n.

Let us fix a partition $\pi = (n_i)_{i \geq 1}$ of n, where n_i denotes the number of occurrences of the integer i, so that $n = \sum_{i \geq 1} i n_i$. We are going to give a geometric construction of the contribution to the coefficient of degree n corresponding to to the partition π.

First of all, to correspond to this partition, we need to choose the term of degree i in n_i factors, so we start with the product

$$\prod_{i \geq 1} X_i^{n_i}.$$

Since each variety X_i comes with a morphism to X, this has a natural morphism

$$\prod_{i \geq 1} X_i^{n_i} \to \prod_{i \geq 1} X^{n_i}.$$

Moreover, different factors correspond to different points $x \in X$ so we need to restrict to points above the complement

$$\left(\prod_{i \geq 1} X^{n_i} \right)_*$$

of the diagonal of $\prod_{i \geq 1} X^{n_i}$, that is, points with pairwise distinct coordinates. This gives us an open subset

$$\left(\prod_{i \geq 1} X_i^{n_i} \right)_* \to \left(\prod_{i \geq 1} X^{n_i} \right)_*.$$

Finally, there is no particular order between factors where we chose a term of the same degree, so that we will take the quotient with respect to the natural action of the product $\prod_{i \geq 1} \mathfrak{S}_{n_i}$ of symmetric groups. This gives us

$$S^\pi \mathscr{X} := \left(\prod_{i \geq 1} X^{n_i} \right)_* / \prod_{i \geq 1} \mathfrak{S}_{n_i}.$$

This is a generalization of the notion of symmetric power $S^n X = X^n / \mathfrak{S}_n$. Moreover, this variety comes with a natural morphism to

$$S^\pi X := \left(\prod_{i \geq 1} X^{n_i} \right)_* / \prod_{i \geq 1} \mathfrak{S}_{n_i},$$

that is, the variety constructed in the same way, but in the special case where $X_i = X$ for all i.

We then put

$$\prod_{x \in X} \left(1 + X_{1,x} t + X_{2,x} t^2 + \ldots \right) = 1 + \sum_{n \geq 1} \left(\sum_{\substack{\pi \text{ partition} \\ \text{of } n}} [S^\pi \mathscr{X}] \right) t^n \in \mathrm{KVar}_k[[t]].$$

This construction is a generalization of the "motivic power" of Gusein-Zade, Luengo and Melle from [GZLM]. For the moment this is just a notation, and one has to show that this actually behaves like a product. For example, a generalization of the formula

$$S^n X = \sum_{i=0} S^i Z S^{n-i} U$$

for a variety X with closed subscheme Z and open complement U provides a multiplicativity property

$$\prod_{x\in X}\left(1+X_{1,x}t+X_{2,x}t^2+\ldots\right)=\prod_{x\in Z}\left(1+X_{1,x}t+X_{2,x}t^2+\ldots\right)\prod_{x\in U}\left(1+X_{1,x}t+X_{2,x}t^2+\ldots\right).$$

3 Manin's problem

3.1 Statement of the theorem

The setting is the following: let C be a smooth connected projective curve over an algebraically closed field k of characteristic zero. We will denote by F the function field $F := k(C)$.

Let $\mathcal{X}$ be a smooth projective variety with a non-constant morphism $\pi : \mathcal{X} \to C$, and let $\mathcal{L}$ be a line bundle on X. Denote $X = \mathcal{X}_F$ and $L = \mathcal{L}_X$, and assume the following conditions on these generic fibres:

- X is an equivariant compactification of $G = G_{a,F}^n$, that is, X has an open subset isomorphic to G, and the addition law $G \times G \to G$ extends to an action $G \times X \to X$. This covers projective spaces and all possible blowups of those at hypersurfaces of the hyperplane at infinity.
- The boundary $X \setminus G$ has strict normal crossings.
- The line bundle L is equal to the *anticanonical divisor* $-K_X$ of X. (This corresponds to the idea of counting points with respect to the anti-canonical height, which is the relevant height in Manin's conjecture.)

We are interested in

$$M_d = \{\text{ sections } \sigma : C \to \mathcal{X} \text{ such that } \sigma(\eta_C) \in G \text{ and } \deg(\sigma^*\mathcal{L}) = d\}$$

where η_X is the generic point of the curve C. By the theory of Hilbert schemes, this moduli space exists as a quasi-projective k-scheme, and we therefore may consider the motivic height zeta function

$$Z(T) = \sum_{d\geq 0}[M_d]T^d \in \mathrm{KVar}_k[[T]],$$

and ask for its convergence properties (for some topology that has yet to be defined.) Under an additional assumption of local existence of these sections, we have:

Theorem 3.1. *Put $r = \mathrm{rk\ Pic}\ X$ (this is exactly the number of irreducible components of the boundary divisor). There exists an integer $a \geq 1$ and a real number $\delta > 0$ such that the series*

$$(1-(\mathbf{L}T)^a)^r Z(T)$$

converges for $|T| < \mathbf{L}^{-(1-\delta)}$ and takes a non-zero effective value at $\mathbf{L}^{-1}$

In other words: $Z(T)$ has a pole at $\mathbf{L}^{-1}$, of order exactly r, which is what one gets also in the classical Manin's conjecture. We will say how this convergence is defined in a second. Let us just state a corollary which deduces the growth of the dimension and number of irreducible components of M_d as d goes to infinity. Because of the exponent a in the statement of the theorem, we actually need to make d go to infinity in congruence classes modulo a:

Corollary 3.2. *For any $p \in \{0, \dots, a-1\}$ one of the following cases occur when d tends to infinity in the congruence class of p modulo a:*

1. *Either* $\dim M_d - d \to -\infty$.
2. *Or* $\dim M_d - d$ *has finite limit and*

$$\frac{\log(\kappa(M_d))}{\log(d)} \longrightarrow e \in \{0, \dots, r-1\}.$$

Moreover, the second case happens for at least one integer p.

This condition on congruence classes may not be avoided: if the line bundle $\mathcal{L}$ is by chance equal to 2 times some other line bundle, then all M_d with d odd will be empty, and the interesting things happen for d even.

Previous results Counting sections $C \to \mathscr{X}$ is the same as counting $k(C)$-rational points on X. Thus this is a motivic analogue of Chambert-Loir and Tschinkel's proof of Manin's conjecture for equivariant compactifications of vector groups over number fields, which I am going to give more details on in a minute. They also treated integral points in a subsequent paper, and the motivic analogue of this is due to Chambert-Loir and Loeser.

3.2 Sketch of proof and definition of the topology

The proof of the theorem is essentially modeled on the proof of the corresponding result over number fields by Chambert-Loir and Tschinkel. However (and this is the main difficulty!), most of the main tools, which are classical in the arithmetic setting, need to be constructed and proved from scratch. Let us sketch the main ideas of Chambert-Loir and Tschinkel's proof, in the case where the base field is $\mathbf{Q}$.

We start with X an equivariant compactification of $G = \mathbf{G}_a^n$ over $\mathbf{Q}$, endowed with the anti-canonical height H. The aim is to prove that the height zeta function

$$\zeta_H(s) = \sum_{x \in G(\mathbf{Q})} H(x)^{-s}$$

converges for $\mathrm{Re}(s) > 1$, has a pole of order $r = \mathrm{rkPic}X$ at 1, and a meromorphic continuation for $\mathrm{Re}(s) > 1$. Using the Poisson summation formula for the discrete subgroup $G(\mathbf{Q}) = \mathbf{Q}^n$ inside the locally compact group of adeles $G(\mathbf{A}_{\mathbf{Q}})$ to which the height function may be extended, one may rewrite all of this in the following way:

$$\zeta_H(s) = \sum_{y \in \mathbf{Z}^n} \mathscr{F}(H^{-s})(y),$$

where $\mathscr{F}(H^{-s})$ is the Fourier transform of the function H^{-s}. This rearranges the terms in a way that makes the term corresponding to $y = 0$ responsible for the first pole of $\zeta_H(s)$, with the other terms having poles of strictly smaller order. To prove this, one uses a decomposition into local factors:

$$H(x)^{-s} = \prod_{p \leq \infty} H_p(x)^{-s}$$

so that

$$\mathscr{F}(H^{-s}) = \prod_{p \leq \infty} \mathscr{F}(H_p^{-s})$$

One gives estimates for each of the local factors by reduction modulo p, which imply the expected estimates on the product.

More precisely, in each factor we have terms of the form

$$D_\alpha(\mathbf{F}_p) - p^{n-1}$$

for each boundary component D_α, where $n-1$ is exactly the dimension of D_α. To achieve the desired convergence, it is absolutely crucial to use the Lang-Weil estimates, and bound this by

$$|D_\alpha(\mathbf{F}_p) - p^{n-1}| \leq cp^{n-\frac{3}{2}}.$$

In the motivic setting, after using a *motivic Poisson formula* and the above notion of Euler product, in the same way, we get terms of the form

$$[\mathscr{D}_{\alpha,v}] - \mathbf{L}^{n-1}$$

and this, as an element of KVar_k, will in general will be of dimension exactly $n-1$. Thus, the usual topology on KVar_k given by the dimensional filtration, is too coarse to be able to give the same kind of bound. To drop the size of this element, we need to use a finer (twice as fine) topology, coming from Hodge theory.

This is defined in the following manner. In the same way as we defined the Grothendieck ring of varieties, one can also define a Grothendieck ring $K_0(HS)$ of Hodge structures. The Grothendieck ring of varieties over k is generated by smooth and projective varieties, and therefore is a well-defined ring morphism

$$\chi^{\mathrm{Hdg}} : \mathrm{KVar}_k \to K_0(HS),$$

called the *virtual Hodge realization*, such that for all smooth projective varieties X over k,

$$\chi^{\mathrm{Hdg}}(X) = \sum_{i=0}^{2\dim X} [H^i_{\mathrm{sing}}(X(\mathbf{C}), \mathbf{Q})]$$

where $[H^i_{\mathrm{sing}}(X(\mathbf{C}), \mathbf{Q})]$ is the pure Hodge structure of weight $2i$ on the singular cohomology group $H^i_{\mathrm{sing}}(X(\mathbf{C}), \mathbf{Q})$.

Note that for irreducible X, we have $H^{2\dim X}(X(\mathbf{C}), \mathbf{Q}) = \mathbf{Q}(-\dim X)$, the pure Hodge structure of dimension 1 and weight $2\dim X$. Therefore, the terms of highest weight $2(n-1)$ in

$$\chi^{\mathrm{Hdg}}(\mathscr{D}_{\alpha,v} - \mathbf{L}^{n-1})$$

cancel out, and so

$$\mathrm{weight}(\mathscr{D}_{\alpha,v} - \mathbf{L}^{n-1}) \leq 2(n-1) - 1 = 2n - 3 = 2\left(n - \frac{3}{2}\right)$$

which corresponds to the exponent $n - \frac{3}{2}$ we got after applying the Lang-Weil estimates.

The radius of convergence of a series $F(T) = \sum_{i \geq 0} f_i t^i \in \mathrm{KVar}_k[[t]]$ will be

$$\sigma_F := \limsup \frac{\mathrm{weight}(f_i)}{2i},$$

and for any $s \leq \sigma_F$, one can say that F converges for $|T| < \mathbf{L}^{-s}$.

3.3 The motivic Poisson formula

To finish, let me give some extra details on the motivic tools used in the proof. To make sense of Fourier transforms in the motivic setting, the Grothendieck ring of varieties is not sufficient, one has to consider the more sophisticated Grothendieck ring of varieties with exponentials $\mathrm{KExpVar}_k$. As a group, it is defined by generators and relations. Generators are pairs (X, f) with X a variety over k and $f : X \to \mathbf{A}^1$ a morphism. The relations are:

$$(X, f) - (y, f \circ u)$$

if $u : Y \to X$ is an isomorphism.

$$(X, f) - (Y, f_{|Y}) - (U, f_{|U})$$

where Y is a closed subscheme of X and U its open complement, and a new relation:

$$(X \times \mathbf{A}^1, \mathrm{pr}_2).$$

There is a product, given by $[X, f][Y, g] = [X \times Y, f \circ \mathrm{pr}_1 + g \circ \mathrm{pr}_2]$.

To understand why this ring is what we need, let me illustrate how one should think about it. For the usual Grothendieck ring, for k finite, remember we had the counting measure. Here, for k finite, we may define a ring morphism

$$\mathrm{KExpVar}_k \to \mathbf{C}$$

sending the class $[X, f]$ to $\sum_{x \in X(k)} \exp(2i\pi f(x))$. To check this is well-defined one has to verify that the relations defining $\mathrm{KExpVar}_k$ map to zero. The first relation just translates the fact that we may do changes of variables, the second one the fact that we may cut the sum into two, and the last relation the fact that

$$\sum_{x \in k} \exp(2i\pi x) = 0$$

which is crucial to make Fourier analysis work. There is an injective morphism $\mathrm{KVar}_k \to \mathrm{KExpVar}_k$. To extend the above weight topology to $\mathrm{KExpVar}_k$, we use the notion of *motivic vanishing cycles*.

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