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Produits eulériens motiviques

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Résumé

L'objectif de cette thèse est l'étude de la fonction zêta des hauteurs motivique associée à un problème de comptage de courbes sur les compactifications équivariantes d'espaces affines, résolvant au chapitre 6 l'analogue motivique de la conjecture de Manin pour celles-ci.

La fonction zêta des hauteurs provenant du problème de comptage considéré est récrite convenablement à l'aide d'une formule de Poisson motivique démontrée au cinquième chapitre, qui généralise celle de Hrushovski-Kazhdan. Chaque terme est alors décomposé sous la forme d'un produit eulérien motivique, dont la définition et les propriétés sont établies au chapitre 3. La convergence de ces produits eulériens doit être comprise pour une topologie des poids que nous introduisons au quatrième chapitre et qui repose d'une part sur la théorie des modules de Hodge de Saito, et d'autre part sur une mesure motivique sur l'anneau de Grothendieck des variétés avec exponentielles, construite dans le chapitre 2 à l'aide de la notion de cycles évanescents motiviques.

On en déduit ainsi une description de l'asymptotique d'une proportion positive des coefficients du polynôme de Hodge-Deligne des espaces de modules des courbes sur la compactification équivariante donnée, lorsque le degré tend vers l'infini.

Mots-clefs : Anneaux de Grothendieck des variétés, hauteurs, compactifications équivariantes d'espaces affines, problème de Manin, théorie de Hodge, cycles évanescents.

Abstract

The goal of this thesis is the study of the motivic height zeta function associated to the problem of counting curves on equivariant compactifications of vector groups, solving in chapter 6 the motivic analogue of Manin's conjecture for such varieties.

The motivic height zeta function coming from this counting problem is rewritten in a convenient way using a Poisson summation formula proved in chapter 5, and which generalises Hrushovski and Kazhdan's motivic Poisson formula. Each term is then expressed as a motivic Euler product, the definition and properties of the latter being established in chapter 3. The convergence of these Euler products must be understood for a weight topology which we introduce in the fourth chapter and which relies both on Saito's theory of mixed Hodge modules and on a motivic measure on the Grothendieck ring of varieties with exponentials, constructed in chapter 2 using the notion of motivic vanishing cycles.

We deduce from this a description of the asymptotic of a positive proportion of the coefficients of the Hodge-Deligne polynomial of the moduli spaces of curves on the given equivariant compactification, when the degree goes to infinity.

Key words : Grothendieck rings of varieties, heights, equivariant compactifications of vector groups, Manin's problem, Hodge theory, vanishing cycles.

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Table des matières

Chapitre 1

Introduction

1.1 Le problème de Manin classique

L'une des grandes préoccupations de la théorie des nombres et de la géométrie algébrique dans les dernières décennies est le lien subtil qui existe entre la répartition des points rationnels ou entiers d'une variété algébrique définie sur un corps de nombres et certains de ses invariants géométriques. Soient F un corps de nombres et X une variété projective sur F, munie d'un fibré en droites ample L. Un tel fibré en droites fournit (à une fonction bornée près) une fonction hauteur $H : X(F) \to \mathbf{R}_+$ telle que pour tout réel B, l'ensemble

$$\{x \in X(F), H(x) \leq B\}$$

soit fini. On notera alors, pour tout ouvert de Zariski U de X,

$$N_{U,H}(B) := \#\{x \in U(F), H(x) \leq B\}.$$

Lorsque l'ensemble $U(F)$ est lui-même infini, on peut s'interroger sur le comportement asymptotique de $N_{U,H}(B)$ lorsque B tend vers $+\infty$. Dans tous les cas connus, il est de la forme

$$N_{U,H}(B) \underset{B\to\infty}{\sim} CB^a(\log B)^{b-1}$$

avec $C > 0$ et $a \geq 0$ des réels, et $b \in \frac{1}{2}\mathbf{Z}$, $b \geq 1$.

C'est avec des conjectures, ou plutôt, des questions, énoncées par Manin et al. dans [FMT] et [BM], à la fin des années 80, que fut initié un vaste programme cherchant à donner une interprétation géométrique des exposants a et b, en termes des classes du fibré en droites L et du faisceau canonique K_X dans le groupe de Néron-Severi de X, ainsi que du cône des classes des diviseurs effectifs. Pour pouvoir produire un énoncé vraisemblable, il faut bien entendu faire quelques restrictions sur les variétés considérées. En particulier, on ne peut espérer avoir une asymptotique décrivant correctement la géométrie de X que si l'ensemble $X(F)$ est Zariski-dense dans X. La conjecture de Lang, qui affirme que cela ne devrait pas être le cas pour les variétés de type général, fait que la plupart des questions considérées se restreignent de plus aux variétés dites de Fano, c'est-à-dire de fibré anticanonique ample. On considère parfois une classe plus large de variétés, les variétés *presque de Fano*, pour lesquelles, notamment, K_X^{-1} n'est plus nécessairement ample, mais gros, c'est-à-dire appartenant à l'intérieur du cône effectif, ce qui est suffisant pour que la hauteur correspondante satisfasse la propriété de Northcott dans un ouvert non vide de la variété. Pour ces variétés, les conjectures prennent une forme particulièrement simple si on compte les points par rapport à la hauteur anticanonique :

Problème 1. Soit X une variété presque de Fano sur un corps de nombres F, telle que $X(F)$ soit Zariski dense dans X. Soit H une hauteur relative au faisceau anticanonique sur X. Existe-t-il un ouvert dense U de X vérifiant

$$N_{U,H}(B) \underset{B\to\infty}{\sim} C_H B(\log B)^{r-1} \tag{1.1}$$

où r est le rang du groupe de Picard de X, et C_H une constante strictement positive ?

Notons qu'il est nécessaire d'autoriser la restriction à un ouvert, pour prendre en compte des phénomènes d'accumulation de points rationnels sur des fermés stricts, occultant leur répartition sur l'ouvert complémentaire. Mentionnons également la forme plus précise de cette conjecture par Peyre, qui proposa dans [Pey95] une interprétation pour la constante C_H en termes de certains volumes d'espaces adéliques. La définition de cette constante fut ajustée par Batyrev et Tschinkel, qui y ajoutèrent, dans [BT98] un facteur cohomologique.

Le premier résultat de ce type a été obtenu par Schanuel [Sch], bien avant la première formulation des conjectures, dans le cas où X et U sont l'espace projectif $\mathbf{P}^n_F$, avec de plus une formule explicite pour la constante C_H. Dans l'article [FMT], il est montré que la formule (1.1) est vérifiée pour les variétés de drapeaux (avec $U = X$). Depuis, une réponse affirmative a été apportée à la question 1, et souvent même à son raffinement par Peyre, dans de nombreux cas particuliers, par diverses approches, parmi lesquelles on peut citer l'analyse harmonique, les torseurs universels ainsi que la méthode du cercle. Il existe cependant également des contre-exemples, dont celui de Batyrev et Tschinkel [BT98], qui expliquent pourquoi nous avons préféré énoncer la conjecture ci-dessus sous forme de question. L'outil principal, commun à la plupart des approches, est la fonction zêta des hauteurs

$$\zeta_{U,H}(s) = \sum_{x\in U(F)} H(x)^{-s},$$

qui est une fonction définie sur une partie du plan complexe, dont les propriétés de convergence (abscisse de convergence, ordre du premier pôle, coefficient en ce pôle) sont liées, par des théorèmes taubériens, aux exposants et à la constante intervenant dans l'asymptotique cherchée. Plus précisément, le problème 1 se reformule alors essentiellement de la manière suivante :

Problème 2. Soit X une variété presque de Fano sur un corps de nombres F, telle que $X(F)$ soit Zariski dense dans X. Soit H une hauteur relative au faisceau anticanonique sur X. Existe-t-il un ouvert dense U de X tel que la série $\zeta_{U,H}(s)$ converge absolument pour $\mathrm{Re}(s) > 1$, et se prolonge en une fonction méromorphe sur l'ouvert $\{\mathrm{Re}(s) > 1 - \delta\}$ pour un certain réel $\delta > 0$, avec un unique pôle d'ordre $r = \mathrm{rg}\ \mathrm{Pic}(X)$ en $s = 1$?

De même que le problème 1, cette question peut être raffinée en demandant de plus que le coefficient du terme principal de $\zeta_{U,H}(s)$ en 1 soit la constante prédite par Peyre.

1.2 Le problème de Manin par l'analyse harmonique

Le premier cas traité par des techniques d'analyse harmonique fut celui, cité plus haut, des variétés de drapeaux : il s'agissait de remarquer que la fonction zêta des hauteurs était dans ce cas une série d'Eisenstein, et d'appliquer les résultats de Langlands pour déduire les propriétés analytiques de celle-ci. Ensuite, au milieu des années 90, Batyrev et Tschinkel traitèrent le cas des variétés toriques, l'ouvert U dans ce cas étant l'orbite ouverte de l'action du tore sur la variété. Leur argument, fondé sur la formule sommatoire de Poisson, utilise de manière cruciale cette action, et a pu être généralisé à de nombreuses autres variétés munies d'une action d'un groupe algébrique avec une orbite ouverte, par exemple certaines compactifications équivariantes de groupes algébriques. Une *compactification équivariante* d'un groupe algébrique G est une variété (projective et lisse) X ayant un ouvert dense isomorphe à G, et munie d'une action $G \times X \to X$ de G qui étend la loi de groupe $G \times G \to G$. Outre les variétés toriques qui sont exactement les compactifications équivariantes des tores algébriques, le problème de Manin a été résolu pour les compactifications équivariantes d'espaces affines ([CLT02], [CLT12]), ainsi que pour des compactifications de certains groupes algébriques non-commutatifs ([ShTTB03], [ShTTB07], [ShT04], [ShT16], [TTB], [TT]). Expliquons les grandes lignes de l'argument dans le cas des compactifications équivariantes d'espaces affines traité par Chambert-Loir et Tschinkel, et qui jouera un rôle privilégié dans cette thèse. Soit X une compactification équivariante du groupe $G = \mathbf{G}_a^n$ pour un certain $n \geq 1$, définie sur un corps de nombres F. Pour chaque place v de F, on dispose d'une hauteur locale $H_v : G(F_v) \to \mathbf{R}_+$, la hauteur H étant donnée par la formule

$$H(x) = \prod_v H_v(x),$$

qui fournit une manière d'étendre H au groupe localement compact $G(\mathbb{A}_F)$, où $\mathbb{A}_F$ est le groupe des adèles du corps F. Le groupe $G(\mathbb{A}_F)$ est autodual, et le groupe $G(F)$ peut se voir comme un sous-groupe discret de $G(\mathbb{A}_F)$ d'orthogonal identifié à lui-même, de sorte qu'en appliquant la formule de Poisson à la fonction zêta des hauteurs (après vérification des hypothèses d'intégrabilité nécessaires), on obtient :

$$\zeta_{H,G}(s) = \sum_{x \in G(F)} H(x)^{-s} = \sum_{\xi \in G(F)} \mathscr{F}(H^{-s})(\xi), \tag{1.2}$$

où $\mathscr{F}$ désigne la transformation de Fourier. Cette égalité vaut dès que la partie réelle de s est assez grande. De plus, la fonction H^{-s} étant invariante modulo un sous-groupe compact de $G(\mathbb{A}_F)$, sa transformée de Fourier est de support contenu dans un tel sous-groupe, ce qui réduit la sommation sur $G(F)$ dans le côté droit à une sommation sur un réseau. Il se trouve que cette procédure réarrange convenablement les termes, et le terme correspondant au caractère trivial (c'est-à-dire ici à $\xi = 0$), dans tous les cas où cette méthode a fonctionné et lorsque H est la hauteur anticanonique, est l'unique porteur du premier pôle avec le bon ordre. Pour le prouver, on utilise de nouveau la décomposition en facteurs locaux

$$\mathscr{F}(H^{-s})(\xi) = \prod_v \mathscr{F}(H_v^{-s})(\xi_v)$$

et on étudie les facteurs séparément. Ceux-ci se présentent sous la forme d'intégrales de type fonction zêta d'Igusa, pour lesquelles on obtient pour presque toute place v une expression explicite par réduction dans le corps résiduel, dont la convergence est étudiée en utilisant des majorations à base d'estimées de Lang-Weil. Aux places archimédiennes, des intégrations par parties permettent d'obtenir des bornes polynomiales en s. Pour le nombre fini de places restantes, des majorations plus grossières suffisent. Dans le cas où la hauteur considérée est la hauteur anticanonique, cela permet de voir que le terme correspondant à $\xi = 0$ a un pôle en $s = 1$ avec le coefficient prédit par Peyre, d'ordre le rang du groupe de Picard, que les autres termes ont des pôles d'ordre strictement plus petit, et que la fonction $s \mapsto (s-1)^{\mathrm{rg}\mathrm{Pic}(X)}\zeta_{H,G}(s)$ admet un prolongement holomorphe à $\Re(s) > 1 - \delta$ pour un certain $\delta > 0$.

Si la question du dénombrement des solutions entières d'équations diophantiennes est naturelle, personne n'avait jamais abordé cette question sous l'angle géométrique suggéré par Manin, lorsque dans [CLT12], Chambert-Loir et Tschinkel proposèrent également une solution au problème de Manin pour les points entiers dans le cas des compactifications équivariantes *partielles* d'espaces affines. Une telle compactification partielle U est vue comme le complémentaire, dans une compactification équivariante X, d'un diviseur D qui géométriquement est à croisements normaux stricts. De même que précédemment, elle contient un ouvert dense G isomorphe au groupe additif $\mathbf{G}_a^n$. On fixe des modèles de X, U, D au-dessus de l'anneau des entiers $\mathcal{O}_F$, un ensemble fini de places S incluant les places archimédiennes, et on cherche à compter les points de $G(F)$ qui sont S-entiers par rapport au modèle de U choisi. Notons que ce problème inclut le problème précédent, car dans le cas où $D = \varnothing$, si le modèle choisi pour D est vide également, alors tous les points de $G(F)$ seront S-entiers, par projectivité de X. Pour ce problème de comptage, la bonne hauteur à considérer n'est plus la hauteur anticanonique, mais la hauteur *log-anticanonique*, c'est-à-dire associée au fibré en droites $-(K_X + D)$, qui est gros. Chambert-Loir et Tschinkel prouvent alors que le nombre $N(B)$ de points S-entiers de hauteur log-anticanonique plus petite que B satisfait l'asymptotique

$$N(B) \sim CB(\log B)^{b-1}$$

où C est une constante réelle strictement positive, et où l'exposant b est donné par la formule

$$b = \mathrm{rg}\ \mathrm{Pic}(U) + \sum_{v \in S}(1 + \dim \mathscr{C}_{F_v}^{\mathrm{an}}(D)). \tag{1.3}$$

Ici, pour chaque v, $\mathscr{C}^{\mathrm{an}}_{F_v}(D)$ est un complexe simplicial qui encode les propriétés d'incidence des composantes de D au-dessus de la place v. Le terme $1+\dim \mathscr{C}^{\mathrm{an}}_{F_v}(D)$ correspond exactement au nombre maximal de composantes de D sur F_v dont l'intersection a des F_v-points. Bien entendu, dans le cas particulier $U = X$, tous ces termes sont nuls, et on retrouve le résultat précédent pour les points rationnels.

1.3 Le problème de Manin sur les corps de fonctions

Pour le moment, nous nous sommes toujours restreints au cas où F est un corps de nombres. Le cas où F est le corps de fonctions $k(C)$ d'une courbe projective lisse C sur un corps fini k de cardinal q fut évoqué pour la première fois par Batyrev et Manin ([BM], paragraphe 3.13). Dans ce cas, les hauteurs utilisées (convenablement normalisées) vont prendre leurs valeurs dans l'ensemble $q^{\mathbf{Z}}$, ce qui interdit l'existence d'une asymptotique telle que celle annoncée dans la question 1. Cela dit, la question 2 reste valide, à la petite modification près que la fonction zêta correspondante sera dans ce cas $\frac{2i\pi}{\log(q)}$-périodique, et qu'il faut donc autoriser, en plus du pôle en 1, des pôles en $1 + \frac{2i\pi m}{\log(q)}$ pour tout entier m. Autrement dit, il sera plus judicieux ici de considérer la fonction zêta des hauteurs comme une fonction de la variable $t = q^{-s}$, ce qui permet la reformulation suivante du problème 2 qui tient compte de cette périodicité :

Problème 3. Soit C une courbe projective lisse connexe sur $\mathbf{F}_q$, dont on note F le corps des fonctions. Soit X une variété presque de Fano sur le corps F, telle que $X(F)$ soit Zariski dense dans X. Soit H la hauteur relative au fibré anticanonique sur X. Existe-t-il un ouvert dense U de X tel que la série $\zeta_{U,H}(t)$ converge absolument sur le disque défini par $|t| < q^{-1}$, et se prolonge en une fonction méromorphe sur le disque défini par $|t| < q^{-1+\delta}$ pour un certain réel $\delta > 0$, avec un unique pôle d'ordre $r = \mathrm{rg}\ \mathrm{Pic}(X)$ en $t = q^{-1}$?

Précisons qu'en général on peut cependant s'attendre à d'autres pôles d'ordre $< r$ sur le cercle $|t| = q^{-1}$, en particulier dans le cas de certaines variétés toriques déployées. Le problème acquiert dans le cas fonctionnel une interprétation géométrique supplémentaire : si on choisit un modèle $\mathcal{X}$ de X au-dessus de la courbe C, les points rationnels d'un ouvert U de X correspondent aux sections $\sigma : C \to \mathcal{X}$ du morphisme structurel $\pi : \mathcal{X} \to C$ telles que, en notant η_C le point générique de C, $\sigma(\eta_C) \in U(F)$. Si on se donne un fibré en droites $\mathcal{L}$ sur $\mathcal{X}$ (génériquement ample, ou du moins gros), la hauteur d'une telle section par rapport à $\mathcal{L}$ est donnée par q^d, où d est le degré du fibré en droites $\sigma^*\mathcal{L}$ sur C. La fonction zêta des hauteurs s'écrira alors

$$\zeta_{U,\mathcal{L}}(s) = \sum_{x\in U(F)} H(x)^{-s} = \sum_{d\geq 0} m_d q^{-ds},$$

avec

$$m_d = |\{\sigma : C \to \mathcal{X},\ \sigma(\eta_C) \in U(F),\ \deg \sigma^*\mathcal{L} = d\}|.$$

Ainsi, des informations sur la convergence de la fonction zêta donneront l'asymptotique du cardinal m_d des points de hauteur q^d donnée, qui sera l'analogue dans ce cadre de la formule 1.1.

Le problème de Manin sur les corps de fonctions sur un corps fini reste à ce jour assez peu étudié : il faut citer néanmoins à ce sujet les travaux de Bourqui [Bou02, Bou03, Bou11], qui traitent entièrement le cas des variétés toriques (en recourant à l'analyse harmonique d'une part, et à la méthode des torseurs universels d'autre part), ainsi que les articles [LY] et [Pey12] traitant indépendamment le cas des variétés de drapeaux généralisées, l'article de Peyre donnant de plus l'interprétation de la constante. La méthode de Peyre est analogue à celle de Franke, Manin et Tschinkel pour les variétés de drapeaux sur les corps de nombres, l'utilisation des travaux de Langlands étant remplacée par celle des résultats de Morris au sujet des séries d'Eisenstein sur les corps de fonctions.

Les progrès récents de l'*intégration motivique* suggèrent enfin la généralisation suivante du problème de Manin sur les corps de fonctions : les sections $\sigma : C \to \mathcal{X}$ telles que $\sigma(\eta_C) \in G(F)$ et $\deg \sigma^*\mathcal{L} = d$ admettent un *espace des modules* M_d, qui est un k-schéma quasi-projectif. L'intérêt de l'étude géométrique de tels espaces de modules en lien avec les conjectures de Manin avait déjà été souligné par Batyrev. Plus précisément, suivant une idée de Peyre, on peut s'interroger non seulement sur l'asymptotique du cardinal m_d de $M_d(k)$, mais plus précisément aux propriétés, lorsque d est grand, de la classe de M_d dans l'anneau de Grothendieck des variétés KVar_k sur k : en tant que groupe, celui-ci est défini comme le quotient du groupe abélien libre sur les classes d'isomorphismes de variétés sur k par les relations de la forme

$$X - U - Z$$

pour toute variété X et tout sous-schéma fermé Z de X de complémentaire ouvert U. La structure d'anneau provient du produit des variétés : en notant les classes dans KVar_k entre crochets, on a $[X][Y] = [X \times_k Y]$ pour toutes les k-variétés X et Y. On note $\mathbf{L} = [\mathbf{A}^1_k]$ la classe de la droite affine, et on considère par ailleurs souvent l'anneau de Grothendieck localisé $\mathscr{M}_{\mathbf{C}} = \mathrm{KVar}_k[\mathbf{L}^{-1}]$.

La classe d'une variété dans l'anneau de Grothendieck contient un grand nombre d'informations géométriques sur cette variété : on dispose en effet de nombreuses *mesures motiviques*, c'est-à-dire de morphismes d'anneaux de KVar_k vers d'autres anneaux, associant à une classe $[X]$ divers invariants géométriques de X. Parmi celles-ci, dans le cas où le corps est fini, on peut citer la *mesure de comptage*

$$\begin{array}{ccc} \mathrm{KVar}_k & \to & \mathbf{Z} \\ [X] & \mapsto & |X(k)| \end{array}$$

qui permet de récupérer le nombre de points rationnels de la variété. Pour un corps quelconque k, en fixant une clôture séparable k^s de k et ℓ un nombre premier inversible dans k, le polynôme d'Euler-Poincaré (associé à la cohomologie à coefficients dans $\mathbf{Q}_\ell$) définit également une mesure motivique

$$\begin{array}{ccc} \mathrm{KVar}_k & \to & \mathbf{Z}[t] \\ [X] & \mapsto & EP(X)(t) \end{array}$$

qui pour une variété projective et lisse X vaut

$$EP(X)(t) = \sum_{i=0}^{2\dim X} (-1)^i \dim_{\mathbf{Q}_\ell} H^i_{\text{ét}}(X \otimes_k k^s, \mathbf{Q}_\ell) t^i.$$

Un autre exemple du même genre, et qui sera important pour nous, est le polynôme de Hodge-Deligne :

$$\begin{array}{ccc} \mathrm{KVar}_{\mathbf{C}} & \to & \mathbf{Z}[u,v] \\ [X] & \mapsto & HD(X)(u,v) \end{array},$$

envoyant la classe d'une variété complexe projective et lisse X sur le polynôme

$$HD(X)(u,v) = \sum_{0\leq p,q\leq \dim X} (-1)^{p+q} h^{p,q}(X) u^p v^q$$

défini à partir des nombres de Hodge $h^{p,q}(X)$ de X. En remarquant que $HD(\mathbf{L}) = uv$, on peut en outre prolonger cette mesure en un morphisme d'anneaux

$$HD : \mathscr{M}_{\mathbf{C}} \to \mathbf{Z}[u,v,(uv)^{-1}].$$

L'observation fondamentale qu'on peut faire ici est que, maintenant que nous avons posé ces définitions, il est possible de donner un sens à une version du problème de Manin au-dessus d'un corps fonctions d'une courbe $k(C)$ même lorsque le corps de base k n'est plus nécessairement fini, voire est de caractéristique nulle : il s'agirait alors de s'interroger, par exemple, sur certains invariants géométriques de l'espace M_d lorsque d tend vers l'infini, comme sa dimension, ou son nombre de composantes irréductibles de dimension maximale. Dans ce cadre, que nous appellerons *motivique*, la notion de fonction zêta des hauteurs prend la forme d'une série

$$Z_{U,\mathcal{L}}(T) = \sum_{d\geq 0} [M_d] T^d \in \mathrm{KVar}_k[[T]] \tag{1.4}$$

à coefficients dans l'anneau de Grothendieck, appelée *fonction zêta des hauteurs motivique*. Ce genre de fonctions ont été étudiées par Bourqui dans [Bou09] pour certaines variétés toriques. Une première difficulté qui apparaît dans ce cadre est la question de la convergence d'une telle série. L'anneau de Grothendieck KVar_k, ou, plutôt, l'anneau $\mathscr{M}_k = \mathrm{KVar}_k[\mathbf{L}^{-1}]$ obtenu en inversant la classe $\mathbf{L}$ de la droite affine dans KVar_k, admet une topologie naturelle induite par la *filtration dimensionnelle* : pour tout $n \in \mathbf{Z}$ on définit $F_n\mathscr{M}_k$ comme le sous-groupe de $\mathscr{M}_k$ engendré par les classes de la forme $[X]\mathbf{L}^{-m}$ où X est une k-variété telle que $\dim(X) - m \leq n$. Il est alors naturel de dire que la série ci-dessus converge en $\mathbf{L}^{-s}$ si $\dim[M_d] - ds \to -\infty$ lorsque $d \to +\infty$. Malheureusement, en particulier à cause du caractère assez grossier de la notion de dimension, cette convergence est peu maniable. Dans cette thèse, nous allons utiliser une topologie un peu plus fine, provenant de la théorie des poids en cohomologie. Notons que dans [Bou09] Bourqui obtient néanmoins une convergence par rapport à la filtration dimensionnelle (dans l'anneau de Grothendieck des motifs de Chow), car la filtration dimensionnelle perçoit, dans le cas des variétés toriques déployées dont il y est question, l'annulation des poids maximaux nécessaire à la convergence. Puis, dans [Bou10], Bourqui introduit une topologie similaire à la notre, dont l'idée apparaît également dans [Ekedahl].

1.4 Résultat principal de cette thèse

Nous allons maintenant décrire le contenu de cette thèse. L'objet principal en est l'obtention d'un analogue motivique du théorème de Chambert-Loir et Tschinkel sur le comptage des points entiers sur les compactifications partielles d'espaces affines décrit ci-dessus. Il s'agit, comme dans le cas arithmétique, de l'établir à l'aide de l'analyse harmonique, ce qui demande l'élaboration d'outils adaptés à cet effet dans le cadre motivique. Commençons par décrire le résultat obtenu.

Hypothèse 1. *Soit C_0 une courbe lisse quasi-projective connexe sur un corps k algébriquement clos de caractéristique nulle, soit C sa compactification projective et lisse, et soit $S = C \setminus C_0$. On pose $F = k(C)$ le corps des fonctions de C. On se donne un k-schéma projectif irréductible $\mathcal{X}$ muni d'un morphisme non-constant $\pi : \mathcal{X} \to C$, $\mathcal{U}$ un ouvert de Zariski de $\mathcal{X}$, et $\mathcal{L}$ un fibré en droites sur $\mathcal{X}$. Nous faisons les hypothèses suivantes sur les fibres génériques $X = \mathcal{X}_F$, $U = \mathcal{U}_F$ et le fibré en droites $L = \mathcal{L}_F$:*

- *X est lisse, l'ouvert U de X contient un ouvert dense G isomorphe à $\mathbf{G}_{a,F}^n$, et U et X sont munis d'une action de G qui étend la loi de groupe de G. Autrement dit X (resp. U) est une compactification (resp. compactification partielle) équivariante du groupe additif $\mathbf{G}_a^n$.*
- *le bord $\partial X = X \setminus U$ est un diviseur D à croisements normaux stricts.*
- *le fibré en droites L sur X est le fibré log-anticanonique $-K_X(\partial X)$.*

Comme ci-dessus, nous nous intéressons aux espaces des modules des sections $\sigma : C \to \mathcal{X}$ telles que $\sigma(\eta_C) \in \mathbf{G}_a^n(F)$ (où η_C est le point générique de C), mais nous nous restreignons à celles qui correspondent à des points S-entiers, ce qui revient à imposer de plus $\sigma(C_0) \subset \mathcal{U}$. Du point de vue géométrique, la première condition signifie qu'une telle section $\sigma : C \to \mathcal{X}$ ne sort de G qu'en un nombre fini de points de C, appelés *pôles*, et la seconde qu'en les $v \in C_0$, $\sigma(v)$ reste toujours contenu dans $\mathcal{U}$. Si on note $(D_\alpha)_{\alpha\in\mathscr{A}}$ les composantes irréductibles de $X \setminus G$, le diviseur log-anticanonique s'écrit sous la forme $\sum_{\alpha\in\mathscr{A}} \rho'_\alpha D_\alpha$ pour des entiers ρ'_α strictement positifs. Le fibré en droites $\mathcal{L}$ étant génériquement log-anticanonique, il s'écrit sous la forme $\mathcal{L} = \sum_{\alpha\in\mathscr{A}} \rho'_\alpha \mathcal{L}_\alpha$ où pour tout $\alpha \in \mathscr{A}$, la restriction du fibré $\mathcal{L}_\alpha$ à la fibre générique est D_α. Pour $d \in \mathbf{Z}$ notons M_d l'espace des modules des sections σ qui vérifient de plus $\deg \sigma^*\mathcal{L} = d$. Au vu de la description de $\mathcal{L}$ ci-dessus, à un nombre fini de places près (car $\mathcal{L}$ peut avoir un nombre fini de composantes verticales), seuls les pôles de σ contribuent à ce degré, l'apport de chaque pôle étant la somme des degrés d'intersection de σ en ce pôle avec les diviseurs $\mathcal{L}_\alpha$ (l'*ordre* du pôle par rapport à $\mathcal{L}_\alpha$), pondérés par les entiers ρ'_α. On vérifie que les espaces M_d sont vides pour $d \ll 0$, de sorte qu'on peut définir la fonction zêta des hauteurs motivique par

$$Z(T) = \sum_{d\in\mathbf{Z}} [M_d] T^d \in \mathrm{KVar}_k[[T]][T^{-1}]. \tag{1.5}$$

En plus de l'hypothèse générique ci-dessus, il faut imposer quelque chose sur le modèle $\mathcal{U}$ pour que les espaces M_d soient non-vides. Il se trouve qu'une hypothèse de type « principe de Hasse » est suffisante :

Hypothèse 2. *Nous supposons qu'il n'y a pas d'obstruction locale à l'existence de telles sections, c'est-à-dire que pour tout point fermé* $v \in C_0$ *nous avons* $G(F_v) \cap \mathcal{U}(\mathcal{O}_v) \neq \varnothing$*, où le corps* F_v *est le complété de* F *au point* v*, et* $\mathcal{O}_v$ *son anneau des entiers.*

Pour comprendre cette hypothèse, il est utile de reformuler la condition sur les sections dans un langage adélique. Chaque section correspond à un unique point $\sigma(\eta_C)$ de $G(F)$, qui par plongement diagonal fournit un point de l'ensemble $G(\mathbb{A}_F)$ des points de G à valeurs dans les adèles du corps F. A un nombre fini de places près, pour un point $v \in C_0$ qui n'est pas un pôle de σ, la composante de σ dans $G(F_v)$ est un élément de $G(\mathcal{O}_v)$. La condition d'intégralité $\sigma(C_0) \subset \mathcal{U}$ dit alors que pour tout $v \in C_0$, la composante de σ dans $G(F_v)$ est un élément de $\mathcal{U}(\mathcal{O}_v)$: la non-vacuité de l'intersection $G(F_v) \cap \mathcal{U}(\mathcal{O}_v)$ est donc une condition nécessaire à l'existence d'une telle section.

Sous ces hypothèses, ainsi que sous une troisième hypothèse concernant les produits eulériens motiviques, on obtient la convergence souhaitée pour une topologie sur l'anneau de Grothendieck $\mathscr{M}_{\mathbf{C}}$ que nous expliciterons plus bas :

Théorème 1. *On suppose* $k = \mathbf{C}$*, on se place sous les hypothèses 1 et 2 ainsi que sous l'hypothèse 3 expliquée plus bas, et on note* b *l'entier donné par la formule (1.3). Il existe un entier* $a \geq 1$ *et un nombre réel* $\delta > 0$ *tels que la série de Laurent* $(1 - (\mathbf{L}T)^a)^b Z(T)$ *converge pour* $|T| < \mathbf{L}^{-1+\delta}$ *et prend une valeur effective non-nulle en* $T = \mathbf{L}^{-1}$*.*

Ainsi, l'ordre du pôle de la fonction zêta des hauteurs motivique est donné par la même formule que dans le résultat de Chambert-Loir et Tschinkel mentionné à la fin de la section 1.2. La valeur effective non-nulle en $\mathbf{L}^{-1}$ mentionnée dans l'énoncé est un élément du complété $\widehat{\mathscr{M}_{\mathbf{C}}}$ pour la topologie considérée : elle se présente comme un produit infini de volumes locaux, et est un analogue motivique de la constante de Peyre.

Pour tout ensemble k-constructible M, nous notons $\kappa(M)$ le nombre de composantes irréductibles de dimension maximale de M. Le polynôme de Hodge-Deligne HD se prolonge en une mesure motivique

$$\widehat{\mathscr{M}_{\mathbf{C}}} \to \mathbf{Z}[[(uv)^{-1}]][u, v]$$

sur le complété sus-cité, et le théorème 1, via l'utilisation de cette mesure motivique, fournit une description du comportement asymptotique de $\dim(M_d)$ et $\kappa(M_d)$, la présence de l'exposant a imposant une distinction selon la classe de congruence de d modulo a. De plus, un raisonnement de type principe de Lefschetz permet de s'affranchir de l'hypothèse $k = \mathbf{C}$ présente dans le théorème.

Corollaire 2. *Pour tout* $p \in \{0, \ldots, a-1\}$*, l'un des cas suivants se produit lorsque* d *tend vers l'infini dans la classe de congruence de* p *modulo* a *:*

(i) Soit $\limsup \frac{\dim(M_d)}{d} < 1$.

(ii) Soit $\dim(M_d) - d$ *a une limite finie et*

$$\frac{\log(\kappa(M_d))}{\log d}$$

converge vers un élément de $\{0, \ldots, b-1\}$*.*

De plus, le second cas de figure se produit pour au moins un entier $p \in \{0, \ldots, a-1\}$.

Une condition sur les classes de congruences ne peut pas être évitée en général : par exemple, si le diviseur log-anticanonique est le multiple $(L')^a$ d'une classe dans $\mathrm{Pic}(\mathcal{X})$, alors $M_d = \varnothing$ pour $d \nmid a$. Remarquons d'autre part que nous obtenons en fait même un résultat plus précis, qui décrit une proportion positive des coefficients du polynôme de Hodge-Deligne de M_d : nous renvoyons à la proposition 4.7.3.1 du chapitre 4 pour plus de détails.

Il est important de signaler deux cas particuliers importants de ce résultat : lorsque $\mathcal{X} = \mathcal{U}$ nous obtenons un analogue motivique du résultat de Chambert-Loir et Tschinkel dans [CLT02] pour les points rationnels sur les compactifications équivariantes d'espaces affines. Dans ce cas, il n'y a aucune condition sur les pôles des sections. A l'inverse, le cas $U = G_F$ où l'on n'autorise que les sections ayant des pôles dans l'ensemble fini $C \setminus C_0$ a été traité dans l'article [ChL] de Chambert-Loir et Loeser en suivant la même idée de preuve que celle décrite ci-dessus pour le problème de Manin classique. A cause de la restriction sur les pôles, seul un ensemble fini fixé de places contribue à la hauteur. De plus, à degré d fixé, les ordres des pôles des sections comptabilisées dans l'espace M_d sont essentiellement bornés par d. En revenant à la description adélique ci-dessus, on voit alors que la fonction caractéristique des sections paramétrées par M_d est une fonction sur les adèles $G(\mathbb{A}_F)$ dont la restriction à $G(F_v)$ est, pour presque tout v, la fonction caractéristique de $G(\mathcal{O}_v)$. Plus précisément, à cause de la borne sur les ordres de pôles et de l'équivariance de la compactification, cette fonction caractéristique est une *fonction de Schwartz-Bruhat motivique*. L'outil principal utilisé par Chambert-Loir et Loeser est la *formule de Poisson motivique* de Hrushovski et Kazhdan, qui est justement vérifiée pour ce genre de fonctions, et permet, en l'appliquant à la fonction caractéristique de M_d dans $G(\mathbb{A}_F)$ pour chaque d, de réécrire la fonction zêta $Z(T)$ sous la forme

$$Z(T) = \sum_{\xi \in G(F)} Z(T, \xi),$$

où les $Z(T, \xi)$ sont des séries à coefficients dans un anneau de Grothendieck décrit plus loin. Cette égalité est l'analogue, dans ce cadre, de l'identité (1.2). Chambert-Loir et Loeser étudient alors les fonctions $Z(T, \xi)$ séparément : puisque, comme expliqué plus haut, seul un nombre fini de places contribue, celles-ci sont des produits *finis* de facteurs locaux (alors que les décompositions $H(x) = \prod_v H_v(x)$ chez Chambert-Loir et Tschinkel avaient, elles, un nombre infini de facteurs différents de 1) qui peuvent être réécrits comme des intégrales motiviques sur l'espace d'arcs de la variété $\mathcal{X}$, de type fonctions zêta d'Igusa motiviques, dont l'étude remonte à Denef et Loeser ([DL98]). Par une méthode analogue à l'analyse de Chambert-Loir et Tschinkel, la réduction au corps résiduel étant remplacée par la réduction de l'espace d'arcs à la fibre spéciale, Chambert-Loir et Loeser prouvent que chaque facteur est une fonction rationnelle. Ainsi, dans cette situation la fonction zêta des hauteurs se trouve être rationnelle, et dans [ChL] l'équivalent du théorème 1 s'énonce en décrivant ses dénominateurs. Le coefficient en $\mathbf{L}^{-1}$ est dans ce cas un produit fini de volumes motiviques.

Notre approche dans ce texte est grandement inspirée de celle de Chambert-Loir et Loeser, mais demande cependant plusieurs adaptations majeures. Tout d'abord, puisque nous imposons moins de contraintes sur les pôles des sections que nous comptons, ceux-ci ne vivent plus dans un ensemble fini fixé : les produits de facteurs locaux, finis chez Chambert-Loir et Loeser, n'ont donc plus de raison de l'être dans notre cas de figure, et pour traiter ce problème, nous avons besoin de donner un sens à des analogues motiviques des produits infinis $H(x) = \prod_v H_v(x)$ utilisés par Chambert-Loir et Tschinkel. D'autre part, la fonction caractéristique des sections paramétrées par l'espace M_d est dans notre cadre une fonction adélique compliquée, et une application directe de la formule de Poisson de Hrushovski et Kazhdan n'est a priori pas possible. Enfin, l'apparition de produits infinis pose la question de leur convergence, ce qui demandera l'introduction d'une topologie adaptée sur les anneaux de Grothendieck considérés.

1.5 Outils et schéma de la preuve

1.5.1 Produit eulérien motivique

Il est bien connu que la fonction zêta de Riemann

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

admet une décomposition en *produit eulérien*

$$\zeta(s) = \prod_p (1-p^{-s})^{-1} = \prod_p \left(1 + p^{-s} + p^{-2s} + \dots\right),$$

où le produit se fait sur l'ensemble de tous les nombres premiers. Chaque facteur, séparément, converge pour $\Re(s) > 0$, mais le produit converge seulement pour $\Re(s) > 1$. Cette propriété de décomposition en produit eulérien est vérifiée plus généralement pour les séries de Dirichlet

$$\sum_{n\geq 1} a(n)n^{-s} = \prod_p (1 + a(p)p^{-s} + a(p^2)p^{-2s} + \dots)$$

où $a : \mathbf{N} \to \mathbf{C}$ est une fonction multiplicative. Un autre exemple, plus géométrique, est donné par la fonction zêta d'une variété X sur un corps fini $\mathbf{F}_q$:

$$\zeta_X(s) := \exp\left(\sum_{m\geq 1} \frac{|X(\mathbf{F}_{q^m})|}{m} q^{-ms}\right).$$

En notant X_{cl} l'ensemble des points fermés de la variété X, celle-ci peut en effet être écrite sous la forme d'un produit infini

$$\zeta_X(s) = \prod_{x\in X_{cl}} (1 - q^{-s\deg x})^{-1}. \tag{1.6}$$

Ici, comme pour la fonction zêta de Riemann, chaque facteur local converge pour $\Re(s) > 0$, mais la série $\zeta_X(s)$ converge seulement pour $\Re(s) > \dim X$: le fait de faire le produit avance l'abscisse de convergence de la dimension du schéma sur laquelle on fait le produit. L'exemple de la fonction zêta de Riemann peut aussi être compris de cette manière, si on voit l'ensemble des nombres premiers comme l'ensemble des points fermés du schéma arithmétique $\mathrm{Spec}\,(\mathbf{Z})$ de dimension 1.

La méthode de preuve esquissée ci-dessus pour le problème de Manin classique au-dessus d'un corps de nombres montre que les deux outils principaux pour l'aborder sont la possibilité de décomposer en un produit infini de facteurs locaux, et la théorie de Fourier. Le chapitre 3 de cette thèse introduit à cet effet une notion de *produit eulérien motivique* qui donne un sens, pour toute variété quasi-projective X et toute famille $\mathscr{X} = (X_i)_{i\geq 1}$ de variétés quasi-projectives au-dessus de X (ou, plus généralement, de classes dans un anneau de Grothendieck relatif au-dessus de X), à un produit de la forme

$$\prod_{x\in X}\left(1 + X_{1,x}t + X_{2,x}t^2 + \ldots\right), \tag{1.7}$$

où on peut voir chaque $X_{i,x}$ comme la fibre de X_i au-dessus du point $x \in X$. Par analogie avec les exemples ci-dessus issus de la théorie des nombres, il faut penser à la variable t comme correspondant à $q^{-s\deg x}$, du moins lorsque le corps k est algébriquement clos.

Pour définir (1.7), nous commençons par construire les coefficients de la série qui devrait être le développement d'un tel produit. En tentant de développer (1.7) naïvement, on observe que toute contribution au coefficient de degré n consiste à choisir un certain terme dans chaque facteur, de sorte à ce que la somme des degrés des termes choisis soit n, ce qui induit une certaine partition de l'entier n. Nous construisons séparément la partie du coefficient de degré n correspondant à une partition π de n fixée. En notant $\pi = (n_i)_{i\geq 1}$ où n_i est le nombre d'occurrences de l'entier i dans la partition π, de sorte que $\sum_{i\geq 1} in_i = n$, nous définissons le *produit symétrique* $S^\pi\mathscr{X}$ de la famille $\mathscr{X}$ de la manière suivante : tout d'abord, puisque nous voulons construire la partie du coefficient de degré n correspondant à la partition π, il s'agit de choisir chaque terme $X_{i,v}t^i$ dans exactement n_i parenthèses, ce qui nous amène à considérer le produit

$$\prod_{i\geq 1} X_i^{n_i}. \tag{1.8}$$

D'autre part, ces termes ont été choisis dans des facteurs distincts, c'est-à-dire se trouvant au-dessus de points $v \in X$ distincts. Ainsi, en considérant le morphisme

$$\prod_{i\geq 1} X_i^{n_i} \to \prod_{i\geq 1} X^{n_i}$$

induit par les morphismes structurels $X_i \to X$, nous devons nous restreindre aux points du produit (1.8) d'image dans $\prod_{i\geq 1} X^{n_i}$ ayant toutes ses coordonnées distinctes, autrement dit, se trouvant dans le complémentaire de la grande diagonale. En notant

$$\left(\prod_{i\geq 1} X_i^{n_i}\right)_{*,X}$$

l'ouvert ainsi obtenu, il ne reste plus qu'à observer qu'il n'y a pas d'ordre particulier entre les facteurs du produit eulérien ci-dessus et qu'il convient donc de quotienter par l'action naturelle du produit de groupes symétriques $\prod_{i\geq 1}\mathfrak{S}_{n_i}$ agissant par permutation. Nous posons donc

$$S^{\pi}\mathscr{X} = \left(\prod_{i\geq 1} X_i^{n_i}\right)_{*,X} / \prod_{i\geq 1}\mathfrak{S}_{n_i},$$

qui existe comme variété sous les hypothèses de quasi-projectivité supposées ci-dessus. Le produit eulérien (1.7) sera ainsi introduit dans la section 3.8 tout d'abord comme une *notation* pour la série

$$1 + \sum_{n\geq 1}\left(\sum_{\substack{\pi \text{ partition} \\ \text{de } n}} [S^{\pi}\mathscr{X}]\right) t^n \in \mathrm{KVar}_k[[t]].$$

En montrant diverses propriétés de la construction géométrique que nous venons de décrire, nous mettrons en évidence dans la section 3.8 des règles de calcul avec cette notion de produit eulérien, dont la propriété de multiplicativité

$$\prod_{x\in X}\left(1 + X_{1,x}t + X_{2,x}t^2 + \ldots\right)$$
$$= \prod_{x\in U}\left(1 + X_{1,x}t + X_{2,x}t^2 + \ldots\right)\prod_{x\in Y}\left(1 + X_{1,x}t + X_{2,x}t^2 + \ldots\right)$$

pour tout sous-schéma fermé Y de X d'ouvert complémentaire U, qui montre que nous avons effectivement défini quelque chose qui se comporte comme un produit.

Un exemple important de série à coefficients dans KVar_k est la fonction zêta de Kapranov, introduite par Kapranov dans [Kapr]. Pour une variété quasi-projective X sur k, on note S^nX, pour tout $n \geq 0$ sa n-ième puissance symétrique $X^n/\mathfrak{S}_n$, qui est également une variété quasi-projective, et on définit

$$Z_X(t) = \sum_{n\geq 0}[S^nX]t^n \in \mathrm{KVar}_k[[t]].$$

C'est l'analogue motivique de la fonction $\zeta_X(s)$ ci-dessus, au sens où, pour k fini, elle se spécialise vers cette dernière via la mesure de comptage. Sa décomposition en produit eulérien motivique est donnée par

$$Z_X(t) = \prod_{x\in X}(1 + t + t^2 + \ldots) = \prod_{x\in X}\frac{1}{1-t},$$

ce qui est l'analogue motivique de la décomposition en produit eulérien (1.6) de la fonction zêta d'une variété sur un corps fini. Notons que cette écriture de $Z_X(t)$ était déjà possible dans le cadre de la notion de *puissance motivique* due à Guisein-Zade, Luengo et Melle ([GZLM]), dont nos produits eulériens sont une généralisation.

1.5.2 Formule de Poisson de Hrushovski et Kazhdan

La formule de Poisson de Hrushovski et Kazhdan, démontrée dans [HK09], est un analogue motivique (d'une version affaiblie) de la formule de Poisson sur les adèles d'un corps de nombres telle qu'elle intervient dans la démonstration du théorème de Chambert-Loir et Tschinkel ci-dessus. Tout d'abord, pour pouvoir faire de l'analyse de Fourier dans un cadre motivique, il convient de travailler dans un groupe de Grothendieck plus grand, le groupe de Grothendieck des variétés avec exponentielles $\mathrm{KExpVar}_k$ d'un corps k. Il est donné comme le quotient du groupe abélien libre sur les classes d'isomorphisme de couples (X, f) avec X une variété sur k et $f : X \to \mathbf{A}^1$ un morphisme, par des relations de découpage analogues à celles de l'anneau KVar_k classique, ainsi que par la relation supplémentaire

$$(X \times \mathbf{A}^1, \mathrm{pr}_2) \tag{1.9}$$

pour toute variété X sur k, avec $\mathrm{pr}_2 : X \times \mathbf{A}^1 \to \mathbf{A}^1$ la seconde projection. Il y a également un produit sur $\mathrm{KExpVar}_k$ tel que dans le cas où le corps k est fini, la mesure de comptage sur KVar_k s'étende, pour tout caractère non-trivial $\psi : k \to \mathbf{C}^*$ du corps k, en une mesure motivique

$$\begin{array}{ccc} \mathrm{KExpVar}_k & \to & \mathbf{C} \\ [X, f] & \to & \sum_{x \in X(k)} \psi(f(x)) \end{array} \tag{1.10}$$

la relation (1.9) traduisant le fait que, le caractère ψ étant non-trivial, on ait

$$\sum_{x \in k} \psi(x) = 0,$$

propriété essentielle au fonctionnement de l'analyse de Fourier. Les fonctions « motiviques » que nous considèrerons prendront leurs valeurs dans l'anneau $\mathrm{KExpVar}_k$, ou plutôt dans son localisé $\mathscr{E}xp\mathscr{M}_k$ obtenu en inversant la classe $[\mathbf{A}^1, 0]$. L'application naturelle $\mathrm{KVar}_k \to \mathrm{KExpVar}_k$ donnée par $[X] \mapsto [X, 0]$ est un morphisme d'anneaux injectif.

La formule de Poisson classique sur les adèles d'un corps de nombres F s'énonce souvent pour des *fonctions de Schwartz-Bruhat*. Ce sont des combinaisons linéaires de fonctions $f : \mathbb{A}_F \to \mathbf{C}$ s'écrivant sous la forme de produit

$$f = \prod_v f_v$$

tel que pour toute place v, f_v soit une fonction $F_v \to \mathbf{C}$ sur le complété F_v en v, lisse à décroissance rapide si v est archimédienne, localement constante à support compact si v est non-archimédienne, et égale à la fonction caractéristique $\mathbf{1}_{\mathcal{O}_v}$ de l'anneau des entiers de F_v pour presque toutes les places non-archimédiennes. Au-dessus d'un corps de fonctions, nous n'avons que des places non-archimédiennes, et Hrushovski et Kazhdan proposent un analogue géométrique d'une fonction de Schwartz-Bruhat pour celles-ci.

Pour une fonction de Schwartz-Bruhat $f : F \to \mathbf{C}$ définie sur un corps local non-archimédien F (dont on note $\mathcal{O}$ l'anneau des entiers, t une uniformisante et k le corps résiduel), on dispose de deux entiers $M, N \geq 0$ tels que f soit nulle en dehors de $t^{-M}\mathcal{O}$, et invariante modulo le sous-groupe $t^N\mathcal{O}$. Par conséquent, une telle fonction peut être vue comme une fonction sur le quotient $t^{-M}\mathcal{O}/t^N\mathcal{O}$, qui se trouve être un k-espace vectoriel de dimension $M+N$. Ainsi, dans le cadre motivique, une fonction de Schwartz-Bruhat locale (de niveau $(-M,N)$) sera une fonction définie sur un espace affine $\mathbf{A}_k^{M+N}$ (noté $\mathbf{A}_k^{(-M,N)}$ pour garder en mémoire les valeurs de M et N) et à valeurs dans l'anneau $\mathrm{KExpVar}_k$. Plus précisément, elles sont introduites comme les éléments de l'anneau de Grothendieck relatif $\mathrm{KExpVar}_{\mathbf{A}_k^{(-M,N)}}$.

Cette construction peut également être effectuée pour produire un analogue motivique pour les fonctions localement constantes et à support compact définies sur un produit fini de corps locaux, et la formule de Poisson de Hrushovski et Kazhdan est démontrée pour ces fonctions. Ainsi, cette formule est l'analogue de la formule de Poisson sur les adèles d'un corps de nombres F, mais appliquée à une fonction adélique $f:\mathbb{A}_F\to\mathbf{C}$ *dont la restriction au complété F_v pour presque toute place v est la fonction caractéristique de l'anneau des entiers $\mathcal{O}_v$*.

Comme nous l'avons déjà mentionné plus haut, cette formule était suffisante pour les besoins du travail de Chambert-Loir et Loeser [ChL], car la fonction hauteur pour le problème de comptage considéré vérifiait cette hypothèse très restrictive. Dans le cas général que nous traitons, les pôles des sections que nous comptons ne sont pas fixes. Il s'agit alors d'établir un cadre dans lequel on peut appliquer la formule de Hrushovski et Kazhdan en famille, en faisant varier le lieu des pôles des sections. Pour cela, nous utilisons la notion de produit symétrique introduite plus haut. Pour expliquer cela dans un cas simple, considérons, pour tout $i\geq 1$ et pour deux entiers $M_i,N_i\geq 0$ la variété

$$\mathbf{A}_C^{(-M_i,N_i)} := C\times\mathbf{A}_k^{(-M_i,N_i)}$$

au-dessus de la courbe C. On peut former, pour tout entier $m\geq 0$ le produit symétrique

$$S^m((\mathbf{A}_C^{(-M_i,N_i)})_{i\geq 1})$$

de la famille $\left(\mathbf{A}_C^{(-M_i,N_i)}\right)_{i\geq 1}$. Ce produit symétrique est naturellement muni d'un morphisme vers la puissance symétrique S^mC. On observe que pour tout zéro-cycle effectif $D=\sum_v m_v v\in S^mC(k)$, la fibre de $S^m((\mathbf{A}_C^{(-M_i,N_i)})_{i\geq 1})$ est de la forme

$$\prod_{v\in C}\mathbf{A}_k^{(-M_{m_v},N_{m_v})},$$

c'est-à-dire que c'est le domaine de définition d'une fonction de Schwartz-Bruhat au sens de Hrushovski et Kazhdan, dont le support et les pôles sont contrôlés, respectivement, par les zéro-cycles

$$-\sum_v M_{m_v}v \quad\text{et}\quad \sum_v N_{m_v}v.$$

Ainsi, en faisant varier le zéro-cycle D, on peut de cette manière paramétrer des fonctions de pôles variables (si les M_i sont suffisamment grands) et de supports variables (si les N_i sont suffisamment grands). Nous expliquons ensuite que toutes les opérations de la théorie de Hrushovski et Kazhdan peuvent être effectuées en famille au-dessus de S^mC pour tout $m\geq 1$, et justifions en particulier la validité de la formule de Poisson dans ce cadre.

1.5.3 Filtration par les poids et convergence

Revenons maintenant à la fonction zêta des hauteurs motivique (1.5). Grâce à un découpage des espaces de modules M_d suivant les valeurs et ordres des pôles des sections, on peut lui appliquer la formule de Poisson dans le cadre du paragraphe précédent. Cela nous permet, de manière analogue à ce qui a été expliqué dans la section 1.2, de réécrire la fonction zêta sous la forme

$$Z(T) = \sum_{\xi \in k(C)^n} Z(T, \xi) \tag{1.11}$$

pour des séries $Z(T, \xi)$ à coefficients dans $\mathscr{E}xp\mathscr{M}_k$, chacune admettant une décomposition en produit eulérien. Nous arrivons maintenant à la question de la convergence de ces produits eulériens : comme chez Chambert-Loir et Tschinkel, nous souhaitons montrer que la série $Z(T, 0)$ est l'unique responsable du premier pôle de la fonction $Z(T)$ en $\mathbf{L}^{-1}$, et la prolonger méromorphiquement au-delà de celui-ci. Nous avons mentionné plus haut que la filtration dimensionnelle sur l'anneau de Grothendieck des variétés ne peut pas nous fournir la convergence attendue pour la fonction $Z(T)$. Pour comprendre cela, revenons aux majorations de Chambert-Loir et Tschinkel : pour presque tous les facteurs locaux, leurs calculs font apparaître des différences de la forme $D_\alpha(\mathbf{F}_q) - q^{n-1}$ pour chaque composante irréductible D_α du diviseur à l'infini D, q la puissance d'un nombre premier, et n la dimension de X. Afin d'obtenir la convergence souhaitée, il est crucial d'utiliser ici une majoration non triviale, donnée par les estimées de Lang-Weil [LW] :

$$\left| D_\alpha(\mathbf{F}_q) - q^{n-1} \right| \leq c q^{n-\frac{3}{2}}, \tag{1.12}$$

pour une constante $c > 0$. Dans le cadre motivique, les calculs sont complètement analogues, et on se retrouve donc naturellement avec le même genre de différence, à savoir $[D_\alpha] - \mathbf{L}^{n-1} \in \mathscr{M}_k$. En général, la dimension de cet élément de $\mathscr{M}_k$ sera $n-1$: par exemple, si $[D_\alpha]$ est une courbe de genre $g \geq 1$, alors, le polynôme de Hodge-Deligne

$$HD([D_\alpha] - \mathbf{L}^{n-1}) = (1 + gu + gv + uv) - uv = 1 + gu + gv,$$

est de degré 1, ce qui montre que $[D_\alpha] - \mathbf{L}^{n-1}$ ne peut être une combinaison linéaire d'éléments de dimension ≤ 0. La filtration dimensionnelle fournit donc une borne plus faible que celle de Lang-Weil dans le cas arithmétique, et on est par conséquent conduit à utiliser une topologie un peu plus fine sur l'anneau de Grothendieck des variétés, construite à l'aide de la théorie de Hodge.

Supposons temporairement que $k = \mathbf{C}$. Il existe une mesure motivique

$$\chi^{\mathrm{Hdg}} : \mathscr{M}_{\mathbf{C}} \to K_0(HS),$$

appelée *réalisation de Hodge*, à valeurs dans l'anneau de Grothendieck des structures de Hodge, qui à la classe d'une variété complexe Y associe

$$\chi(Y) = \sum_{i=0}^{2\dim Y} (-1)^i [H^i_c(Y(\mathbf{C}), \mathbf{Q})], \tag{1.13}$$

où $[H^i_c(Y(\mathbf{C}),\mathbf{Q})]$ est la classe dans $K_0(HS)$ de la structure de Hodge mixte sur le i-ème groupe de cohomologie singulière à support compact de $Y(\mathbf{C})$. Il y a une filtration croissante naturelle $(W_{\leq n}K_0(HS))_{n\in\mathbf{Z}}$ par les poids sur l'anneau $K_0(HS)$, donnée en définissant $W_{\leq n}K_0(HS)$ comme le sous-groupe de $K_0(HS)$ engendré par les classes des structures de Hodge pures de poids $\leq n$. Pour un élément $\mathfrak{a}\in\mathscr{M}_{\mathbf{C}}$, on définit son poids par

$$w(\mathfrak{a}) = \inf\{n\in\mathbf{Z}, \chi^{\mathrm{Hdg}}(\mathfrak{a})\in W_{\leq n}K_0(HS).\}.$$

La formule (1.13) donnant χ^{Hdg} implique alors directement que pour la classe d'une variété Y, nous avons $w(Y) = 2\dim Y$. Plus précisément, nous avons

$$H^{2\dim Y}(Y(\mathbf{C}),\mathbf{Q}) \simeq \mathbf{Q}(-\dim Y)^{\kappa(Y)},$$

où $\kappa(Y)$ est le nombre de composantes irréductibles de dimension maximale de Y et $\mathbf{Q}(-\dim Y)$ est l'unique structure de Hodge pure de poids $2\dim Y$ et de dimension 1. Ainsi, on observe que dans l'expression de

$$\chi^{\mathrm{Hdg}}([D_\alpha] - \mathbf{L}^{n-1})$$

les termes correspondant aux groupes de cohomologie de degré maximal se simplifient, et que par conséquent

$$w([D_\alpha]-\mathbf{L}^{n-1}) \leq 2n-3 = 2\left(n-\frac{3}{2}\right).$$

Cette borne est l'analogue, dans le cadre motivique, de l'inégalité (1.12).

1.5.4 Cycles évanescents motiviques

Rappelons-nous que dans la décomposition (1.11) donnée par la formule de Poisson, les différentes séries $Z(T,\xi)$ sont à coefficients dans $\mathscr{E}xp\mathscr{M}_{\mathbf{C}}$ (on suppose toujours $k=\mathbf{C}$). Ainsi, nous avons besoin d'étendre la topologie des poids, définie dans le paragraphe précédent sur $\mathscr{M}_{\mathbf{C}}$, à l'anneau avec exponentielles $\mathscr{E}xp\mathscr{M}_{\mathbf{C}}$. De plus, il est souhaitable, pour que l'analogie avec le cas arithmétique continue à fonctionner, que la fonction de poids ainsi étendue vérifie *l'inégalité triangulaire* : pour toute variété complexe X munie d'un morphisme $f : X\to\mathbf{A}^1$, on souhaiterait que

$$w([X,f]) \leq w([X]). \tag{1.14}$$

Via la mesure motivique (1.10) sur l'anneau de Grothendieck avec exponentielles, on voit que l'inégalité (1.14) est l'analogue motivique de l'inégalité

$$\left|\sum_{x\in X(\mathbf{F}_q)}\psi(f(x))\right| \leq |X(\mathbf{F}_q)|$$

issue de l'inégalité triangulaire classique, pour une variété X sur $\mathbf{F}_q$, un morphisme $f : X\to\mathbf{A}^1$ et un caractère non-trivial $\psi:\mathbf{F}_q\to\mathbf{C}^*$.

La solution à ce problème utilise les *cycles évanescents motiviques* de Denef et Loeser. Pour une variété lisse X sur un corps k de caractéristique nulle et un morphisme $f : X \to \mathbf{A}^1_k$, les cycles évanescents $\varphi_{f,a}$ de f au voisinage de $a \in k$ sont un élément de l'anneau de Grothendieck localisé $\mathscr{M}^{\hat{\mu}}_{f^{-1}(a)}$ au-dessus de la fibre $f^{-1}(a) \subset X$, avec action du groupe $\hat{\mu}$, limite projective des groupes des racines n-ièmes de l'unité pour tout $n \geq 1$. Lorsque la fibre $f^{-1}(a)$ est nulle part dense dans X, Denef et Loeser donnent une formule pour calculer $\varphi_{f,a}$ en termes des composantes du diviseur exceptionnel d'une log-résolution du couple $(X, f^{-1}(a))$.

En utilisant les travaux de Denef et Loeser à ce sujet, ainsi que l'article de Guibert, Loeser et Merle [GLM], Lunts et Schnürer ont prouvé dans [LS16b], que dans le cas où k est algébriquement clos de caractéristique nulle, en combinant les cycles évanescents motiviques en tous les points de k, on pouvait définir un morphisme d'anneaux, appelé la mesure des cycles évanescents motiviques,

$$\Phi : \mathscr{E}xp\mathscr{M}_k \to (\mathscr{M}^{\hat{\mu}}_k, *)$$

où $*$ est un *produit de convolution* dont la définition est due à Looijenga, Denef et Loeser. Pour une classe $[X, f]$ avec X lisse et $f : X \to \mathbf{A}^1_k$ propre, nous avons

$$\Phi([X, f]) = \sum_{a \in k} f_! \varphi_{f,a}.$$

L'ingrédient principal dans cette preuve est un théorème de Thom-Sebastiani pour les cycles évanescents motiviques, dû à Denef et Loeser, qui est l'analogue motivique du théorème correspondant dans la théorie classique. Dans le chapitre 2 de cette thèse, nous avons étendu la définition de la mesure Φ à k de caractéristique nulle non nécessairement algébriquement clos, puis au-dessus d'une k-variété S quelconque.

Les cycles évanescents motiviques nous fournissent ainsi une manière de passer de $\mathscr{E}xp\mathscr{M}_{\mathbf{C}}$ à $\mathscr{M}^{\hat{\mu}}_{\mathbf{C}}$. Afin de pouvoir étendre la filtration par le poids à l'anneau $\mathscr{E}xp\mathscr{M}_{\mathbf{C}}$, nous utilisons alors le fait que la réalisation de Hodge évoquée plus haut se généralise pour donner un morphisme

$$\mathscr{M}^{\hat{\mu}}_{\mathbf{C}} \to K_0(HS^{\mathrm{mon}}) \tag{1.15}$$

de l'anneau de Grothendieck des variétés complexes avec $\hat{\mu}$-action, vers l'anneau de Grothendieck des structures de Hodge avec action par un opérateur linéaire d'ordre fini (appelé *monodromie*), qui est également un morphisme d'anneaux lorsque les deux groupes sont munis de produits appropriés. Composer cette réalisation de Hodge avec le morphisme Φ revient alors à regarder la structure de Hodge sur les cycles évanescents (au sens classique), avec l'action naturelle de la monodromie autour de la singularité considérée. Nous étendons alors la filtration par le poids au groupe $K_0(HS^{\mathrm{mon}})$, en définissant $W_{\leq n}K_0(HS^{\mathrm{mon}})$ pour tout $n \in \mathbf{Z}$ comme le sous-groupe engendré par les structures de Hodge pures de poids $\leq n$ avec monodromie triviale, ainsi que les structures de Hodge pures de poids $\leq n-1$ avec monodromie non-triviale. En utilisant la formule de Denef-Loeser évoquée plus haut, on voit alors que l'inégalité triangulaire est vérifiée.

La dernière difficulté qui se présente est d'étudier la convergence des produits eulériens en utilisant cette filtration par le poids, ce qui nécessite en fait d'avoir les résultats ci-dessus *en famille* : nous utiliserons donc notre construction de la mesure des cycles évanescents motiviques au-dessus d'une base S, ainsi que la théorie des modules de Hodge mixtes de Saito, dans le chapitre 4 de ce texte.

1.6 Plan et quelques énoncés

Pour finir, récapitulons le contenu de cette thèse : le chapitre 2 contient les définitions des divers anneaux de Grothendieck utilisés et rappelle le formalisme des cycles évanescents motiviques. Nous y donnons une définition, pour une variété X sur corps k de caractéristique nulle, des cycles évanescents totaux φ_f^{tot} d'un morphisme $f : X \to \mathbf{A}^1$: c'est un élément de $\mathscr{M}_{\mathbf{A}^1_k}^{\hat{\mu}}$ caractérisé par le fait que la fibre au-dessus de tout point $a \in \mathbf{A}^1$ est donnée par les cycles évanescents $f_! \varphi_{f,a} \in \mathscr{M}_{k(a)}^{\hat{\mu}}$ de f en a (poussés vers le point a par $f^{-1}(a) \to \{a\}$). Nous montrons que les cycles évanescents totaux peuvent servir à construire une mesure motivique, généralisant le résultat de Lunts et Schnürer au-dessus d'un corps de caractéristique nulle quelconque, puis en famille.

Théorème 3 (Théorème 2.3.5.1)**.** *Soit k un corps de caractéristique nulle.*

1. *Il existe un unique morphisme de $\mathscr{M}_k$-algèbres*

$$\Phi_k : \mathscr{E}xp\mathscr{M}_k \to (\mathscr{M}_k^{\hat{\mu}}, *),$$

appelé mesure des cycles évanescents motiviques, tel que pour toute variété lisse X et tout morphisme propre $f : X \to \mathbf{A}^1_k$, on ait

$$\Phi_k([X \xrightarrow{f} \mathbf{A}^1_k]) = \epsilon_!(\varphi_f^{\mathrm{tot}})$$

où $\epsilon : \mathbf{A}^1_k \to k$ est le morphisme structurel.

2. *Soit S une variété sur k. Il existe un unique morphisme de $\mathscr{M}_k$-algèbres*

$$\Phi_S : \mathscr{E}xp\mathscr{M}_S \to (\mathscr{M}_S^{\hat{\mu}}, *)$$

tel que, pour tout $s \in S$, le diagramme

$$\begin{array}{ccc} \mathscr{E}xp\mathscr{M}_S & \xrightarrow{\Phi_S} & \mathscr{M}_S^{\hat{\mu}} \\ \downarrow & & \downarrow \\ \mathscr{E}xp\mathscr{M}_{\kappa(s)} & \xrightarrow{\Phi_{\kappa(s)}} & \mathscr{M}_{\kappa(s)}^{\hat{\mu}} \end{array}$$

commute.

De plus pour toute variété S sur k, la restriction de Φ_S à $\mathscr{M}_S$ coïncide avec l'inclusion naturelle $\mathscr{M}_S \to \mathscr{M}_S^{\hat{\mu}}$.

Les produits eulériens motiviques sont définis dans le chapitre 3 et leurs propriétés de calcul y sont explicitées. Pour les besoins de la suite, les définitions sont données dans un cadre très général, au-dessus d'une base R et pour un système d'indéterminées $(t_i)_{i\in I}$ indexé par un ensemble I qui typiquement sera de la forme $\mathbf{N}^p\setminus\{0\}$ pour un entier $p\geq 1$. Voici un condensé des propriétés de ces produits eulériens :

Théorème 4. *Soit $A\in\{\mathrm{KVar},\mathscr{M},\mathrm{KExpVar},\mathscr{E}xp\mathscr{M}\}$. Soit R une variété sur un corps parfait k. Soit X une variété sur R et $\mathscr{X}=(X_i)_{i\in I}$ une famille d'éléments de A_X. Alors on a les égalités suivantes dans l'anneau $A_R[[(t_i)_{i\in I}]]$:*

1. *(Produit à un facteur) Quand $X=R$ on a*

$$\prod_{u\in R/R}\left(1+\sum_{i\in I}X_{i,u}t_i\right)=1+\sum_{i\in I}X_it_i.$$

2. *(Associativité) Soit $X=U\cup Y$ une partition de X en un sous-schéma fermé Y et son complémentaire U, et $\mathscr{U}=(U_i)_{i\in I}$ (resp. $\mathscr{Y}=(Y_i)_{i\in I}$) la restriction de $\mathscr{X}$ à U (resp. à Y). Alors*

$$\prod_{u\in X/R}\left(1+\sum_{i\in I}X_{i,u}t_i\right)=\prod_{u\in U/R}\left(1+\sum_{i\in 1}U_{i,u}t_i\right)\prod_{u\in Y/R}\left(1+\sum_{i\in I}Y_{i,u}t_i\right).$$

3. *(Produits finis) Si X est une union disjointe de m variétés $Y_1,\ldots,Y_m$ isomorphes à R,*

$$\prod_{v\in X/R}\left(1+\sum_{i\in I}X_{i,v}t_i\right)=\prod_{j=1}^{m}\left(1+\sum_{i\in I}X_{i,j}t_i\right)\in\mathrm{KVar}_R[[\mathbf{t}]]$$

   où pour tout $i\in I$, $X_{i,j}$ est la restriction de X_i à $Y_j\simeq R$, et le produit dans le côté droit est un produit fini de séries formelles au sens classique.

4. *(Changement de variables de la forme $\mathbf{t}\mapsto\mathbf{L}^{\mathbf{m}}\mathbf{t}$) Pour tout $(m_i)_{i\in I}\in\mathbf{N}^I$,*

$$\prod_{u\in X/R}\left(1+\sum_{i\in I}X_{i,u}(\mathbf{L}^{m_i}t_i)\right)=\prod_{u\in X/R}\left(1+\sum_{i\in I}(X_{i,u}\mathbf{L}^{m_i})t_i\right).$$

5. *On suppose que la variété R est elle-même une R'-variété pour une k-variété R', et que les X_i sont des variétés (avec exponentielles) au-dessus de X. Alors le double produit*

$$\prod_{v\in R/R'}\left(\prod_{u\in X/R}\left(1+\sum_{i\in I}X_{i,u}t_i\right)\right)_v$$

   a un sens et est égal à

$$\prod_{v\in R/R'}\left(\prod_{u\in X/R}\left(1+\sum_{i\in I}X_{i,u}t_i\right)\right)_v=\prod_{u\in X/R'}\left(1+\sum_{i\in I}X_{i,u}t_i\right)$$

   dans $\mathrm{KExpVar}_R[[(t_i)_{i\in I}]]$.

6. *(Compatibilité avec les produits finis) Pour des familles de variétés (avec exponentielles) $(Y_i)_{i\in I}$ and $(Z_i)_{i\in I}$ au-dessus de X qui est lui-même supposé être une réunion de deux copies de R, on a :*

$$\prod_{v\in R/R'}\left(1+\sum_{i\in I}Y_{i,v}t_i\right)\left(1+\sum_{i\in I}Z_{i,v}t_i\right)$$

$$=\prod_{v\in R/R'}\left(1+\sum_{i\in I}Y_{i,v}t_i\right)\prod_{v\in R/R'}\left(1+\sum_{i\in I}Z_{i,v}t_i\right)$$

dans $\mathrm{KExpVar}_R[[(t_i)_{i\in I}]]$.

Une remarque sur cet énoncé : les deux dernières propriétés ne sont pas suffisantes pour l'usage que nous ferons des produits eulériens, mais à l'heure de la soumission de ce manuscrit, nous n'avons pu établir les deux dernières propriétés que pour des séries à coefficients *effectifs* (c'est-à-dire, s'écrivant comme combinaisons linéaires à coefficients positifs de classes de variétés dans l'anneau de Grothendieck des variétés). C'est de là que vient l'hypothèse supplémentaire que nous introduisons :

Hypothèse 3. *Soit X une variété sur un corps algébriquement clos de caractéristique nulle k, soit $(X_i)_{i\geq 1}$ une famille de classes effectives dans $\mathscr{E}xp\mathscr{M}_X$, et soient $a\geq 0$, $b\geq 1$ des entiers. Alors on a*

$$\prod_{x\in X}(1+X_{1,x}t+X_{2,x}t^2+\dots)(1-\mathbf{L}^a t^b)=\prod_{x\in X}(1+X_{1,x}t+\dots)\prod_{x\in X}(1-\mathbf{L}^a t^b)$$

dans $\mathscr{E}xp\mathscr{M}_k[[t]]$.

Le chapitre 4, après un bref survol de quelques faits utiles de la théorie des modules de Hodge de Saito, fournit la construction de la fonction de poids. Nous commençons par la construire sur l'anneau de Grothendieck des modules de Hodge mixtes avec monodromie $K_0(\mathrm{MHM}_S^{\mathrm{mon}})$ sur une variété complexe S. Ensuite nous la composons avec la réalisation de Hodge (généralisation au-dessus de S de la réalisation de Hodge (1.15))

$$\chi_S^{\mathrm{Hdg}}:\mathscr{M}_S^{\hat{\mu}}\to K_0(\mathrm{MHM}_S^{\mathrm{mon}}),$$

puis avec la mesure des cycles évanescents $\Phi_S:\mathscr{E}xp\mathscr{M}_S\to\mathscr{M}_S^{\hat{\mu}}$ ci-dessus. Il s'agit alors, d'une part, de vérifier que cette composition conserve les propriétés de la filtration par le poids, et d'autre part, de montrer diverses compatibilités avec les produits symétriques qui assurent que cette topologie permettra de contrôler convenablement la convergence des produits eulériens.

Théorème 5. *Soit S une variété complexe. Il existe une application*

$$w_S:\mathscr{E}xp\mathscr{M}_S\to\mathbf{Z}\cup\{-\infty\},$$

appelée la fonction de poids, qui vérifie les propriétés suivantes : pour S,T des variétés complexes et pour tous $\mathfrak{a},\mathfrak{a}'\in\mathscr{E}xp\mathscr{M}_S$, $\mathfrak{b}\in\mathscr{E}xp\mathscr{M}_T$:

1. $w_S(0) = -\infty$.
2. $w_S(\mathfrak{a} + \mathfrak{a}') \leq \max\{w_S(\mathfrak{a}), w_S(\mathfrak{a}')\}$ *avec égalité si* $w_S(\mathfrak{a}) \neq w_S(\mathfrak{a}')$.
3. $w_{S\times T}(\mathfrak{a} \boxtimes \mathfrak{b}) \leq w_S(\mathfrak{a}) + w_T(\mathfrak{b})$.
4. Si $f : S \to T$ *est un morphisme avec des fibres de dimension* $\leq d$*, alors*

$$w_T(f_!(\mathfrak{a})) \leq w_S(\mathfrak{a}) + d.$$

5. Si $f : S \to T$ *est un morphisme avec des fibres de dimension* $\leq d$*, alors*

$$w_S(f^*(\mathfrak{b})) \leq w_T(\mathfrak{b}) + d.$$

6. Soit I *un ensemble et* $\pi = (n_i)_{i\in I} \in \mathbf{N}^{(I)}$*. Soit* X *une variété complexe et* $\mathscr{A} = (\mathfrak{a}_i)_{i\in I}$ *une famille d'éléments de* $\mathscr{E}xp\mathscr{M}_X$*. Alors*

$$w_{S^\pi X}(S^\pi \mathscr{A}) \leq \sum_{i\in I} n_i w_X(\mathfrak{a}_i).$$

7. Si X *est une variété sur* S*,* $w_S(X) = 2\dim_S X + \dim S$.
8. (Inégalité triangulaire) Soit X *une variété sur* S *et* $f : X \to \mathbf{A}^1$ *un morphisme. Alors*

$$w_S([X, f]) \leq w_S(X).$$

9. Soient $p : X \to S$*,* $q : Y \to S$ *des morphismes lisses de fibres de dimension* $d \geq 0$*, avec* X *et* Y *irréductibles. Alors*

$$w_S([X \xrightarrow{p} S] - [Y \xrightarrow{q} S]) \leq 2d + \dim S - 1.$$

Nous définissons le rayon de convergence d'une série $F(T) = \sum_{i\geq 0} X_i T^i \in \mathscr{E}xp\mathscr{M}_X[[T]]$ par

$$\sigma_F = \limsup_{i\geq 1} \frac{w_X(X_i)}{2i}$$

et nous disons que F converge pour $|T| < \mathbf{L}^{-r}$ si $r \geq \sigma_F$. La propriété 6 du théorème 5 est fondamentale dans l'établissement du résultat de convergence suivant :

Proposition 6 (Proposition 4.7.2.1)**.** *Soit* $F(T) = 1 + \sum_{i\geq 1} X_i T^i \in \mathscr{E}xp\mathscr{M}_X[[T]]$ *une série telle qu'il existe un entier* $M \geq 0$ *et des nombres réels* $\epsilon > 0$*,* $\alpha < 1$ *et* β *tels que*

— *pour tout* $i \in \{1, \ldots, M\}$*,* $w_X(X_i) \leq (i - \frac{1}{2} - \epsilon)w(X)$
— *pour tout* $i \geq M + 1$*,* $w_X(X_i) \leq (\alpha i + \beta - \frac{1}{2})w(X)$*;*

Alors il existe $\delta > 0$ *tel que le produit eulérien* $\prod_{v\in X} F_v(T) \in \mathscr{E}xp\mathscr{M}_{\mathbf{C}}[[T]]$

— *converge pour* $|T| < \mathbf{L}^{-\frac{w(X)}{2}\left(1-\delta+\frac{\beta}{M+1}\right)}$
— *pour tout* $0 \leq \eta < \delta$*, prenne des valeurs non-nulles pour* $|T| \leq \mathbf{L}^{-\frac{w(X)}{2}\left(1-\eta+\frac{\beta}{M+1}\right)}$ *(c'est-à-dire, pour tout* $\mathfrak{a} \in \mathscr{E}xp\mathscr{M}_{\mathbf{C}}$ *tel que* $w(\mathfrak{a}) < -w(X)\left(1 - \eta + \frac{\beta}{M+1}\right)$*).*

Nous terminons le chapitre 4 par un résultat permettant d'estimer, pour une série à coefficients effectifs $(M_n)_{n\geq 0}$, la croissance d'une proportion strictement positive des coefficients de $HD(M_n)$, à partir de la localisation et de l'ordre de son premier pôle.

Proposition 7 (Proposition 4.7.3.1)**.** *Soit* $Z(T) = \sum_{n\geq 0}[M_n]T^n \in \mathrm{KVar}^+_{\mathbf{C}}[[T]]$ *une série formelle à coefficients effectifs, telle qu'il existe des entiers* $a, r \geq 1$*, un nombre réel* $\delta > 0$ *et une série formelle* $F(T) = \sum_{i\geq 0} f_i T^i \in \mathscr{M}_{\mathbf{C}}[[T]]$ *convergeant pour* $|T| < \mathbf{L}^{-1+\delta}$ *et prenant une valeur effective non-nulle en* $T = \mathbf{L}^{-1}$*, telle que*

$$Z(T) = \frac{F(T)}{(1-\mathbf{L}^a T^a)^r}.$$

Alors pour tout $p \in \{0, \ldots, a-1\}$*, l'un des cas suivants se produit lorsque* n *tend vers l'infini dans la classe de congruence de* p *modulo* a *:*

(i) Soit $\limsup \frac{\dim(M_n)}{n} < 1$.

(ii) Soit $\dim(M_n) - n$ *a une limite finie* $d_0 \in \mathbf{Z}$ *et* $\frac{\log(\kappa(M_n))}{\log n}$ *converge vers un élément de l'ensemble* $\{0, \ldots, r-1\}$*. Plus généralement, pour tout nombre réel* η *tel que* $0 < \eta < \delta$ *et pour* n *assez grand dans la classe de congruence de* p *modulo* a*, les coefficients du polynôme de Hodge-Deligne* $HD(M_n)$ *de degrés appartenant à l'intervalle*

$$[2(1-\eta)n + 2d_0, 2n + 2d_0]$$

sont des polynômes en $\frac{n-p}{a}$ *de degré inférieur ou égal à* $r-1$*.*

De plus, le second cas de figure se produit pour au moins une valeur de p*.*

La formule de Poisson de Hrushovski et Kazhdan est rappelée et généralisée convenablement dans le chapitre 5. Enfin, le chapitre 6 contient la preuve du théorème 1, suivant la méthode de Chambert-Loir et Tschinkel dans sa version motivique due à Chambert-Loir et Loeser. Nous y introduisons le contexte géométrique du problème et la fonction zêta des hauteurs. Nous appliquons la formule de Poisson dans sa formulation du chapitre 5 à celle-ci. Comme évoqué plus haut, nous obtenons une décomposition

$$Z(T) = \sum_{\xi\in G(F)} Z(T,\xi) = \sum_{\xi\in G(F)} \prod_{v\in C} Z_v(T,\xi)$$

de la fonction zêta en somme de séries à coefficients dans $\mathscr{E}xp\mathscr{M}_k[[T]]$, qui elles-mêmes ont chacune une décomposition en produit eulérien. Nous étudions alors la convergence des facteurs $Z_v(T,\xi)$, d'abord pour les $v \in C_0$, puis pour les $v \in C \setminus C_0$, pour en déduire la convergence des séries $Z(T,\xi)$.

Notons $\mathscr{A}$ l'ensemble des composantes irréductibles du diviseur $X \setminus G$, et $\mathscr{A}_D \subset \mathscr{A}$ le sous-ensemble de celles qui sont contenues dans le diviseur à croisements normaux stricts $D = X \setminus U$. Le diviseur log-anticanonique associé à la paire (X, D) s'écrit sous la forme

$$-(K_X + D) = \sum_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D} \rho_\alpha D_\alpha + \sum_{\alpha\in\mathscr{A}_D} (\rho_\alpha - 1) D_\alpha$$

pour des entiers $\rho_\alpha \geq 2$. Par un calcul d'intégrales motiviques et l'application des propriétés de la fonction de poids (en particulier la propriété 9 de la proposition 6), nous obtenons :

Proposition 8. *Il existe un nombre réel $\delta > 0$ tel que le produit*

$$\prod_{v\in C_0}\left(Z_v(T,0)\prod_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}(1-\mathbf{L}^{\rho_\alpha-1}T^{\rho_\alpha})\right)$$

converge pour $|T| < \mathbf{L}^{-1+\delta}$ et prenne une valeur effective non-nulle dans $\widehat{\mathscr{M}_{\mathbf{C}}}$ en $T = \mathbf{L}^{-1}$.

Un élément de l'anneau complété $\widehat{\mathscr{M}_{\mathbf{C}}}$ est dit *effectif* s'il est limite d'éléments effectifs. La non-nullité de la valeur en $\mathbf{L}^{-1}$ est ici obtenue grâce à l'hypothèse 2. Cette valeur s'exprimera comme un produit eulérien.

Pour tout $\xi \in G(F)$, la forme linéaire $x \mapsto \langle x, \xi\rangle$ sur $G(F_v)$ se prolonge en une fonction méromorphe f_ξ sur X, dont le diviseur des pôles est inclus dans $X \setminus G$, et est donc de la forme

$$\mathrm{div}_\infty f_\xi = \sum_{\alpha\in\mathscr{A}} d_\alpha(\xi) D_\alpha.$$

En notant $\mathscr{A}_0(\xi)^D$ le sous-ensemble de $\mathscr{A} \setminus \mathscr{A}_D$ des α tels que $d_\alpha(\xi) = 0$, nous avons :

Proposition 9. *Il existe un nombre réel $\delta > 0$ tel que le produit*

$$\prod_{v\in C_0}\left(Z_v(T,\xi)\prod_{\alpha\in\mathscr{A}_0^D(\xi)}(1-\mathbf{L}^{\rho_\alpha-1}T^{\rho_\alpha})\right)$$

converge pour $|T| < \mathbf{L}^{-1+\delta}$.

Pour les places $v \in C \setminus C_0$, nous notons $d_v = 1 + \dim \mathrm{Cl}_v^{\mathrm{an}}(X, D)$ le terme combinatoire évoqué plus haut, défini à l'aide du complexe de Clemens analytique de (X, D). On a alors :

Proposition 10. *Il existe un réel $\delta > 0$ tel que pour tout multiple commun non-nul a des entiers $\rho_\alpha - 1$, $\alpha \in \mathscr{A}_D$, la série de Laurent $(1-\mathbf{L}^a T^a)^{d_v} Z_v(T, 0)$ converge pour $|T| < \mathbf{L}^{-1+\delta}$ et prenne une valeur effective non-nulle en $\mathbf{L}^{-1}$.*

Sous l'hypothèse 3 (que nous n'avons pas utilisée jusqu'ici) nous obtenons la convergence souhaitée pour $Z(T, 0)$ en combinant les propositions 8 et 10 : on trouve un prolongement méromorphe avec un pôle en $\mathbf{L}^{-1}$, d'ordre

$$|\mathscr{A} \setminus \mathscr{A}_D| + \sum_{v\in C\setminus C_0} d_v$$

égal à l'entier (1.3) obtenu par Chambert-Loir et Tschinkel dans l'analogue arithmétique de ce travail. Pour décrire la convergence des $Z(T, \xi)$ pour $\xi \neq 0$, nous remarquons d'abord que la sommation

$$\sum_{\xi\in G(F)} Z(T,\xi)$$

est en fait une sommation sur un k-espace vectoriel V de dimension finie. Nous combinons la proposition 9 avec un résultat, issu de [CLT12] et [ChL], qui exprime en chaque $v \in$

$C \setminus C_0$, de manière uniforme sur les strates d'une partition constructible finie de $V \setminus \{0\}$, l'ordre du pôle de la fonction $Z_v(T,\xi)$ en $\mathbf{L}^{-1}$ comme la dimension du sous-complexe du complexe simplicial $\mathrm{CL}_v^{\mathrm{an}}(X,D)$ obtenu en gardant seulement les sommets correspondant aux composantes suivant lesquelles la fonction f_ξ induite par ξ sur X n'a pas de pôle. Un argument dû à Chambert-Loir et Tschinkel permet de conclure que le pôle total de $Z(T,\xi)$ en $\mathbf{L}^{-1}$ est d'ordre strictement plus petit que celui de $Z(T,0)$.

Chapter 2

Grothendieck rings of varieties and the motivic vanishing cycles measure

Introduction

The *Grothendieck group of varieties* was defined by Grothendieck in a letter to Serre dated 16 August 1964, in which he also developed the idea of motives. He wrote:

> Soit k un corps, algébriquement clos pour fixer les idées, et soit $\mathrm{L}(k)$ le "groupe $\mathbf{K}$" défini par les schémas de type fini sur k, avec comme relations celles qui proviennent d'un découpage en morceaux (...).

Though Grothendieck does not mention it in the above quotation, this group is in fact a ring, which we will denote by KVar_k, the product being given by the product of varieties over k. The ring KVar_k became particularly prominent when, in a celebrated lecture in 1995 at Orsay, Kontsevich sketched the basics of *motivic integration*, a theory of integration with values in this ring, and showed how one could use it to prove that birationally equivalent Calabi-Yau varieties had the same Hodge numbers. This gave a generalisation of a result of Batyrev stating equality of Betti numbers under the same conditions. Batyrev's proof relied on reduction to positive characteristic and p-adic integration, and Kontsevich's groundbreaking idea consisted in noticing that one could replace the latter by a more geometric theory of integration, with values in the Grothendieck ring of varieties, and thus avoid the former. Moreover, the identities one obtains are valid in a suitable completion of the Grothendieck ring $\mathrm{KVar}_{\mathbf{C}}$ (localised at the class of the affine line), which gives rise to equality of more precise geometric invariants, e.g. Hodge numbers as stated above.

Since then, several theories of motivic integration have been constructed, and have found a wide range of applications. The first names to quote here are those of Denef and Loeser, who formalised Kontsevich's idea and extended it to singular varieties in [DL99a]. They also gave numerous applications of motivic integration to singularity theory, generalising to the motivic framework many results obtained previously via p-adic integration. In particular, they studied motivic versions of Igusa zeta functions in [DL98], which provide a notion of motivic nearby fibre and motivic vanishing cycles ([DL98, DL99b, DL01, DL02]).

Other theories of motivic integration, based on model theory of valued fields, are due to Cluckers-Loeser ([ClL08, ClL10]) and Hrushovski-Kazhdan ([HK06]): they allow to encompass integrals with parameters, as well as integrals involving additive characters, opening the path to a motivic version of Fourier analysis. The key idea for the latter is to work in larger Grothendieck rings called *Grothendieck rings with exponentials*.

Section 2.1 of this chapter contains definitions and properties of various types of Grothendieck rings of varieties, in particular Grothendieck rings with exponentials and Grothendieck rings with $\hat{\mu}$-actions. Section 2.2 introduces Denef and Loeser's notion of motivic nearby and vanishing cycles, generalises them to the relative setting and recalls a result due to Guibert, Loeser and Merle from [GLM] which combines various nearby cycles over a fixed variety X over a field of characteristic zero into a group morphism defined on the Grothendieck ring of varieties over X. Section 2.3 recalls Denef, Loeser and Looijenga's Thom-Sebastiani theorem for motivic vanishing cycles together with the convolution product $*$ it involves, and uses it to construct a ring morphism

$$\mathscr{E}xp\mathscr{M}_X \to (\mathscr{M}_X^{\hat{\mu}}, *)$$

from the Grothendieck ring of varieties with exponentials over X to the Grothendieck ring of varieties with $\hat{\mu}$-action over X localised at the class of the affine line, endowed with this product. The construction for $X = \operatorname{Spec} k$ with k an algebraically closed field of characteristic zero is due to Lunts and Schnürer [LS16b]. Finally, section 2.4 contains an explicit computation checking the Thom-Sebastiani theorem for vanishing cycles of the morphism $\mathbf{A}^1 \to \mathbf{A}^1$, $x \mapsto x^2$.

2.1 Grothendieck rings of varieties

References for this section are e.g. [CNS] for most classical definitions and properties of Grothendieck rings, and Hrushovski and Kazdan's [HK09], or Cluckers and Loeser's [ClL10] for Grothendieck rings with exponentials. We follow mostly Chambert-Loir and Loeser's [ChL] containing a short introduction to Grothendieck rings with exponentials and their main properties.

2.1.1 Grothendieck semirings

Let R be a noetherian scheme. By a variety over R we mean a R-scheme of finite type. For a point $r \in R$, we denote by $\kappa(r)$ its residue field.

The *Grothendieck monoid of varieties* KVar_R^+ over R is a commutative monoid defined by generators and relations. Generators are all varieties over R, and relations are

$$X \sim Y$$

whenever X and Y are isomorphic as R-varieties,

$$\varnothing \sim 0$$

and

$$X \sim Y + U$$

whenever X is a R-variety, Y a closed subscheme of X and U its open complement. We will write $[X]$ to denote the class of a variety X in KVar_R^+. The product $[X][Y] = [X \times_R Y]$ endows the monoid KVar_R^+ with a semiring structure. Two R-varieties X and Y have the same class in KVar_R^+ if and only if they are piecewise isomorphic over R, that is, one can partition them into locally closed subsets $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_m$ such that for all i, X_i and Y_i are isomorphic as R-schemes with their induced reduced structures (see [CNS], Chapter 1, Corollary 1.4.9).

The *Grothendieck monoid of varieties with exponentials* $\mathrm{KExpVar}_R^+$ is defined by generators and relations as well. Generators are pairs (X, f), where X is a variety over R and $f\colon X \to \mathbf{A}^1 = \mathrm{Spec}\,(\mathbf{Z}[T])$ is a morphism. Relations are the following:

$$(X, f) \sim (Y, f \circ u)$$

whenever X, Y are R-varieties, $f\colon X \to \mathbf{A}^1$ a morphism, and $u\colon Y \to X$ an R-isomorphism;

$$(\varnothing, f) \sim 0$$

where $f : \varnothing \to \mathbf{A}^1$ is the empty morphism;

$$(X, f) \sim (Y, f|_Y) + (U, f|_U)$$

whenever X is an R-variety, $f\colon X \to \mathbf{A}^1$ a morphism, Y a closed subscheme of X and $U = X \setminus Y$ its open complement;

$$(X \times_{\mathbf{Z}} \mathbf{A}^1, \mathrm{pr}_2)$$

where X is an R-variety and pr_2 is the second projection. We will write $[X, f]$ to denote the class in $\mathrm{KExpVar}_R^+$ of a pair (X, f). The product $[X, f][Y, g] = [X \times_R Y, f \circ \mathrm{pr}_1 + g \circ \mathrm{pr}_2]$ endows $\mathrm{KExpVar}_R^+$ with a semiring structure, the class $[R \xrightarrow{\mathrm{id}} R]$ being the unit element.

2.1.2 Grothendieck rings

Let R be a noetherian scheme.

The *Grothendieck group of varieties* KVar_R is defined by generators and relations. Generators are all varieties over R, and relations are

$$X - Y$$

whenever X and Y are isomorphic as R-varieties, and

$$X - Y - U$$

whenever X is a R-variety, Y a closed subscheme of X and U its open complement. It is the group associated to the monoid KVar_R^+. Every constructible set X over R (that is, every constructible subset of a variety over R) has a class $[X]$ in the group KVar_R (see [CNS], chapter 1, corollaries 1.3.5 and 1.3.6). It will sometimes be denoted by $[X]_R$ when different base schemes are in play. The product $[X][Y] = [X \times_R Y]$ endows the group KVar_R with a ring structure with unit element the class $1 = [R \xrightarrow{\mathrm{id}} R]$. Let $\mathbf{L}$, or $\mathbf{L}_R$, be the class of the affine line $\mathbf{A}^1_R$ in KVar_R. We define the Grothendieck ring of varieties localised at $\mathbf{L}$ to be $\mathscr{M}_R = \mathrm{KVar}_R[\mathbf{L}^{-1}]$.

The *Grothendieck group of varieties with exponentials* $\mathrm{KExpVar}_R$ is defined by generators and relations as well. Generators are pairs (X, f), where X is a variety over R and $f\colon X \to \mathbf{A}^1 = \mathrm{Spec}\,(\mathbf{Z}[T])$ is a morphism. Relations are the following:

$$(X, f) - (Y, f \circ u)$$

whenever X, Y are R-varieties, $f\colon X \to \mathbf{A}^1$ a morphism, and $u\colon Y \to X$ an R-isomorphism;

$$(X, f) - (Y, f|_Y) - (U, f|_U)$$

whenever X is a R-variety, $f\colon X \to \mathbf{A}^1$ a morphism, Y a closed subscheme of X and $U = X \setminus Y$ its open complement,

$$(X \times_{\mathbf{Z}} \mathbf{A}^1, \mathrm{pr}_2)$$

where X is an R-variety and pr_2 is the second projection. We will write $[X, f]$ (or $[X, f]_R$ if we want to keep track of the base scheme R) to denote the class in $\mathrm{KExpVar}_R$ of a pair (X, f). The product $[X, f][Y, g] = [X \times_R Y, f \circ \mathrm{pr}_1 + g \circ \mathrm{pr}_2]$ endows $\mathrm{KExpVar}_R$ with a ring structure. We will use the notation $f \oplus g$ for the morphism $f \circ \mathrm{pr}_1 + g \circ \mathrm{pr}_2$. We denote by $\mathbf{L}$, or $\mathbf{L}_R$, the class of $[\mathbf{A}^1_R, 0]$ in $\mathrm{KExpVar}_R$. As for KVar_R, we may invert $\mathbf{L}$, which gives us a ring denoted by $\mathscr{E}xp\mathscr{M}_R$.

There are obvious morphisms of semirings

$$\mathrm{KVar}^+_R \to \mathrm{KVar}_R$$

and

$$\mathrm{KExpVar}^+_R \to \mathrm{KExpVar}_R$$

which identify KVar_R (resp. $\mathrm{KExpVar}_R$) with the ring obtained from the semiring KVar^+_R (resp. $\mathrm{KExpVar}^+_R$) by adding negatives. An element in the image of one of these morphisms is said to be *effective*.

There are ring morphisms $\mathrm{KVar}_R \to \mathrm{KExpVar}_R$ and $\mathscr{M}_R \to \mathscr{E}xp\mathscr{M}_R$ sending the class of X to the class $[X, 0]$. According to lemma 1.1.3 in [ChL] together with lemma 2.1.3.1 below, they are injective.

Let X be an R-variety. A piecewise morphism $f : X \to \mathbf{A}^1$ is the datum of a partition $X_1, \ldots, X_m$ of X into locally closed subsets and of morphisms $f_i : X_i \to \mathbf{A}^1$ for every i. Any pair (X, f) consisting of an R-variety X and of a piecewise morphism $f\colon X \to \mathbf{A}^1$ has a class $[X, f]$ in $\mathrm{KExpVar}_R$.

Remark 2.1.2.1. The localisation morphism $\mathrm{KVar}_R \to \mathscr{M}_R$ is not injective in general: it was proved by Borisov in [Bor] that $\mathbf{L}$ is a zero-divisor in the Grothendieck ring KVar_k over a field k of characteristic zero. Borisov's argument also implies that two varieties X and Y having the same class in the Grothendieck ring of varieties are not necessarily piecewise isomorphic, so that the above morphism $\mathrm{KVar}_R^+ \to \mathrm{KVar}_R$ is not injective either.

Let $A \in \{\mathrm{KVar}, \mathscr{M}, \mathrm{KExpVar}, \mathscr{E}xp\mathscr{M}\}$. For noetherian schemes R and S over some noetherian scheme T, there is an *exterior product* morphism

$$A_R \otimes_{A_T} A_S \xrightarrow{\boxtimes_T} A_{R\times_T S} \tag{2.1}$$

of A_T-algebras. For $A = \mathrm{KExpVar}$, it is given by sending a pair $([X, f], [Y, g])$ to the class $[X \times_T Y, f \circ \mathrm{pr}_1 + g \circ \mathrm{pr}_2]$.

2.1.3 Functoriality and interpretation as functions

Let R and S be noetherian schemes. A morphism $u : R \to S$ induces morphisms $u_!$ and u^* between the corresponding Grothendieck groups. For example, for KExpVar, these morphisms are defined in the following manner: the *proper pushforward*

$$u_! : \mathrm{KExpVar}_R \to \mathrm{KExpVar}_S$$

sends a class $[X, f]_R$ of a R-variety X in $\mathrm{KExpVar}_R$ to the class $[X, f]_S$ of the pair (X, f) with X viewed as an S-variety through the morphism u. This is a morphism of rings in the case when u is an immersion, but not in general.

In the other direction, there is a morphism of rings

$$u^* : \mathrm{KExpVar}_S \to \mathrm{KExpVar}_R$$

called the *pullback*, given by sending the class of a pair (X, f) with X a S-variety to the class of the pair $(X \times_S R, f \circ \mathrm{pr}_1)$, where R is viewed as an S-variety through the morphism u. In particular, $\mathrm{KExpVar}_S$ may be seen as a $\mathrm{KExpVar}_S$-algebra via this morphism.

Elements of the Grothendieck rings over R may be interpreted as motivic functions on R. When $r \in R$ is a point, and $\mathfrak{a} \in \mathrm{KExpVar}_R$, we will denote by $\mathfrak{a}(r) = r^*(\mathfrak{a})$ (or $\mathfrak{a}_r$) the image of $\mathfrak{a}$ in $\mathrm{KExpVar}_{\kappa(r)}$ by the morphism r^* induced by $r : \mathrm{Spec}\,(\kappa(r)) \to R$. This will be interpreted as evaluation of the motivic function $\mathfrak{a}$ at r. More generally, the above morphism u^* is just composition with u. As for $u_!$, it may be interpreted as "summation over rational points" in the fibres of u, as explained in the following paragraph. We recall Lemma 1.1.8 from [ChL], which, with this functional interpretation, means that a motivic function on R is determined by its values at points of R.

Lemma 2.1.3.1. *Let* $\mathfrak{a} \in \mathrm{KVar}_R$ *(resp.* $\mathscr{M}_R$, $\mathrm{KExpVar}_R$, $\mathscr{E}xp\mathscr{M}_R$*). If* $\mathfrak{a}(r) = 0$ *for every* $r \in R$*, then* $\mathfrak{a} = 0$.

2.1.4 Exponential sum notation

We start with the following lemma:

Lemma 2.1.4.1. *Let k be a finite field and let $\psi : k \to \mathbf{C}^*$ be a non-trivial additive character. Then there is a motivic measure*

$$\begin{array}{ccc} \mathrm{KExpVar}_k & \to & \mathbf{C} \\ [X, f] & \mapsto & \sum_{x \in X(k)} \psi(f(x)) \end{array}$$

extending the counting measure

$$\begin{array}{ccc} \mathrm{KVar}_k & \to & \mathbf{Z} \\ [X] & \mapsto & |X(k)| \end{array}$$

Proof. We may define a group morphism on the free abelian group of generators (X, f) of $\mathrm{KExpVar}_k$, by sending (X, f) to $\sum_{x \in X(k)} \psi(f(x))$. This passes to the quotient modulo the first relation defining $\mathrm{KExpVar}_k$ because it corresponds to being able to perform changes of variables in such a sum:

$$\sum_{x \in X(k)} \psi(f(x)) = \sum_{y \in Y(k)} \psi(f \circ u(y))$$

whenever $u : Y \to X$ is an isomorphism over k. It also passes to the quotient by the second relation because the latter corresponds to cutting up the sum into smaller sums:

$$\sum_{x \in X(k)} \psi(f(x)) = \sum_{x \in Y(k)} \psi(f(x)) + \sum_{x \in U(k)} \psi(f(x))$$

whenever Y is a closed subscheme of X and $U = X \setminus Y$ its open complement. The third relation is also satisfied, and comes from the fact that if ψ is a non-trivial character, then

$$\sum_{x \in \mathbf{A}^1_k(k)} \psi(x) = 0.$$

Finally, we observe that the product of two classes $[X, f]$ and $[Y, g]$ is sent to

$$\sum_{x \in X} \sum_{y \in Y} \psi(f(x))\psi(g(y)) = \sum_{(x,y) \in X \times Y} \psi(f(x) + g(y))$$

which is exactly the image of the product $[X \times Y, f \oplus g]$ of these classes, so this map is a ring morphism. □

This lemma is meant as a motivation of the fact that, as explained in [ChL] 1.1.9, the class of a pair (X, f) in $\mathrm{KExpVar}_k$ must be thought of as an analogue of the exponential sum

$$\sum_{x \in X(k)} \psi(f(x))$$

where k is a finite field and $\psi : k \to \mathbf{C}^*$ is a non-trivial additive character. This is why, when doing calculations with it, we will denote the class $[X, f] \in \mathrm{KExpVar}_k$, even for a not necessarily finite field k, by

$$\sum_{x \in X} \psi(f(x)).$$

As noted in the proof of the lemma, when using this notation, the three relations occurring in the definition of $\mathrm{KExpVar}_k$ translate respectively as the possibility of performing change of variables, additivity and the property that

$$\sum_{x \in k} \psi(x) = 0,$$

the latter being essential to make Fourier analysis work.

More generally, let R be a k-variety. For any morphism $h : R \to \mathbf{A}^1$ and any element $\theta \in \mathrm{KExpVar}_R$, we may define

$$\sum_{r \in R} \theta(r)\psi(h(r)) = \theta \cdot [R, h]$$

where the product is taken in $\mathrm{KExpVar}_R$, and its result is viewed in $\mathrm{KExpVar}_k$. In the case when $h = 0$, the map $\theta \mapsto \sum_{r \in R} \theta(r)$ is exactly $u_!$ for $u : R \to k$ the structural morphism. In the same manner, if $u : R \to S$ is a morphism, then for any $\theta \in \mathrm{KExpVar}_R$, the motivic function $u_!\theta$ sends $s \in S$ to $\sum_{r \in u^{-1}(s)} \theta(s)$.

2.1.5 Grothendieck rings of varieties with action

Grothendieck rings with actions were introduced by Denef and Loeser in [DL02] to be able to take into account monodromy actions on the motivic nearby fibre. Other references are [DL01] and [GLM].

Let k be a field of characteristic zero. We start by giving some general definitions about group actions on varieties. Let G be a finite group acting on a variety X over k. We say that the action of G is *good* if every G-orbit is contained in an affine open subset of X. If X and Y are two varieties with good G-action, we denote by $X \times^G Y$ the quotient of the product $X \times Y$ by the equivalence relation $(gx, y) \sim (x, gy)$. It exists as a variety, and there is a good G-action on $X \times^G Y$ induced by the action of G on the first factor of $X \times Y$.

Let S be a variety over k and X a variety over S. When we speak of a good action of G on the S-variety X we require it to leave the fibres of the structural morphism $X \to S$ invariant, i.e., the structural morphism must be equivariant if S is equipped with the trivial G-action.

For $n \geq 1$, let $\mu_n = \mathrm{Spec}\,(k[x]/(x^n - 1))$ be the group scheme of n-th roots of unity, and let $\hat{\mu}$ be the projective limit $\varprojlim \mu_n$ of the projective system with transition morphisms $\mu_{nd} \to \mu_n$, $x \mapsto x^d$. For any integer $r \geq 1$, a good $\hat{\mu}^r$-action on a variety X is an action of $\hat{\mu}^r$ that factors through a good action of μ_n^r for some integer $n \geq 1$.

Let S be a variety over k and $r \geq 1$ an integer. The Grothendieck group of varieties with $\hat{\mu}^r$-action $\mathrm{KVar}_S^{\hat{\mu}^r}$ is defined in a similar way to KVar_S: generators are pairs (X, σ) where X is a variety over S and σ a good $\hat{\mu}^r$-action on X, and relations are

$$(X, \sigma) - (Y, \tau)$$

whenever X and Y are S-varieties with good $\hat{\mu}^r$-actions σ, τ such that there exists an equivariant S-isomorphism $u : (X, \sigma) \to (Y, \tau)$,

$$(X, \sigma) - (Y, \sigma_{|Y}) - (U, \sigma_{|U})$$

where X is a variety over S with good $\hat{\mu}^r$-action σ, Y a closed $\hat{\mu}^r$-invariant subscheme of X, and U its open complement, as well as an additional relation saying that

$$(X \times \mathbf{A}_k^n, \sigma) - (X \times \mathbf{A}_k^n, \sigma') \tag{2.2}$$

whenever σ and σ' are two liftings of the same $\hat{\mu}^r$-action on X to an affine action on $X \times \mathbf{A}_k^n$ (that is, a good action the restriction of which to all fibres of the affine bundle $X \times \mathbf{A}_k^n \to X$ is affine). The fibred product with the diagonal $\hat{\mu}^r$-action endows $\mathrm{KVar}_S^{\hat{\mu}^r}$ with a ring structure.

Remark 2.1.5.1. We won't use this product much, because in the context of vanishing cycles, products defined via convolution are more relevant, see sections 2.2.1, 2.2.2.

The class of the pair (X, σ) will be denoted $[X, \sigma]$, or even $[X, \hat{\mu}^r]$ or $[X]$ when it is clear from the context what σ is.

We denote by $\mathbf{L}_S$, or $\mathbf{L}$, the class $\mathbf{A}_S^1$ with trivial $\hat{\mu}^r$-action. The last relation (2.2) says in particular that $\mathbf{L}$ is equal to the class of the affine space endowed with any affine $\hat{\mu}^r$-action. As for KVar_S, we may define the ring $\mathscr{M}_S^{\hat{\mu}^r} = \mathrm{KVar}_S^{\hat{\mu}^r}[\mathbf{L}^{-1}]$.

Remark 2.1.5.2 (Trivial action and forgetful morphism). There is a ring morphism

$$\mathrm{KVar}_S \to \mathrm{KVar}_S^{\hat{\mu}^r} \tag{2.3}$$

sending the class of a variety X to the class of X endowed with the trivial action. There is also a forgetful ring morphism

$$\mathrm{KVar}_S^{\hat{\mu}^r} \to \mathrm{KVar}_S$$

which sends a class $[X, \sigma]$ of a variety X with action σ to the class $[X]$, well defined because all relations defining $\mathrm{KVar}_S^{\hat{\mu}^r}$ go to zero in KVar_S. The composition of the two gives the identity of KVar_S, which shows that (2.3) is injective. This remains valid with KVar replaced by $\mathscr{M}$.

Remark 2.1.5.3. Since KVar_S is a KVar_k-module, the morphism (2.3) endows $\mathrm{KVar}_S^{\hat{\mu}^r}$ with a KVar_k-module structure. It is given, for any k-variety X and any S-variety Y with $\hat{\mu}^r$-action σ by $[X][Y, \sigma] = [X \times_k Y, \sigma']$, where σ' is acts trivially on X and by σ on Y.

If $u : R \to S$ is a morphism of k-varieties, there are, as in 2.1.3, a group morphism

$$u_! : \mathrm{KVar}_R^{\hat{\mu}^r} \to \mathrm{KVar}_S^{\hat{\mu}^r}$$

and a ring morphism

$$u^* : \mathrm{KVar}_S^{\hat{\mu}^r} \to \mathrm{KVar}_R^{\hat{\mu}^r},$$

defined similarly. These morphisms exist also when KVar is replaced with $\mathscr{M}$.

Exterior products in the flavour of (2.1) also exist for Grothendieck rings of varieties with action. For a k-variety T and T-varieties R and S, the one we are going to use is a morphism

$$\mathrm{KVar}_R^{\hat{\mu}^r} \otimes_{\mathrm{KVar}_T} \mathrm{KVar}_S^{\hat{\mu}^r} \xrightarrow{\boxtimes_T} \mathrm{KVar}_{R\times_T S}^{\hat{\mu}^r\times\hat{\mu}^r}$$

of KVar_T-algebras sending a pair $([X,\sigma],[Y,\tau])$ to the class of $X \times_T Y$ endowed with the product action $\sigma \times_T \tau$ given by $(s,t).(x,y) = (\sigma(s)(x), \tau(t)(y))$.

2.1.6 The dimensional filtration

A reference for this section is [CNS], chapter 1, sections 4.1 and 4.2.

Definition 2.1.6.1. Let S be a scheme. The relative dimension of a variety X over S, denoted by $\dim_S(X)$, is defined to be

$$\dim_S X := \sup_{s\in S} \dim_{\kappa(s)} X_s.$$

Let S be a noetherian scheme. The above notion of relative dimension gives rise to a natural filtration on the ring KVar_S: we define $F_d\mathrm{KVar}_S$ to be the subgroup of KVar_S generated by classes of varieties of relative dimension $\leq d$.

We may now define a function

$$\dim_S : \mathrm{KVar}_S \to \mathbf{Z} \cup \{-\infty\}$$

by sending a class $\mathfrak{a}$ to $\inf\{d \in \mathbf{Z},\ \mathfrak{a} \in F_d\mathrm{KVar}_S\}$. This function has the following elementary properties:

Let $\mathfrak{a}, \mathfrak{a}' \in \mathrm{KVar}_S$. Then

(*i*) $\dim_S(0) = -\infty$.

(*ii*) $\dim_S(\mathfrak{a} + \mathfrak{a}') \leq \min\{\dim_S(\mathfrak{a}), \dim_S(\mathfrak{a}')\}$, with equality whenever $\dim_S(\mathfrak{a}) \neq \dim_S(\mathfrak{a}')$

(*iii*) $\dim_S(\mathfrak{a}\mathfrak{a}') \leq \dim_S(\mathfrak{a}) + \dim_S(\mathfrak{a}')$.

Moreover, using the Euler-Poincaré polynomial, one may prove that for every variety X over S, $\dim_S([X]) = \dim_S X$ (see lemma 4.1.3 in [CNS], chapter 1).

For every $d \in \mathbf{Z}$, we define $F_d\mathscr{M}_S$ to be the subgroup of $\mathscr{M}_S$ generated by elements of the form $[X]\mathbf{L}^{-n}$ where X is an S-variety, $n \in \mathbf{Z}$ is an integer and $\dim_S X - n \leq d$. This gives us an increasing and exhaustive filtration on the ring $\mathscr{M}_S$. In the same manner as above, we may define a function

$$\dim_S : \mathscr{M}_S \to \mathbf{Z} \cup \{-\infty\}$$

satisfying the same properties.

The same definition gives rise to dimensional filtrations and functions on Grothendieck rings with actions or with exponentials. Thus, for example, on $\mathrm{KExpVar}_S$, we define $F_d\mathrm{KExpVar}_S$ as the subgroup of $\mathrm{KExpVar}_S$ generated by classes of the form $[X,f]$ where X is of relative dimension $\leq d$.

2.1.7 Some presentations of Grothendieck groups in characteristic zero

We start by recalling a classical result (it follows for example from [Bit], Theorem 3.1).

Lemma 2.1.7.1. *Let X be a variety over a field k of characteristic zero. The Grothendieck group KVar_X is generated by classes of the form $[Y \xrightarrow{p} X]$ where p is proper and Y is smooth over k.*

This may be generalised in the following form:

Lemma 2.1.7.2. *Let S be a variety over a field k of characteristic zero, and X a variety over S with structural morphism $u : X \to S$. Then the group KVar_X is generated by classes $[Y \xrightarrow{p} X]$ such that $U = u \circ p(Y)$ is a locally closed subset of S, Y is smooth over U and such that the morphism $p \times_S \mathrm{id}_U : Y \times_S U \to X \times_S U$ is proper.*

Proof. First of all, KVar_X is generated by classes of quasi-projective morphisms. Let therefore $p : Y \to X$ be a quasi-projective morphism, and define T to be the closure of $u \circ p(Y)$ in S. We will argue by induction on $\dim T$. Let $T_1, \ldots, T_n$ be the irreducible components of T. For each $i \in \{1, \ldots, n\}$ it suffices to write $[(u \circ p)^{-1}(T_i) \to X]$ as a sum of classes as in the statement of the theorem, and use induction on $\dim T$ to conclude. We may therefore assume that T is irreducible. Compactify p into a proper morphism $\overline{p} : \overline{Y} \to X$, where $\overline{Y}$ is a variety containing Y as a dense open subset. The closure of $u \circ \overline{p}(\overline{Y})$ in S is again T. An additional induction on $\dim Y$ (with $\dim T$ fixed), initialised at $\dim Y = \dim T$ (in which case we use the induction hypothesis for $\dim T - 1$) then allows us to work with $\overline{p}$ instead of p. Let η be the generic point of T. Consider a resolution of singularities $\widetilde{\overline{Y}_\eta} \to \overline{Y}_\eta$ above $\kappa(\eta)$. Above some dense open subset T' of T, this spreads out to a proper birational morphism $f : \widetilde{\overline{Y}} \to \overline{Y} \times_T T'$ with $\widetilde{\overline{Y}}$ smooth over T'. If $\dim T = 0$, then $T' = T$, otherwise by induction on $\dim T$, we may replace $\overline{p} : \overline{Y} \to X$ with its restriction $p' : \overline{Y} \times_T T' \to X$. Finally, by induction on $\dim Y$ (and on $\dim T$ if $\dim Y = \dim T$), the morphism $p' : Y \times_T T' \to X$ may be replaced with the composition $g = p' \circ f : \widetilde{\overline{Y}} \to X$. The morphism g then satisfies the condition in the statement of the theorem. Indeed, the morphism $u \circ g : \widetilde{\overline{Y}} \to T'$ is smooth, and therefore open, so that $U := u \circ g\left(\widetilde{\overline{Y}}\right)$ is open in its closure and therefore is locally closed is S, and $\widetilde{\overline{Y}}$ is smooth over U. Finally, $g \times_S \mathrm{id}_U$ is the composition of the morphisms $\widetilde{\overline{Y}} \times_S U \to \overline{Y} \times_S U$ and $\overline{Y} \times_S U \to X \times_S U$ which are both proper by base change. □

2.2 Motivic vanishing cycles

Throughout this section, k will be a field of characteristic zero.

2.2.1 Convolution

In this section we recall the definition of Looijenga's convolution from [Loo], in its generalised form due to Guibert, Loeser and Merle ([GLM]), following section 2.3 in [LS16b].

Notation 2.2.1.1. Let $n \geq 1$ be an integer. We denote by F_0^n (resp. F_1^n) the Fermat curve with equation $x^n + y^n = 0$ (resp. $x^n + y^n = 1$) inside $\mathbf{G}_m \times \mathbf{G}_m$ (with coordinates x, y), with the obvious $\mu_n \times \mu_n$-action.

Let S be a variety over a field k of characteristic zero. There is a KVar_k-linear morphism

$$\Psi : \mathrm{KVar}_S^{\hat{\mu}\times\hat{\mu}} \to \mathrm{KVar}_S^{\hat{\mu}}$$

given in the following manner: let $Z \xrightarrow{p} S$ be an S-variety with a $\hat{\mu} \times \hat{\mu}-$action. Then there is an integer n such that this action factors through $\mu_n \times \mu_n$. One defines

$$\Psi(Z \xrightarrow{p} S) = [Z \times^{\mu_n\times\mu_n} F_0^n \xrightarrow{p\circ \mathrm{pr}_1} S] - [Z \times^{\mu_n\times\mu_n} F_1^n \xrightarrow{p\circ \mathrm{pr}_1} S].$$

This gives an element of $\mathrm{KVar}_S^{\hat{\mu}}$: indeed, as we said in section 2.1.5, for $i = 0, 1$, the variety

$$Z \times^{\mu_n\times\mu_n} F_i^n$$

is endowed with an action of μ_n, given by

$$t.(z, x, y) = ((t, t).z, x, y) = (z, tx, ty).$$

As explained in the discussion after remark 2.13 in [LS16b], this construction does not depend on the integer n, and therefore Ψ is well-defined. By remark 2.8 in [LS16b], for every morphism $f : R \to S$ of k-varieties, Ψ commutes with $f_!$ and f^*.

We may define the convolution product on $\mathrm{KVar}_S^{\hat{\mu}}$ by the KVar_k-linear composition

$$* : \mathrm{KVar}_S^{\hat{\mu}} \otimes_{\mathrm{KVar}_k} \mathrm{KVar}_S^{\hat{\mu}} \xrightarrow{\boxtimes_S} \mathrm{KVar}_S^{\hat{\mu}\times\hat{\mu}} \xrightarrow{\Psi} \mathrm{KVar}_S^{\hat{\mu}},$$

(see definition of the exterior product $\boxtimes_S$ at the end of section 2.1.5) so that for two S-varieties X, Y with actions σ and τ, we have

$$[X, \sigma] * [Y, \tau] = \Psi([X \times_S Y, \sigma \times_S \tau]).$$

By proposition 2.12 in [LS16b] (or proposition 5.2 in [GLM]), for any k-variety S, the convolution product $*$ endows $\mathrm{KVar}_S^{\hat{\mu}}$ with an associative, commutative KVar_k-algebra structure, with unit element the class of the identity $[S \xrightarrow{\mathrm{id}} S]$ (with trivial $\hat{\mu}$-action). From now on, the ring structure on $\mathrm{KVar}_S^{\hat{\mu}}$ we are going to consider will be the convolution product $*$. Thus, in what follows, both $\mathrm{KVar}_S^{\hat{\mu}}$ and $(\mathrm{KVar}_S^{\hat{\mu}}, *)$ denote the same thing.

The n-fold convolution product of $\mathbf{L}_S$ with itself is $\mathbf{L}_S^n$, and therefore localising the KVar_k-algebra $(\mathrm{KVar}_S^{\hat{\mu}}, *)$ at the multiplicative set $\{1, \mathbf{L}_S, \mathbf{L}_S * \mathbf{L}_S, \ldots\}$ yields an $\mathscr{M}_k$-algebra $(\mathscr{M}_S^{\hat{\mu}}, *)$ with same underlying $\mathscr{M}_k$-module structure as the usual localisation $\mathscr{M}_S^{\hat{\mu}}$ of $\mathrm{KVar}_S^{\hat{\mu}}$. By $\mathscr{M}_k$-linearity, Ψ and $*$ extend to localised Grothendieck rings. Since Ψ commutes with pullbacks, for any morphism $f : R \to S$ of k-varieties, there are pullback morphisms

$$f^* : \mathrm{KVar}_S^{\hat{\mu}} \to \mathrm{KVar}_R^{\hat{\mu}}$$

(resp. $f^* : \mathscr{M}_S^{\hat{\mu}} \to \mathscr{M}_R^{\hat{\mu}}$) of KVar_k-algebras (resp. of $\mathscr{M}_k$-algebras).

Remark 2.2.1.2. By remark 2.11 in [LS16b], whenever the action τ on Y is trivial, we have

$$[X, \sigma] * [Y, \tau] = [X, \sigma][Y, \tau],$$

where on the right we consider the usual product on $\mathrm{KVar}_S^{\hat{\mu}}$ from section 2.1.5. In particular, there is a ring morphism

$$\mathrm{KVar}_S \to (\mathrm{KVar}_S^{\hat{\mu}}, *)$$

sending the class of an S-variety X to the class $[X, 1]$ where 1 denotes the trivial action. Such a morphism exists also at the level of localised Grothendieck rings.

2.2.2 Grothendieck rings over the affine line

Let S be a k-variety. By section 2.2.1, the Grothendieck group $\mathrm{KVar}_{\mathbf{A}_S^1}^{\hat{\mu}}$ has a natural $(\mathrm{KVar}_S^{\hat{\mu}}, *)$-algebra structure given by the ring morphism $\epsilon_S^* : (\mathrm{KVar}_S^{\hat{\mu}}, *) \to (\mathrm{KVar}_{\mathbf{A}_S^1}^{\hat{\mu}}, *)$ where $\epsilon_S : \mathbf{A}_S^1 \to S$ is the structural morphism.

On the other hand, there is a well-defined addition morphism

$$\mathrm{add} : \mathbf{A}_S^1 \times_S \mathbf{A}_S^1 \to \mathbf{A}_S^1$$

on the group scheme $\mathbf{A}_S^1$, and this endows $\mathrm{KVar}_{\mathbf{A}_S^1}^{\hat{\mu}}$ with a product $\star$ given by the composition

$$\star : \mathrm{KVar}_{\mathbf{A}_S^1}^{\hat{\mu}} \otimes_{\mathrm{KVar}_S} \mathrm{KVar}_{\mathbf{A}_S^1}^{\hat{\mu}} \xrightarrow{\boxtimes_S} \mathrm{KVar}_{\mathbf{A}_S^1 \times_S \mathbf{A}_S^1}^{\hat{\mu} \times \hat{\mu}} \xrightarrow{\mathrm{add}_!} \mathrm{KVar}_{\mathbf{A}_S^1}^{\hat{\mu} \times \hat{\mu}} \xrightarrow{\Psi} \mathrm{KVar}_{\mathbf{A}_S^1}^{\hat{\mu}},$$

so that, for $\mathbf{A}_S^1$-varieties X and Y with $\hat{\mu}$-actions σ and τ, we have

$$[X \xrightarrow{f} \mathbf{A}_S^1, \sigma] \star [Y \xrightarrow{g} \mathbf{A}_S^1, \tau] = \Psi([X \times_S Y \xrightarrow{f \oplus g} \mathbf{A}_S^1, \sigma \times_S \tau])$$

where $f \oplus g = f \circ \mathrm{pr}_1 + g \circ \mathrm{pr}_2$.

Lemma 2.2.2.1. *Let $\iota_S : S \to \mathbf{A}_S^1$ be the morphism given by the zero-section of the trivial affine bundle $\mathbf{A}_S^1 \to S$. Then*

$$\begin{array}{rccc} (\iota_S)_! : & (\mathrm{KVar}_S^{\hat{\mu}}, *) & \to & (\mathrm{KVar}_{\mathbf{A}_S^1}^{\hat{\mu}}, \star) \\ & \left[X \xrightarrow{u} S\right] & \mapsto & [X \xrightarrow{(u,0)} \mathbf{A}_S^1] \end{array}$$

is a ring morphism.

Proof. Let $X \xrightarrow{u} S$ and $Y \xrightarrow{v} S$ be two S-varieties and let σ (resp. τ) be a $\hat{\mu}$-action on X (resp. Y). Using the fact that Ψ commutes with proper pushforwards, we get

$$\begin{aligned}
(\iota_S)_!([X \xrightarrow{u} S, \sigma] * [Y \xrightarrow{v} S, \tau]) &= (\iota_S)_! \circ \Psi([X \times_S Y \xrightarrow{u \circ \mathrm{pr}_1} S, \sigma \times_S \tau]) \\
&= \Psi \circ (\iota_S)_!([X \times_S Y \xrightarrow{u \circ \mathrm{pr}_1} S, \sigma \times_S \tau]) \\
&= \Psi([X \times_S Y \xrightarrow{(u \circ \mathrm{pr}_1, 0)} \mathbf{A}^1_S, \sigma \times_S \tau]) \\
&= \Psi \circ \mathrm{add}_!([X \times_S Y \xrightarrow{(u \circ \mathrm{pr}_1, 0)} \mathbf{A}^1_S \times_S \mathbf{A}^1_S, \sigma \times_S \tau]) \\
&= \Psi \circ \mathrm{add}_!([X \xrightarrow{(u,0)} \mathbf{A}^1_S, \sigma] \boxtimes_S [Y \xrightarrow{(v,0)} \mathbf{A}^1_S, \tau]) \\
&= [X \xrightarrow{(u,0)} \mathbf{A}^1_S, \sigma] \star [Y \xrightarrow{(v,0)} \mathbf{A}^1_S, \tau] \\
&= (\iota_S)_!([X \xrightarrow{u} S, \sigma]) \star (\iota_S)_!([Y \xrightarrow{v} \mathbf{A}^1_S, \tau]).
\end{aligned}$$

□

Thus, there is another $\mathrm{KVar}^{\hat{\mu}}_S$-algebra structure on $\mathrm{KVar}^{\hat{\mu}}_{\mathbf{A}^1_S}$, given by $(i_S)_!$, the latter becoming a ring morphism if one replaces the product $*$ by $\star$.

Denote by $\widetilde{\mathrm{KVar}}^{\hat{\mu}}_{\mathbf{A}^1_S}$ the $\mathrm{KVar}^{\hat{\mu}}_S$-module structure on $\mathrm{KVar}^{\hat{\mu}}_{\mathbf{A}^1_S}$ given by $(\iota_S)_!$.

Lemma 2.2.2.2. *The identity* $\mathrm{id} : \mathrm{KVar}^{\hat{\mu}}_{\mathbf{A}^1_S} \to \widetilde{\mathrm{KVar}}^{\hat{\mu}}_{\mathbf{A}^1_S}$ *is an isomorphism of* $\mathrm{KVar}^{\hat{\mu}}_S$-*modules.*

Proof. For all varieties X, Y over S with $\hat{\mu}$-actions σ, τ and any morphism $f : Y \to \mathbf{A}^1_S$, we have (denoting by u the structural morphism $u : X \to S$ and by 1 the trivial $\hat{\mu}$-action on $\mathbf{A}^1_S$)

$$\begin{aligned}
\epsilon^*_S([X \xrightarrow{u} S, \sigma]) * [Y \xrightarrow{f} \mathbf{A}^1_S, \tau] &= \Psi([X \times_S \mathbf{A}^1_S \xrightarrow{\mathrm{pr}_2} \mathbf{A}^1_S, \sigma \times_S 1] \times_{\mathbf{A}^1_S} [Y \xrightarrow{f} \mathbf{A}^1_S, \tau]) \\
&= \Psi([(X \times_S \mathbf{A}^1_S) \times_{\mathbf{A}^1_S} Y \xrightarrow{f \circ \mathrm{pr}_2} \mathbf{A}^1_S, (\sigma \times_S 1) \boxtimes_{\mathbf{A}^1_S} \tau]) \\
&= \Psi([X \times_S Y \xrightarrow{f \circ \mathrm{pr}_2} \mathbf{A}^1_S, \sigma \times_S \tau]) \\
&= \Psi \circ \mathrm{add}_!([X \xrightarrow{(u,0)} \mathbf{A}^1_S, \sigma] \boxtimes_S [Y \xrightarrow{f} \mathbf{A}^1_S, \tau]) \\
&= [X \xrightarrow{(u,0)} \mathbf{A}^1_S, \sigma] \star [Y \xrightarrow{f} \mathbf{A}^1_S, \tau] \\
&= (\iota_S)_!([X \xrightarrow{u} S, \sigma]) \star [Y \xrightarrow{f} \mathbf{A}^1_S, \tau].
\end{aligned}$$

□

Thus in fact these two $\mathrm{KVar}^{\hat{\mu}}_S$-module structures are the same, and we will denote this $\mathrm{KVar}^{\hat{\mu}}_S$-module by $\mathrm{KVar}^{\hat{\mu}}_{\mathbf{A}^1_S}$. To distinguish between the two $\mathrm{KVar}^{\hat{\mu}}_S$-algebra structures, we will write them $(\mathrm{KVar}^{\hat{\mu}}_{\mathbf{A}^1_S}, *)$ and $(\mathrm{KVar}^{\hat{\mu}}_{\mathbf{A}^1_S}, \star)$.

Lemma 2.2.2.3. *The pushforward map*

$$(\epsilon_S)_! : (\mathrm{KVar}^{\hat{\mu}}_{\mathbf{A}^1_S}, \star) \to (\mathrm{KVar}^{\hat{\mu}}_S, *)$$

is a morphism of $\mathrm{KVar}^{\hat{\mu}}_S$*-algebras.*

Proof. First of all, we have $(\epsilon_S)_! \circ (\iota_S)_! = \mathrm{id}_{\mathrm{KVar}^{\hat{\mu}}_S}$. Moreover, for all varieties X, Y over S with morphisms $f : X \to \mathbf{A}^1_S$ and $g : Y \to \mathbf{A}^1_S$ and $\hat{\mu}$-actions σ, τ, we have, using that Ψ commutes with proper pushforwards:

$$\begin{aligned}
(\epsilon_S)_!([X \xrightarrow{f} \mathbf{A}^1_S, \sigma] \star [Y \xrightarrow{g} \mathbf{A}^1_S, \sigma]) &= (\epsilon_S)_! \Psi([X \times_S Y \xrightarrow{f \oplus g} \mathbf{A}^1_S, \sigma \times_S \tau]) \\
&= \Psi([X \times_S Y \to S, \sigma \times_S \tau]) \\
&= [X \to S, \sigma] * [Y \to S, \tau] \\
&= (\epsilon_S)_!([X \xrightarrow{f} \mathbf{A}^1_S, \sigma]) * (\epsilon_S)_!([Y \xrightarrow{g} \mathbf{A}^1_S, \tau]).
\end{aligned}$$

□

Remark 2.2.2.4 (Trivial actions). From remark 2.15 in [LS16b], we see that if $f : X \to \mathbf{A}^1_S$ and $g : Y \to \mathbf{A}^1_S$ are $\mathbf{A}^1_S$-varieties with $\hat{\mu}$-actions σ, τ such that τ is the trivial action, then

$$[X \xrightarrow{f} \mathbf{A}^1_S] \star [Y \xrightarrow{g} \mathbf{A}^1_S] = [X \times_S Y \xrightarrow{f \oplus_S g} \mathbf{A}^1_S], \tag{2.4}$$

where $f \oplus_S g$ stands for the morphism $\mathrm{add} \circ (f \times_S g)$. We denote again by $\star$ the restriction of $\star$ to the image of the inclusion $\mathrm{KVar}_{\mathbf{A}^1_S} \to \mathrm{KVar}^{\hat{\mu}}_{\mathbf{A}^1_S}$, given by formula (2.4). Thus, for Grothendieck rings without actions, lemma 2.2.2.1 boils down to the statement that

$$(\iota_S)_! : \mathrm{KVar}_S \to (\mathrm{KVar}_{\mathbf{A}^1_S}, \star)$$

is a ring morphism.

Remark 2.2.2.5 (Convolution induces product on Grothendieck ring with exponentials). By remark 2.2.2.4 and by the definition of the Grothendieck ring with exponentials $\mathrm{KExpVar}_S$, the quotient map $q : (\mathrm{KVar}_{\mathbf{A}^1_S}, \star) \to \mathrm{KExpVar}_S$ is a morphism of KVar_S-algebras.

2.2.3 Localised Grothendieck rings over the affine line

We have

$$\epsilon_S^*(\mathbf{L}_S) = \mathbf{L}_{\mathbf{A}^1_S} = [\mathbf{A}^1_S \times_S \mathbf{A}^1_S \xrightarrow{\mathrm{pr}_2} \mathbf{A}^1_S],$$

whereas

$$(\iota_S)_!(\mathbf{L}_S) = [\mathbf{A}^1_S \xrightarrow{(\epsilon_S, 0)} \mathbf{A}^1_S],$$

which we denote by $\mathbf{L}_0 \in \mathrm{KVar}^{\hat{\mu}}_{\mathbf{A}^1_S}$. Define $(\widetilde{\mathscr{M}}^{\hat{\mu}}_{\mathbf{A}^1_S}, \star)$ as the $\mathscr{M}_S$-algebra obtained by localising $(\mathrm{KVar}^{\hat{\mu}}_{\mathbf{A}^1_S}, \star)$ with respect to the multiplicative set $\{1, \mathbf{L}_0, \mathbf{L}_0 \star \mathbf{L}_0, \ldots\}$. Then we have a canonical isomorphism of $\mathscr{M}_S$-algebras

$$\mathscr{M}_S \otimes_{\mathrm{KVar}_S} (\mathrm{KVar}^{\hat{\mu}}_{\mathbf{A}^1_S}, \star) \xrightarrow{\sim} (\widetilde{\mathscr{M}}^{\hat{\mu}}_{\mathbf{A}^1_S}, \star).$$

Thus, base change along $\mathrm{KVar}_S \to \mathscr{M}_S$ of lemma 2.2.2.2 gives us:

Lemma 2.2.3.1. *There is a canonical isomorphism* $\mathscr{M}^{\hat{\mu}}_{\mathbf{A}^1_S} \xrightarrow{\sim} \widetilde{\mathscr{M}}^{\hat{\mu}}_{\mathbf{A}^1_S}$ *of* $\mathscr{M}_S$*-modules, given by* $\frac{a}{\mathbf{L}^n_{\mathbf{A}^1_S}} \mapsto \frac{a}{\mathbf{L}^n_0}$ *for all* $a \in \mathrm{KVar}^{\hat{\mu}}_{\mathbf{A}^1_S}$ *and* $n \geq 1$*.*

Moreover, lemma 2.2.2.3 and remark 2.2.2.5 remain true with KVar replaced by $\widetilde{\mathscr{M}}$ and KExpVar replaced with $\mathscr{E}xp\mathscr{M}$.

2.2.4 Rational series

Following 2.8 in [GLM], for any k-variety X, define $\mathscr{M}^{\hat{\mu}}_X[[T]]_{\mathrm{rat}}$ to be the $\mathscr{M}^{\hat{\mu}}_X$-subalgebra of $\mathscr{M}^{\hat{\mu}}_X[[T]]$ generated by rational series of the form $p_{e,i}(T) = \frac{\mathbf{L}^e T^i}{1-\mathbf{L}^e T^i}$ where $e \in \mathbf{Z}$ and $i > 0$.

There is a unique morphism of $\mathscr{M}^{\hat{\mu}}_X$-algebras

$$\lim_{T\to\infty} : \mathscr{M}^{\hat{\mu}}_X[[T]]_{\mathrm{rat}} \longrightarrow \mathscr{M}^{\hat{\mu}}_X$$

such that

$$\lim_{T\to\infty} p_{e,i}(T) = -1,$$

for any $e \in \mathbf{Z}$ and $i > 0$.

More generally, for an k-variety S and for a variety X over S, we define $\mathscr{M}^{\hat{\mu}}_X[[T]]_{\mathrm{rat},S}$ to be the $\mathscr{M}^{\hat{\mu}}_S$-subalgebra of $\mathscr{M}^{\hat{\mu}}_X[[T]]$ generated by rational series of the form $p_{e,i}(T) = \frac{\mathbf{L}^e T^i}{1-\mathbf{L}^e T^i}$ where $e \in \mathbf{Z}$ and $i > 0$.

There is a unique morphism of $\mathscr{M}^{\hat{\mu}}_S$-algebras

$$\lim_{T\to\infty} : \mathscr{M}^{\hat{\mu}}_X[[T]]_{\mathrm{rat},S} \longrightarrow \mathscr{M}^{\hat{\mu}}_X$$

such that

$$\lim_{T\to\infty} p_{e,i}(T) = -1,$$

for any $e \in \mathbf{Z}$ and $i > 0$.

2.2.5 Motivic vanishing cycles

In [DL98, DL99b, DL01, DL02] Denef and Loeser defined and studied the notions of *motivic nearby fibre* and *motivic vanishing cycles*. For a smooth connected variety X over$k$ of dimension d and a morphism $f : X \to \mathbf{A}^1_k$, the motivic nearby fibre ψ_f of f at $0 \in \mathbf{A}^1_k$ is an element of $\mathscr{M}^{\hat{\mu}}_{X_0(f)}$ (where $X_0(f)$ is the fibre of X above 0) defined in terms of some motivic zeta function Z_f. More precisely, denoting by $\mathscr{L}_n(X)$ the space of n-jets of X, we define for every $n \geq 1$

$$\mathscr{X}_n(f) := \{\gamma \in \mathscr{L}_n(X) \mid f(\gamma) \equiv t^n (\mathrm{mod}\ t^{n+1})\} \in \mathscr{M}^{\hat{\mu}}_{X_0(f)},$$

the $X_0(f)$-variety structure of $\mathscr{X}_n(f)$ being induced by the truncation morphism $\mathscr{L}_n(X) \to X$, and the $\hat{\mu}$-action being the one induced by the μ_n-action given by $a.\gamma(t) = \gamma(at)$. We then put

$$Z_f(T) = \sum_{n\geq 1} [\mathscr{X}_n(f) \to X_0(f)]\mathbf{L}^{-nd}T^n \in \mathscr{M}^{\hat{\mu}}_{X_0(f)}[[T]].$$

One may write $\mathscr{X}_n(f/k)$, resp. $Z_{f/k}$ if one wants to keep track of the base field k.

Let X be a smooth variety over k, not necessarily connected, and $f : X \to \mathbf{A}^1_k$ a morphism. Let C be the set of connected components of X, which are smooth varieties of pure dimension. Then the above definition extends immediately to the pair (X, f) by putting:

$$Z_f(T) = \sum_{Y\in C} \mathbf{1}_Y Z_{f_{|Y}}(T) \in \mathscr{M}^{\hat{\mu}}_{X_0(f)}[[T]],$$

where $\mathbf{1}_Y$ denotes the element $[Y \cap X_0(f) \to X_0(f)] \in \mathscr{M}^{\hat{\mu}}_{X_0(f)}$ corresponding to the inclusion of the Y-component of $X_0(f)$ into $X_0(f)$, with trivial $\hat{\mu}$ action.

If f is constant, we have $Z_f(T) = 0$. More generally, Denef and Loeser showed in [DL02] that Z_f is a rational function by giving a formula for it in terms of a log-resolution of $(X, X_0(f))$. For this, let X be a smooth variety over k of pure dimension d, and $f : X \to \mathbf{A}^1_k$ a morphism such that $X_0(f)$ is nowhere dense in X. Let $h : X' \to X$ be a log-resolution of the pair $(X, X_0(f))$. Let $(E_i)_{i\in I}$ be the family of irreducible components of $h^{-1}(X_0(f))$, and for every $i \in I$, let a_i be the multiplicity of $f \circ h$ along E_i. For every $J \subset I$ we put $E_J = \cap_{j\in J} E_j$, $E^\circ_J = E_J \setminus \cup_{i\notin J} E_i$ and $a_J = \gcd_{j\in J}(a_j)$. For every $J \subset I$, one defines an unramified Galois cover $\widetilde{E}^\circ_J \to E^\circ_J$ by glueing locally constructed covers obtained as follows: around every point of E°_J, one can find an open subscheme U of X' on which $f \circ h = u \prod_{j\in J} x_j^{a_j}$, where x_j is an equation for $E_j \cap U$ and u is an invertible function on U. One takes the étale cover of $E^\circ_J \cap U$ induced by the étale cover $U' \to U$ obtained by taking the a_J-th root of u^{-1}. There is a natural μ_{a_J}-action on $\widetilde{E_J}^\circ$ which induces a $\hat{\mu}$-action in the obvious way.

Theorem 2.2.5.1 (Denef-Loeser)**.** *Let X be a smooth k-variety of pure dimension d, and $f : X \to \mathbf{A}^1_k$ a morphism such that $X_0(f)$ is nowhere dense in X. Let $h : X' \to X$ be a log-resolution of the pair $(X, X_0(f))$. Let $(E_i)_{i\in I}$ be the family of irreducible components of $h^{-1}(X_0(f))$. For every $i \in I$, let a_i be the multiplicity of $f \circ h$ along E_i and let $\nu_i - 1$ be the multiplicity of the Jacobian ideal of h along E_i. Then one has*

$$Z_f(T) = \sum_{\varnothing\neq J\subset I} (\mathbf{L} - 1)^{|J|-1} \left[\widetilde{E_J}^\circ \to X_0(f), \hat{\mu}\right] \prod_{j\in J} \frac{\mathbf{L}^{-\nu_j}T^{a_j}}{1 - \mathbf{L}^{-\nu_j}T^{a_j}} \in \mathscr{M}^{\hat{\mu}}_{X_0(f)}[[T]],$$

where $\widetilde{E_J}^\circ$ is the Galois cover defined above. In particular $Z_f(T) \in \mathscr{M}^{\hat{\mu}}_{X_0(f)}[[T]]_{\mathrm{rat}}$.

Corollary 2.2.5.2. *Let X be a smooth variety over k and $f : X \to \mathbf{A}^1_k$ a morphism. Then $Z_f(T)$ is an element of $\mathscr{M}^{\hat{\mu}}_{X_0(f)}[[T]]_{\mathrm{rat}}$.*

Proof. Let C be the set of connected components of X. By definition, we have $Z_f = \sum_{Y\in C} \mathbf{1}_Y Z_{f_{|Y}}$, and each $Z_{f_{|Y}}$ is 0 if $f_{|Y}$ is constant, or is an element of $\mathscr{M}^{\hat{\mu}}_{Y_0(f_{|Y})}[[T]]_{\mathrm{rat}}$

(which is naturally a $\mathscr{M}_X^{\hat{\mu}}$-subalgebra of $\mathscr{M}_{X_0(f)}^{\hat{\mu}}[[T]]_{\mathrm{rat}}$) by theorem 2.2.5.1, whence the result. □

By corollary 2.2.5.2, it makes sense to define the *motivic nearby fibre* ψ_f of f at 0 as

$$\psi_f = -\lim_{T\to\infty} Z_f(T) \in \mathscr{M}_{X_0(f)}^{\hat{\mu}}$$

and the *motivic vanishing cycles* φ_f of f at 0 as

$$\varphi_f := [X_0(f) \xrightarrow{\mathrm{id}} X_0(f)] - \psi_f \in \mathscr{M}_{X_0(f)}^{\hat{\mu}}.$$

For the vanishing cycles, we use the definition in [LS16b], which differs from the one by Denef and Loeser by a sign (which will be important for the construction of our motivic measure, see remark 5.5 in [LS16b]).

Under the conditions and with the notations of theorem 2.2.5.1, we have

$$\psi_f = \sum_{\varnothing\neq J\subset I} (1-\mathbf{L})^{|J|-1}\left[\widetilde{E_J}^{\circ} \to X_0(f), \hat{\mu}\right] \in \mathscr{M}_{X_0(f)}^{\hat{\mu}}.$$

Example 2.2.5.3. Here are some important special cases:

(I) Assume $f = a$ is constant. If $a \neq 0$ then we in fact have $X_0(f) = \varnothing$, so that $\mathscr{M}_{X_0(f)}^{\hat{\mu}} = 0$ and $\psi_f = \varphi_f = 0$. If $a = 0$, then $Z_f = 0$, so $\psi_f = 0$, whereas $\varphi_f = [X \xrightarrow{\mathrm{id}} X] \in \mathscr{M}_X^{\hat{\mu}}$ (the action being necessarily trivial).

(II) In the case when f is non-constant and $X_0(f)$ is smooth and nowhere dense in X, then $X \xrightarrow{\mathrm{id}} X$ gives a log-resolution, and we get $\psi_f = [X_0(f) \xrightarrow{\mathrm{id}} X_0(f)]$, and $\varphi_f = 0$.

(III) As a consequence, when f is not constant equal to 0, φ_f lives above $\mathrm{Sing}(f)$, that is, the closed subscheme of X defined by the vanishing of the differential $\mathrm{d}f \in \Gamma(X, \Omega^1_{X/k})$. Thus, φ_f may be seen in a canonical way as an element of $\mathscr{M}_{X_0(f)\,\cap\,\mathrm{Sing}(f)}^{\hat{\mu}}$.

Remark 2.2.5.4. From the formula in theorem 2.2.5.1, it is clear that, if $X_0(f)$ is nowhere dense in X and if $a_X : X \to k$ is the structural morphism, then $\dim((a_X)_!\varphi_f) \leq \dim X - 1$. Without any assumption on $X_0(f)$, we have the weaker inequality $\dim((a_X)_!\varphi_f) \leq \dim X$.

2.2.6 Relative motivic vanishing cycles

The previous definitions also make sense in the relative setting. Let k be a field of characteristic zero, S a k-variety and X a variety over S of relative dimension d, smooth over S, together with a morphism $f : X \to \mathbf{A}^1$. Then we may define

$$\mathscr{X}_n(f/S) := \{\gamma \in \mathscr{L}_n(X/S) |\ f(\gamma) \equiv t^n (\mathrm{mod}\ t^{n+1})\},$$

with action of $\hat{\mu}$ given in the same manner, and $Z_{f/S}(T) = \sum_{n\geq 1}[\mathscr{X}_n(f/S)]\mathbf{L}^{-nd}T^n \in \mathscr{M}^{\hat{\mu}}_{X_0(f)}[[T]]$. Here, $\mathscr{L}_n(X/S)$ is the n-th jet scheme of X relatively to S. By the base-change properties for jet schemes (see [CNS], chapter 2, (2.1.4)), for every $s \in S$ we have

$$\mathscr{L}_n(X/S) \times_S \kappa(s) = \mathscr{L}_n(X \times_S \kappa(s)/\kappa(s))$$

where $\kappa(s)$ is the residue field of s, so that the fibre above s of $Z_{f/S}$ is exactly $Z_{f_s/\kappa(s)}(T) \in \mathscr{M}^{\hat{\mu}}_{X_0(f)_s}[[T]]$, where $f_s : X_s \to \mathbf{A}^1$ is the morphism induced by f.

Lemma 2.2.6.1. *The series $Z_{f/S}(T)$ is an element of $\mathscr{M}^{\hat{\mu}}_{X_0(f)}[[T]]_{\mathrm{rat},S}$.*

Proof. The proof goes by the classical "spreading-out" method. Take a generic point η of S: then by corollary 2.2.5.2, the series Z_{f_η} is an element of $\mathscr{M}^{\hat{\mu}}_{X_0(f)_\eta}[[T]]_{\mathrm{rat}}$, and its coefficients can be spread out over some open subset U of S. One concludes by Noetherian induction.

□

In particular, it makes sense to define the relative versions of the motivic nearby fibre and motivic vanishing cycles , by

$$\psi_{f/S} = -\lim_{T\to\infty} Z_{f/S}(T) \quad \text{and} \quad \varphi_{f/S} = [X_0(f) \xrightarrow{\mathrm{id}} X_0(f)] - \psi_{f/S} \in \mathscr{M}^{\hat{\mu}}_{X_0(f)}$$

2.2.7 The motivic nearby fibre as a group morphism

In [GLM], Guibert, Loeser and Merle construct, for every smooth variety Y together with a function $h : Y \to \mathbf{A}^1_k$ and every dense open subset U of Y, an object $\mathscr{S}_{h,U} \in \mathscr{M}^{\hat{\mu}}_{Y_0(h)}$, such that $\mathscr{S}_{h,Y} = \psi_h$ is the motivic nearby fibre as defined above and such that these objects fit together into a morphism $\mathscr{S}_h$ as stated in the following theorem:

Theorem 2.2.7.1 ([GLM], Theorem 3.9)**.** *Let Y be a k-variety and $h : Y \to \mathbf{A}^1_k$ a morphism. There exists a unique $\mathscr{M}_k$-linear map $\mathscr{S}_h : \mathscr{M}_Y \to \mathscr{M}^{\hat{\mu}}_{Y_0(h)}$ such that for every proper morphism $p : Z \to Y$ with Z smooth and for every dense open subset U of Z, $\mathscr{S}_h([U \to Y]) = p_!(\mathscr{S}_{h\circ p,U})$.*

When we want to keep track of the base field k, we are going to denote the map in the theorem by $\mathscr{S}_{h/k}$. We are going to need the following corollary, which adapts this to a relative setting:

Corollary 2.2.7.2. *Le S be a k-variety and Y a variety over S, with $h : Y \to \mathbf{A}^1_k$ a morphism. There exists a unique $\mathscr{M}_S$-linear map $\mathscr{S}_{h/S} : \mathscr{M}_Y \to \mathscr{M}^{\hat{\mu}}_{Y_0(h)}$ such that for all $s \in S$ the diagram*

$$\begin{array}{ccc} \mathscr{M}_Y & \xrightarrow{\mathscr{S}_{h/S}} & \mathscr{M}^{\hat{\mu}}_{Y_0(h)} \\ \downarrow & & \downarrow \\ \mathscr{M}_{Y_s} & \xrightarrow{\mathscr{S}_{h/\kappa(s)}} & \mathscr{M}^{\hat{\mu}}_{Y_0(h)_s} \end{array}$$

commutes, where the vertical arrows are induced by the pullback of the inclusion $\{s\} \hookrightarrow S$.

Proof. Uniqueness is immediate by lemma 2.1.3.1. Denote by u the structural morphism $u : Y \to S$. According to lemma 2.1.7.2, it suffices to construct $\mathscr{S}_{h/S}([Z \xrightarrow{p} Y])$ for morphisms of S-schemes $p : Z \to Y$ such that $T = u \circ p(Z)$ is locally closed, $u \circ p : Z \to T$ smooth, and $p \times_S \mathrm{id}_T : Z \times_S T \to Y \times_S T$ proper. For such a class, put

$$\mathscr{S}_{h/S}([Z \xrightarrow{p} Y]) = p_! \left(\psi_{h \circ p/S}\right) \in \mathscr{M}^{\hat{\mu}}_{Y_0(h)}.$$

The group KVar_Y is obtained from the free abelian group A on such generators by taking the quotient with respect to some relations, namely elements of the free abelian group on those generators which belong to the kernel of the canonical surjection $A \to \mathrm{KVar}_Y$. Let $R \in A$ be such a relation. For any element $\mathfrak{a} \in A$, denote by $\mathfrak{a}_s$ its image in $\mathscr{M}_{Y_s}$ through the composition $A \to \mathrm{KVar}_Y \to \mathrm{KVar}_{Y_s} \to \mathscr{M}_{Y_s}$. The definition of $\mathscr{S}_{h/S}$ on elements of A and theorem 2.2.7.1 show that, for any element $\mathfrak{a} \in A$, $(\mathscr{S}_{h/S}(\mathfrak{a}))_s = \mathscr{S}_{h/\kappa(s)}(\mathfrak{a}_s)$. In particular, since for every $s \in S$, $R_s = 0$, we have $(\mathscr{S}_{h/S}(R))_s = 0$, and therefore $\mathscr{S}_{h/S}(R) = 0$ by lemma 2.1.3.1. Thus, $\mathscr{S}_{h/S}$ defines a group morphism $\mathrm{KVar}_Y \to \mathscr{M}^{\hat{\mu}}_{Y_0(h)}$. From the last few lines in the proof of theorem 2.2.7.1 in [GLM] (theorem 3.9), it appears that $\mathscr{S}_{h/\kappa(s)}$ is first constructed on KVar_{Y_s}, and then extended to $\mathscr{M}_{Y_s}$ by $\mathscr{M}_{\kappa(s)}$-linearity. By definition, $\mathscr{S}_{h/S}$ is compatible with $\mathscr{S}_{h/\kappa(s)}$ (seen as a morphism with source KVar_{Y_s}) for all $s \in S$. For any $a \in \mathrm{KVar}_S$, and any $x \in \mathrm{KVar}_Y$, the relation $\mathscr{S}_{h/S}(ax) = a\mathscr{S}_{h/S}(x)$ is seen to be true by lemma 2.1.3.1, because for every $s \in S$, $\mathscr{S}_{h/\kappa(s)}$ is $\mathscr{M}_{\kappa(s)}$-linear. Thus, $\mathscr{S}_{h/S}$ is KVar_S-linear, and we may extend it by $\mathscr{M}_S$-linearity to a $\mathscr{M}_S$-linear morphism $\mathscr{M}_Y \to \mathscr{M}^{\hat{\mu}}_{Y_0(h)}$, which ensures the commutativity of the diagram in the statement of the theorem. □

2.3 The motivic vanishing cycles measure

In [LS16b], Lunts and Schnürer defined, for an algebraically closed field k of characteristic zero, a motivic measure $\Phi^{\mathrm{tot}} : (\widetilde{\mathscr{M}_{\mathbf{A}^1_k}}, \star) \to (\widetilde{\mathscr{M}^{\hat{\mu}}_{\mathbf{A}^1_k}}, \star)$, by the formula

$$\Phi^{\mathrm{tot}} = \sum_{a \in k} (i_a)_! (i_a^* - \mathscr{S}_{\mathrm{id}-a}) \tag{2.5}$$

where $i_a : \{a\} \to \mathbf{A}^1_k$ is the inclusion, and $\mathscr{S}_{\mathrm{id}-a} : \mathscr{M}_{\mathbf{A}^1} \to \mathscr{M}^{\hat{\mu}}_{\{a\}}$ is the morphism from theorem 2.2.7.1 applied with $Y = \mathbf{A}^1$ and $h = \mathrm{id} - a$ (this is the measure denoted by Φ in their paper). Formula (2.5) makes sense because for any class $[X \xrightarrow{f} \mathbf{A}^1]$ with X smooth and f proper, we have, denoting by X_a the fibre of f above a and by $f_a : X_a \to \{a\}$ the constant map induced by f on it, by theorem 2.2.7.1

$$\begin{aligned} (i_a)_!(i_a^* - \mathscr{S}_{\mathrm{id}-a})([X \xrightarrow{f} \mathbf{A}^1]) &= (i_a)_! ([X_a \to \{a\}] - (f_a)_! \psi_{f-a}) \\ &= (i_a)_! (f_a)_! ([X_a \to X_a] - \psi_{f-a}) \\ &= f_! \varphi_{f-a} \end{aligned}$$

which is zero whenever a is not a critical point of f. Thus, the sum is always finite because the Grothendieck ring of varieties is generated by such classes, and because the set of critical points of a morphism $f : X \to \mathbf{A}^1_k$ is finite. The image of a class $[X \xrightarrow{f} \mathbf{A}^1]$ with X smooth and f proper is

$$\Phi^{\mathrm{tot}}([X \xrightarrow{f} \mathbf{A}^1]) = \sum_{a\in k} f_! \varphi_{f-a} =: \varphi_f^{\mathrm{tot}}, \tag{2.6}$$

the sum of all vanishing cycles of f at all $a \in k$. In other words, φ_f^{tot} is the element of $\mathscr{M}^{\hat{\mu}}_{\mathbf{A}^1_k}$ corresponding to the motivic function sending $a \in \mathbf{A}^1_k$ to $f_!\varphi_{f-a}$

In what follows, we are going to need to construct such a measure in families. Therefore, we will give a definition of φ_f^{tot} in terms of vanishing cycles relative to the affine line above a base, which behaves well in such a context. For an algebraically closed field k of characteristic zero, this will give an element of $\mathscr{M}^{\hat{\mu}}_{\mathbf{A}^1_k}$ supported above the critical points of $\mathbf{A}^1_k$, with fibre at every point $a \in k$ given by the vanishing cycles $f_!\varphi_{f-a}$, so that we recover formula (2.6).

2.3.1 Total vanishing cycles

Let k be a field of characteristic zero, R a variety over k, and X be a variety over R, smooth over R, and $f : X \to \mathbf{A}^1_k$ a morphism. We apply the construction of the previous paragraph to the variety $X \times \mathbf{A}^1_k$ over $S = \mathbf{A}^1_R$, together with the morphism $g : X \times \mathbf{A}^1_k \to \mathbf{A}^1_k$ given by $g = f \circ \mathrm{pr}_1 - \mathrm{pr}_2$. We have $(X \times \mathbf{A}^1)_0(g) = \Gamma_f$, where $\Gamma_f \subset X \times \mathbf{A}^1_k$ is the graph of the morphism f, which we may identify with X itself through the first projection.

Notation 2.3.1.1. We denote by $\widetilde{\varphi}_{f/R}^{\mathrm{tot}} := \varphi_{g/\mathbf{A}^1_R}$ and $\widetilde{\psi}_{f/R}^{\mathrm{tot}} := \psi_{g/\mathbf{A}^1_R}$ the corresponding vanishing cycles and nearby fibre, which are naturally defined as elements of $\mathscr{M}^{\hat{\mu}}_X$, and related by the identity

$$\widetilde{\varphi}_{f/R}^{\mathrm{tot}} = [X \xrightarrow{\mathrm{id}} X] - \widetilde{\psi}_{f/R}^{\mathrm{tot}}.$$

Denote by f_R the morphism $X \to \mathbf{A}^1_R = R \times \mathbf{A}^1_k$ given by (u, f) where $u : X \to R$ is the structural map. We define $\varphi^{\mathrm{tot}}_{f/R} := (f_R)_!\widetilde{\varphi}_{f/R}^{\mathrm{tot}}$ and $\psi^{\mathrm{tot}}_{f/R} := (f_R)_!\widetilde{\psi}_{f/R}^{\mathrm{tot}}$ their images in $\mathscr{M}^{\hat{\mu}}_{\mathbf{A}^1_R}$, satisfying

$$\varphi^{\mathrm{tot}}_{f/R} = [X \xrightarrow{f_R} \mathbf{A}^1_R] - \psi^{\mathrm{tot}}_{f/R}.$$

In the case where $R = k$, we will simply write $\widetilde{\psi}_f^{\mathrm{tot}}, \widetilde{\varphi}_f^{\mathrm{tot}}$ etc.

These objects will be called *total nearby fibre* and *total vanishing cycles*, because they take into account the nearby fibre and vanishing cycles of f at all points of $\mathbf{A}^1$: indeed, we see that for any $t \in \mathbf{A}^1$,

$$\left(\widetilde{\psi}_{f/R}^{\mathrm{tot}}\right)_t = \psi_{g_t/R_{\kappa(t)}} = \psi_{(f-t)/R_{\kappa(t)}} \in \mathscr{M}^{\hat{\mu}}_{X_t},$$

where $f - t$ is the function $X \times_k \kappa(t) \to \mathbf{A}^1_{\kappa(t)}$ given by $x \mapsto f(x) - t$ and $R_{\kappa(t)} = R \times_k \kappa(t)$. A similar remark holds for $\widetilde{\varphi}^{\mathrm{tot}}_{f/R}, \psi^{\mathrm{tot}}_{f/R}$ and $\varphi^{\mathrm{tot}}_{f/R}$. The properties of vanishing cycles we recalled above lead to similar properties for total vanishing cycles.

Remark 2.3.1.2. Let X and R be as above, and assume $f : X \to \mathbf{A}^1_k$ is constant. Then property (I) of example 2.2.5.3 together with lemma 2.1.3.1 imply that $\widetilde{\psi}_f^{\text{tot}} = 0$. In particular, we have $\widetilde{\varphi}_f^{\text{tot}} = [X \xrightarrow{\text{id}} X]$ and $\varphi_f^{\text{tot}} = [X \xrightarrow{f_R} \mathbf{A}^1_R]$ (with trivial $\hat{\mu}$-action).

Remark 2.3.1.3. With the above notations, it follows from remark 2.2.5.4 applied above every point of $\mathbf{A}^1_R$, that

$$\dim_{\mathbf{A}^1_R}(\varphi^{\text{tot}}_{f/R}) \leq \dim_{\mathbf{A}^1_R} X.$$

We recall that we denote by $\mathrm{Sing}(f)$ the vanishing locus of the differential form $\mathrm{d}f$, and we define $\mathrm{Crit}(f)$ to be the scheme-theoretic image $f_R(\mathrm{Sing}(f)) \subset \mathbf{A}^1_R$.

Proposition 2.3.1.4. *Let X be a smooth R-variety and $f : X \to \mathbf{A}^1_k$ a morphism. Then $\widetilde{\varphi}^{\text{tot}}_{f/R}$ (resp. $\varphi^{\text{tot}}_{f/R}$) is canonically an element of $\mathscr{M}^{\hat{\mu}}_{\mathrm{Sing}(f)}$ (resp. $\mathscr{M}^{\hat{\mu}}_{\mathrm{Crit}(f)}$). In particular, if f is smooth, then $\widetilde{\varphi}^{\text{tot}}_{f/R} = 0$ and $\varphi^{\text{tot}}_{f/R} = 0$.*

Proof. This is a consequence of property (III) above applied point by point, together with lemma 2.1.3.1. □

2.3.2 The Thom-Sebastiani theorem

The classical theorem proved by Thom and Sebastiani in [ST] is a multiplicativity result for the cohomology of Milnor fibres: for two germs $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ and $g : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ of holomorphic functions with an isolated critical point at 0, it expresses the reduced cohomology of the Milnor fibre of the germ $f \oplus g : (x, y) \mapsto f(x) + g(y)$ as a tensor product of the reduced cohomologies of the Milnor fibres of f and g, together with compatibilities of monodromy actions.

An analogue of this for motivic vanishing cycles was first proved by Denef and Loeser in the completed Grothendieck ring of Chow motives in [DL99b]. Then Looijenga, who in [Loo] introduced an appropriate convolution operation, and Denef Loeser in [DL01], showed that essentially the same proof gave an equality in the Grothendieck ring of varieties with $\hat{\mu}$-action. Finally, in [GLM], Guibert, Loeser and Merle showed how one may recover the motivic Thom-Sebastiani theorem from a formula involving iterated vanishing cycles. In that paper, the theorem is stated using the generalised convolution operator Ψ which we defined in section 2.2.1.

We recall the motivic Thom-Sebastiani theorem, in the form in which it appears in [GLM]:

Theorem 2.3.2.1. *Let Y_1, Y_2 be smooth varieties over k, with morphisms $g_1 : Y_1 \to \mathbf{A}^1_k$, $g_2 : Y_2 \to \mathbf{A}^1_k$. Denote by i the natural inclusion $Y_0 := g_1^{-1}(0) \times g_2^{-1}(0) \to (g_1 \oplus g_2)^{-1}(0)$. Then*

$$i^* \left(\varphi_{g_1 \oplus g_2}\right) = \Psi(\varphi_{g_1} \boxtimes \varphi_{g_2})$$

in $\mathscr{M}^{\hat{\mu}}_{Y_0}$.

This may be globalised in the following manner:

Corollary 2.3.2.2 (Thom-Sebastiani for total vanishing cycles)**.** *Let X_1, X_2 be smooth varieties, with morphisms $f_1 : X_1 \to \mathbf{A}^1_k$ and $f_2 : X_2 \to \mathbf{A}^1_k$. Then we have the equalities*

$$\widetilde{\varphi}^{\text{tot}}_{f_1 \oplus f_2} = \Psi(\widetilde{\varphi}^{\text{tot}}_{f_1} \boxtimes \widetilde{\varphi}^{\text{tot}}_{f_2}) \tag{2.7}$$

in $\mathscr{M}^{\hat{\mu}}_{X_1 \times X_2}$, and

$$\varphi^{\text{tot}}_{f_1 \oplus f_2} = \Psi(\varphi^{\text{tot}}_{f_1} \star \varphi^{\text{tot}}_{f_2}) \tag{2.8}$$

in $\mathscr{M}^{\hat{\mu}}_{\mathbf{A}^1_k}$.

Proof. Let $g_i : X_i \times \mathbf{A}^1 \to \mathbf{A}^1$ be the functions defined by $g_i = f_i \circ \mathrm{pr}_1 - \mathrm{pr}_2$ for $i = 1, 2$, and $g : X_1 \times X_2 \times \mathbf{A}^1 \to \mathbf{A}^1$ be the function defined by $g = f_1 \circ \mathrm{pr}_1 + f_2 \circ \mathrm{pr}_2 - \mathrm{pr}_3$. By definition, the left hand side of the first equality is exactly $\varphi_{g/\mathbf{A}^1}$, whereas the right-hand side is given by $\Psi(\varphi_{g_1/\mathbf{A}^1} \boxtimes \varphi_{g_2/\mathbf{A}^1})$. We have the commutative diagram

$$\begin{array}{ccc} X_1 \times X_2 & & \\ {\scriptstyle (f_1,f_2)}\downarrow & \searrow^{f_1 \oplus f_2} & \\ \mathbf{A}^1 \times \mathbf{A}^1 & \xrightarrow{+} & \mathbf{A}^1 \end{array}$$

Let $t \in \mathbf{A}^1 \times \mathbf{A}^1$ with residue field denoted by K, and put $t_1 = \mathrm{pr}_1(t)$, $t_2 = \mathrm{pr}_2(t)$ and $s = t_1 + t_2$. Let i be the natural inclusion

$$i : f_1^{-1}(t_1) \times_k f_2^{-1}(t_2) \to (f_1 \oplus f_2)^{-1}(s) \times_{\kappa(s)} K.$$

Pulling back via the inclusion of t inside $\mathbf{A}^1 \times \mathbf{A}^1$, we see that the left-hand side $\varphi_{g/\mathbf{A}^1}$ goes to $i^*(\varphi_{(f_1 \oplus f_2 - s)/K})$, whereas the right-hand side $\varphi_{g_1/\mathbf{A}^1} \boxtimes \varphi_{g_2/\mathbf{A}^1}$ goes to

$$\Psi\big(\varphi_{(f_1 - t_1)/\kappa(t_1)} \boxtimes \varphi_{(f_2 - t_2)/\kappa(t_2)}\big)$$

because Ψ commutes with pullbacks. These elements are equal in $\mathscr{M}^{\hat{\mu}}_{f_1^{-1}(t_1) \times f_2^{-1}(t_2)}$ by theorem 2.3.2.1. Since t was arbitrary, we get the first equality in the statement of the theorem. The second equality is then obtained easily by applying $(f_1 \oplus f_2)_!$ on both sides and making use of the commutative diagram above. □

2.3.3 Total vanishing cycles as a motivic measure

We are going to prove the following theorem:

Theorem 2.3.3.1. *Let k be a field of characteristic zero. There is a unique morphism*

$$\Phi^{\text{tot}} : (\mathrm{KVar}_{\mathbf{A}^1_k}, \star) \to (\widetilde{\mathscr{M}^{\hat{\mu}}_{\mathbf{A}^1_k}}, \star)$$

of KVar_k-algebras such that $\Phi([X \xrightarrow{f} \mathbf{A}^1_k]) = \varphi^{\text{tot}}_f$ for any smooth variety X over k and any proper morphism $f : X \to \mathbf{A}^1_k$.

The case where k is algebraically closed was treated by Lunts and Schnürer in [LS16b], and our proof goes along the same lines as theirs, the main difference being that we replace their total vanishing cycles $(\varphi_f)_{\mathbf{A}^1_k}$ by our total vanishing cycles φ_f^{tot} which behave better in relative settings. We start by proving the following result, which is the analogue in our setting of Theorem 5.3 in [LS16b]:

Proposition 2.3.3.2. *There exists a unique morphism $\Phi' : \mathscr{M}_{\mathbf{A}^1_k} \to \mathscr{M}^{\hat{\mu}}_{\mathbf{A}^1_k}$ of $\mathscr{M}_k$-modules such that $\Phi'([X \xrightarrow{f} \mathbf{A}^1_k]) = \varphi_f^{\mathrm{tot}}$ for any smooth variety X over k and any proper morphism $f : X \to \mathbf{A}^1_k$.*

Proof. Uniqueness follows from lemma 2.1.7.1 so it remains to prove existence. Apply corollary 2.2.7.2 to $Y = \mathbf{A}^1_k \times \mathbf{A}^1_k$, seen as a variety over $S = \mathbf{A}^1_k$ via the second projection, together with $h = \mathrm{pr}_1 - \mathrm{pr}_2$: we get an $\mathscr{M}_{\mathbf{A}^1}$-linear map

$$\mathscr{S}_{h/\mathbf{A}^1} : \mathscr{M}_{\mathbf{A}^1_k \times \mathbf{A}^1_k} \to \mathscr{M}^{\hat{\mu}}_{\Delta},$$

where $\Delta \subset \mathbf{A}^1_k \times \mathbf{A}^1_k$ is the diagonal $h^{-1}(0)$, which is isomorphic to $\mathbf{A}^1_k$ via pr_2. Composing this with the pull-back $\mathrm{pr}_2^* : \mathscr{M}_{\mathbf{A}^1_k} \to \mathscr{M}_{\mathbf{A}^1_k \times \mathbf{A}^1_k}$, sending a class $[X \xrightarrow{f} \mathbf{A}^1_k]$ to $[X \times \mathbf{A}^1_k \xrightarrow{(f\circ \mathrm{pr}_1, \mathrm{pr}_2)} \mathbf{A}^1_k \times \mathbf{A}^1_k]$ we get a $\mathscr{M}_k$-linear map $\mathscr{S}_{h/\mathbf{A}^1} \circ \mathrm{pr}_2^* : \mathscr{M}_{\mathbf{A}^1_k} \to \mathscr{M}^{\hat{\mu}}_{\mathbf{A}^1_k}$. We put, for any $\mathfrak{a} \in \mathscr{M}_{\mathbf{A}^1_k}$,

$$\Phi'(\mathfrak{a}) = \mathfrak{a} - \mathscr{S}_{h/\mathbf{A}^1} \circ \mathrm{pr}_2^*(\mathfrak{a}).$$

Let $f : X \to \mathbf{A}^1_k$ be a proper morphism with X smooth. We denote by p the morphism

$$\mathrm{pr}_2^*(f) : X \times \mathbf{A}^1_k \xrightarrow{(f\circ \mathrm{pr}_1, \mathrm{pr}_2)} \mathbf{A}^1_k \times \mathbf{A}^1_k$$

which is again proper by base change. We claim that $\mathscr{S}_{h/\mathbf{A}^1_k} \circ \mathrm{pr}_2^*([X \xrightarrow{f} \mathbf{A}^1]) = \psi_f^{\mathrm{tot}}$. Indeed, by theorems 2.2.7.1 and 2.2.7.2, for every $t \in \mathbf{A}^1$, the fibre above t of this element is given by

$$\begin{aligned}\left(\mathscr{S}_{h/\mathbf{A}^1_k}([X \times \mathbf{A}^1_k \xrightarrow{p} \mathbf{A}^1_k \times \mathbf{A}^1_k])\right)_t &= \mathscr{S}_{h_t/\kappa(t)}([X \times_k \kappa(t) \xrightarrow{p_t} \mathbf{A}^1_{\kappa(t)}]) \\ &= (p_t)_!(\psi_{(h\circ p)_t/\kappa(t)}) \in \mathscr{M}^{\hat{\mu}}_{\kappa(t)}.\end{aligned}$$

because p_t is proper and X_t smooth over $\kappa(t)$. On the other hand, we have $h \circ p = f \circ \mathrm{pr}_1 - \mathrm{pr}_2$, so that for every $t \in \mathbf{A}^1$, $\psi_{(h\circ p)_t/\kappa(t)} = (\psi_{h\circ p/\mathbf{A}^1_k})_t = (\widetilde{\psi}_f{}^{\mathrm{tot}})_t$. Moreover, for every t, we have $p_t = f \times_k \kappa(t)$, so that $(p_t)_!(\psi_{(h\circ p)_t/\kappa(t)}) = \left(f_!(\widetilde{\psi}_f{}^{\mathrm{tot}})\right)_t = \left(\psi_f^{\mathrm{tot}}\right)_t$, which proves the claim. Finally, we may conclude that $\Phi'([X \xrightarrow{f} \mathbf{A}^1_k]) = [X \xrightarrow{f} \mathbf{A}^1_k] - \psi_f^{\mathrm{tot}} = \varphi_f^{\mathrm{tot}}$. □

As in remarks 5.4, 5.6 and 5.7 of [LS16b], the map Φ' has the following properties:

Lemma 2.3.3.3. *For any k-variety Z, we have*

(a) $\Phi'([Z \xrightarrow{0} \mathbf{A}^1_k]) = [Z \xrightarrow{0} \mathbf{A}^1_k]$ (with trivial action), so that in particular $\Phi'(\mathbf{L}_0) = \mathbf{L}_0$.

(b) $\Phi'([Z \times \mathbf{A}^1_k \xrightarrow{\mathrm{pr}_2} \mathbf{A}^1_k]) = 0$*, so that in particular* $\Phi'(\mathbf{L}_{\mathbf{A}^1_k}) = 0$*.*

(c) $\Phi'([Z \xrightarrow{f} \mathbf{A}^1_k]) = 0$ *whenever* f *is a smooth and proper morphism.*

Proof. The subgroup of $\mathrm{KVar}_{\mathbf{A}^1_k}$ generated by classes of the form $[Z \xrightarrow{0} \mathbf{A}^1_k]$ is the image of the morphism $(\iota_k)_! : \mathrm{KVar}_k \to \mathrm{KVar}_{\mathbf{A}^1_k}$ from section 2.2.2. It is therefore generated by such classes where Z is additionally assumed to be smooth and proper, and these are preserved by Φ' by proposition 2.3.3.2 and remark 2.3.1.2. This proves (a). In the same way, the subgroup of $\mathrm{KVar}_{\mathbf{A}^1_k}$ generated by classes of the form $[Z \times \mathbf{A}^1_k \xrightarrow{\mathrm{pr}_2} \mathbf{A}^1_k]$ is the image of the morphism $\epsilon_k^* : \mathrm{KVar}_k \to \mathrm{KVar}_{\mathbf{A}^1_k}$ from section 2.2.2, and we may again assume Z to be smooth and proper to prove (b). The statement then follows from propositions 2.3.3.2 and 2.3.1.4. As for property (c), if $f : Z \to \mathbf{A}^1_k$ is smooth and proper, then Z is smooth, so this follows directly from propositions 2.3.3.2 and 2.3.1.4. □

We then consider the morphism of KVar_k-modules Φ^{tot}, defined as the composition

$$\mathrm{KVar}_{\mathbf{A}^1_k} \to \mathscr{M}_{\mathbf{A}^1_k} \xrightarrow{\Phi'} \mathscr{M}^{\hat{\mu}}_{\mathbf{A}^1_k} \to \widetilde{\mathscr{M}}^{\hat{\mu}}_{\mathbf{A}^1_k},$$

where the first arrow is the localisation morphism, and the last arrow is the isomorphism of $\mathscr{M}_k$-modules from lemma 2.2.3.1. The proof of theorem 2.3.3.1 is complete once we have the following:

Proposition 2.3.3.4. *The morphism* Φ^{tot} *is a morphism*

$$\Phi^{\mathrm{tot}} : (\mathrm{KVar}_{\mathbf{A}^1_k}, \star) \to (\widetilde{\mathscr{M}}^{\hat{\mu}}_{\mathbf{A}^1_k}, \star)$$

of KVar_k*-algebras.*

Proof. Part (a) of lemma 2.3.3.3 shows that Φ^{tot} maps the unit element to the unit element, and that it is compatible with the algebra structure maps. By lemma 2.1.7.1, we may restrict to checking multiplicativity for classes of projective morphisms $f : X \to \mathbf{A}^1_k$ with X a connected quasi-projective k-variety which is smooth over k. Let X, Y therefore be connected quasi-projective smooth k-varieties, together with projective morphisms $f : X \to \mathbf{A}^1_k$ and $g : Y \to \mathbf{A}^1_k$. Then we know by proposition 2.3.3.2 and corollary 2.3.2.2 that

$$\Phi^{\mathrm{tot}}([X \xrightarrow{f} \mathbf{A}^1_k]) \star \Phi^{\mathrm{tot}}([Y \xrightarrow{g} \mathbf{A}^1_k]) = (\varphi_f^{\mathrm{tot}}) \star (\varphi_g^{\mathrm{tot}}) = \varphi_{f \oplus g}^{\mathrm{tot}}.$$

It remains to show that $\varphi^{\mathrm{tot}}_{f\oplus g} = \Phi^{\mathrm{tot}}([X \times Y \xrightarrow{f \oplus g} \mathbf{A}^1_k])$, which does not follow directly from proposition 2.3.3.2 because $f \oplus g$ is not proper in general. As in the proof of theorem 5.9 in [LS16b], we make use of lemma 2.3.6.1 below (and the notation therein) which gives us a compactification $h : Z \to \mathbf{A}^1_k$ of $f \oplus g$ for which we may write,

$$[X \times Y \xrightarrow{f\oplus g} \mathbf{A}^1_k] = [Z \xrightarrow{h} \mathbf{A}^1_k] + \sum_I (-1)^{|I|} [D_I \xrightarrow{h_I} \mathbf{A}^1_k].$$

Since all $h_I : D_I \to \mathbf{A}^1_k$ are projective and smooth, their image by Φ^{tot}, given by their total vanishing cycles, is zero by proposition 2.3.1.4. On the other hand, $\mathrm{Sing}(h) = \mathrm{Sing}(f \oplus g)$, and therefore

$$\Phi^{\mathrm{tot}}([X \times Y \xrightarrow{f \oplus g} \mathbf{A}^1_k]) = \Phi^{\mathrm{tot}}([Z \xrightarrow{h} \mathbf{A}^1_k]) = \varphi^{\mathrm{tot}}_h = \varphi^{\mathrm{tot}}_{f \oplus g}.$$

□

2.3.4 The motivic vanishing cycles measure over a base

We are now going to prove that the above motivic measure may be defined in families. For this purpose, we are going to denote for any field k of characteristic zero by $\Phi^{\mathrm{tot}}_k : \mathrm{KVar}_{\mathbf{A}^1_k} \to \mathscr{M}^{\hat{\mu}}_{\mathbf{A}^1_k}$ the motivic measure from theorem 2.3.3.1 relative to the field k.

Theorem 2.3.4.1. *Let k be a field of characteristic zero and S a variety over k. There is a unique morphism of KVar_S-algebras $\Phi^{\mathrm{tot}}_S : (\mathrm{KVar}_{\mathbf{A}^1_S}, \star) \to (\widetilde{\mathscr{M}}^{\hat{\mu}}_{\mathbf{A}^1_S}, \star)$ such that for every $s \in S$ the diagram*

$$\begin{array}{ccc} \mathrm{KVar}_{\mathbf{A}^1_S} & \xrightarrow{\Phi^{\mathrm{tot}}_S} & \widetilde{\mathscr{M}}^{\hat{\mu}}_{\mathbf{A}^1_S} \\ \downarrow & & \downarrow \\ \mathrm{KVar}_{\mathbf{A}^1_{\kappa(s)}} & \xrightarrow{\Phi^{\mathrm{tot}}_{\kappa(s)}} & \widetilde{\mathscr{M}}^{\hat{\mu}}_{\mathbf{A}^1_{\kappa(s)}} \end{array}$$

commutes.

Proof. Uniqueness is immediate by lemma 2.1.3.1. By lemma 2.1.7.2, denoting by $u : \mathbf{A}^1_S \to S$ the structural morphism, it suffices to construct Φ^{tot}_S on classes $[X \xrightarrow{p} \mathbf{A}^1_S]$ such that $T = u \circ p(X)$ is locally closed in S, X is smooth over T and $p \times_S \mathrm{id}_T : X \times_S T \to \mathbf{A}^1_T$ is proper. For such a class, denoting by f the composition $X \xrightarrow{p} \mathbf{A}^1_S \to \mathbf{A}^1_k$, we put

$$\Phi^{\mathrm{tot}}_S([X \xrightarrow{p} \mathbf{A}^1_S]) = \varphi^{\mathrm{tot}}_{f/S} \in \mathscr{M}^{\hat{\mu}}_{\mathbf{A}^1_S}.$$

Then for every $s \in S$, we have

$$\Phi^{\mathrm{tot}}_S([X \xrightarrow{p} \mathbf{A}^1_S])_s = \varphi^{\mathrm{tot}}_{f_s/\kappa(s)}.$$

For any $s \in T$, $f_s : X_s \to \mathbf{A}^1_k$ is proper and the fibre X_s is smooth over $\kappa(s)$ by the assumption on p. For $s \in S \backslash T$, the fibre X_s is empty and therefore $\Phi^{\mathrm{tot}}_S([X \xrightarrow{p} \mathbf{A}^1_S])_s = 0$ in this case. Thus, by the characterisation of $\Phi^{\mathrm{tot}}_{\kappa(s)}$ in theorem 2.3.3.1, we may conclude, as in the proof of theorem 2.2.7.2, that Φ^{tot}_S is well-defined as a group morphism $\mathrm{KVar}_{\mathbf{A}^1_S} \to \mathscr{M}^{\hat{\mu}}_{\mathbf{A}^1_S}$ and that the diagram in the statement is commutative. As in the proof of theorem 2.2.7.2, the fact that Φ^{tot}_S is a morphism of KVar_S-algebras follows from the fact that $\Phi^{\mathrm{tot}}_{\kappa(s)}$ is a morphism of $\mathrm{KVar}_{\kappa(s)}$-algebras for every $s \in S$, using lemma 2.1.3.1. □

Lemma 2.3.4.2. *Let S be a k-variety, and let X be a variety over S, with structural morphism $u : X \to S$. Denote by u_0 the morphism $X \xrightarrow{0\times_k u} \mathbf{A}^1_k \times_k S \simeq \mathbf{A}^1_S$. Then*

(a) $\Phi^{\mathrm{tot}}_S([X \xrightarrow{u_0} \mathbf{A}^1_S]) = [X \xrightarrow{u_0} \mathbf{A}^1_S]$ *(with trivial action).*

(b) $\Phi^{\mathrm{tot}}_S([X \times_S \mathbf{A}^1_S \xrightarrow{\mathrm{pr}_2} \mathbf{A}^1_S]) = 0$

Proof. This follows from lemma 2.3.3.3 and proposition 2.3.4.1 by lemma 2.1.3.1. □

2.3.5 A motivic measure on the Grothendieck ring of varieties with exponentials

We come to the final form of the motivic vanishing cycles measure, which will be the one we are going to use. For the moment, we have constructed, for every k-variety S, a motivic measure Φ^{tot}_S defined on the ring $(\mathrm{KVar}_{\mathbf{A}^1_S}, \star)$ and with values in the ring $(\widetilde{\mathscr{M}^{\hat{\mu}}_{\mathbf{A}^1_S}}, \star)$. For our purposes, it will be convenient to view Φ^{tot}_S as a motivic measure on the Grothendieck ring with exponentials over S, and to compose it with the pushforward map $(\epsilon_S)_!$ where $\epsilon_S : \mathbf{A}^1_S \to S$ is the structural morphism.

Theorem 2.3.5.1 (Motivic vanishing cycles measure)**.** *Let k be a field of characteristic zero.*

1. *There is a unique morphism*

$$\Phi_k : \mathscr{E}xp\mathscr{M}_k \to (\mathscr{M}^{\hat{\mu}}_k, *)$$

of $\mathscr{M}_k$-algebras, called the motivic vanishing cycles measure, such that, for any proper morphism $f : X \to \mathbf{A}^1_k$, with source a smooth k-variety X, one has

$$\Phi_k([X \xrightarrow{f} \mathbf{A}^1_k]) = \epsilon_!(\varphi^{\mathrm{tot}}_f)$$

where $\epsilon : \mathbf{A}^1_k \to k$ is the structural morphism.

2. *Let S be a k-variety. There is a unique morphism $\Phi_S : \mathscr{E}xp\mathscr{M}_S \to (\mathscr{M}^{\hat{\mu}}_S, *)$ of $\mathscr{M}_S$-algebras such that, for any $s \in S$, the diagram*

$$\begin{array}{ccc} \mathscr{E}xp\mathscr{M}_S & \xrightarrow{\Phi_S} & \mathscr{M}^{\hat{\mu}}_S \\ \downarrow & & \downarrow \\ \mathscr{E}xp\mathscr{M}_{\kappa(s)} & \xrightarrow{\Phi_{\kappa(s)}} & \mathscr{M}^{\hat{\mu}}_{\kappa(s)} \end{array}$$

commutes.

Moreover, for any k-variety S, the restriction of Φ_S to $\mathscr{M}_S$ coincides with the natural inclusion $\mathscr{M}_S \to \mathscr{M}^{\hat{\mu}}_S$.

Proof. Property (b) in lemma 2.3.4.2 says that Φ_S^{tot} sends the additional relation

$$[X \times_S \mathbf{A}^1_S \xrightarrow{\text{pr}_2} \mathbf{A}^1_S]$$

defining $\mathrm{KExpVar}_{\mathbf{A}^1_S}$ to zero. Using remark 2.2.2.5, we see that the morphism Φ_S^{tot} induces a morphism of KVar_S-algebras $\mathrm{KExpVar}_S \to (\widetilde{\mathscr{M}}^{\mu}_{\mathbf{A}^1_S}, \star)$. By property (a) of lemma (2.3.4.2), the class $\mathbf{L}_S$ goes to the invertible element $\mathbf{L}_0 \in \widetilde{\mathscr{M}}^{\mu}_{\mathbf{A}^1_S}$ so that Φ^{tot} extends by $\mathscr{M}_S$-linearity to a morphism of $\mathscr{M}_S$-algebras $\mathscr{E}xp\mathscr{M}_S \to (\widetilde{\mathscr{M}}^{\mu}_{\mathbf{A}^1_S}, \star)$. By the localised version of lemma 2.2.2.3, the pushforward $(\epsilon_S)_!$ is a morphism of $\mathscr{M}_S$-algebras $(\widetilde{\mathscr{M}}^{\mu}_{\mathbf{A}^1_S}, \star) \to (\mathscr{M}_S, *)$, so putting $\Phi_S := (\epsilon_S)_! \circ \Phi_S^{\text{tot}}$ we get a morphism of $\mathscr{M}_S$-algebras $\Phi_S : \mathscr{E}xp\mathscr{M}_S \to (\mathscr{M}_S^{\hat{\mu}}, *)$. Part 1 of the statement then follows immediately from theorem 2.3.3.1, whereas part 2 comes from proposition 2.3.4.1. Finally, the statement about the restriction of Φ_S to $\mathscr{M}_S$ is seen to be true by property (a) of lemma 2.3.4.2. □

The following proposition is the motivic version for the dimensional topology of the triangular inequality

$$\left|\sum_{x \in X(\mathbf{F}_q)} \psi(f(x))\right| \leq |X(\mathbf{F}_q)|$$

for X a variety over $\mathbf{F}_q$, $f : X \to \mathbf{A}^1_{\mathbf{F}_q}$ a morphism and $\psi : \mathbf{F}_q \to \mathbf{C}^*$ a non-trivial character.

Proposition 2.3.5.2 (Triangular inequality)**.** *Let S be a k-variety, X a variety over S and $f : X \to \mathbf{A}^1_k$ a morphism. Then*

$$\dim_S(\Phi_S([X, f])) \leq \dim_S X$$

Proof. It suffices to prove this above every point $s \in S$, so we may assume $S = \operatorname{Spec} k$. Then, up to adding classes of strictly smaller dimension which can be dealt with by induction, we may assume that X is smooth and f is proper. In this case $\Phi([X, f]) = \epsilon_! \varphi_f^{\text{tot}}$ and the result follows from remark 2.3.1.3. □

2.3.6 A compactification lemma

The following lemma, which we used in the proof of proposition 2.3.3.4, was stated and proved in [LS16a] in the case where the field k is assumed to be algebraically closed of characteristic zero. We show here that it remains true even if the field is no longer assumed to be algebraically closed.

We are going to deduce this from the proof of proposition 6.1 in [LS16a], by proving that the construction of Z commutes with base change to any extension of the field k, and that the statements of the lemma remain true over k if they are true over an extension of k. For this, most of the arguments will go by faithfully flat descent (see EGA IV, 2.7.1), using that for any extension k' of k, the structural morphism $\operatorname{Spec} k' \to \operatorname{Spec} k$ is faithfully flat.

For this, we recall that, if k' is an extension of k (which is still assumed to be of characteristic zero), then (see e.g. tag 0C3H in the Stacks project) :

(a) The formation of the singular locus X^{sing} of a variety X over k commutes with base change to k'.

(b) For a morphism $f : X \to \mathbf{A}^1_k$, the formation of the singular locus of f commutes with base change to k', i.e: $\text{Sing} f_{k'} = (\text{Sing} f) \times_k k'$.

Proposition 2.3.6.1. *Let k be a field of characteristic zero. Let X and Y be smooth varieties and let $f : X \to \mathbf{A}^1_k$ and $g : Y \to \mathbf{A}^1_k$ be projective morphisms. Then there exists a smooth quasi-projective k-variety Z with an open embedding $X \times Y \hookrightarrow Z$ and a projective morphism $h : Z \to \mathbf{A}^1_k$ such that the following conditions are satisfied.*

(i) The restriction of h to $X \times Y$ is $f \oplus g$.

(ii) All critical points of h are contained in $X \times Y$, i.e. $\text{Sing}(f \oplus g) = \text{Sing}(h)$.

(iii) The boundary $Z \setminus X \times Y$ is the support of a simple normal crossing divisor with pairwise distinct smooth irreducible components $D_1, \ldots, D_s$.

(iv) For every p-tuple $I = (i_1, \ldots, i_p)$ of indices (with $p \geq 1$) the morphism

$$h_I : D_I := D_{i_1} \cap \ldots \cap D_{i_p} \to \mathbf{A}^1_k$$

induced by h is projective and smooth, so that in particular all D_I are smooth quasi-projective k-varieties.

Proof. All references in what follows are to [LS16a] unless otherwise stated.

Condition (K): Lunts and Schnürer define a condition on a pair (U, I) where U is a scheme and $I \subset \mathcal{O}_U$ an ideal sheaf, called condition (K), under which one may associate to (U, I) a monomialisation $c_I(U) \to U$. Since we will change fields, we view (K) as a condition on the triple (U, I, k) where k is the given base field:

(K) U is a reduced scheme of finite type over k, I is not zero on any irreducible component of U, and the closed subscheme $V(I)$ defined by I contains the singular locus U^{sing} of U.

Claim: Let k' be an extension of k and let U be a scheme over k. If $(U_{k'}, I \otimes_k k', k')$ satisfies condition (K), then so does (U, I, k).

Indeed, the fact that U is of finite type and reduced follows from the faithful flatness of $\operatorname{Spec} k' \to \operatorname{Spec} k$. If I is zero on some irreducible component U, then $I \otimes_k k'$ would be zero on any irreducible component of $U_{k'}$ lying over this component. Finally, by statement (a) above, the condition that $V(I)$ contains U^{sing} descends as well.

Monomialisation; The monomialisation procedure recalled in remark 6.3 and used throughout the proof, may be done for any field of characteristic zero, commutes with extension of scalars, as stated in [Kol], 3.34.2, 3.35 and 3.36.

Compactification of one morphism Proposition 6.4 is a first compactification result, which produces, from a smooth quasi-projective variety X and a morphism $f : X \to \mathbf{A}^1$, a smooth projective variety $\overline{X}$ with an open embedding $X \hookrightarrow \overline{X}$ and a morphism $\overline{f} : X \to \mathbf{P}^1_k$

extending f and such that the fibre at infinity is a strict normal crossing divisor. This may be done over a field that is not necessarily algebraically closed, and the construction commutes with change of fields, since it consists in applying monomialisation and a root extraction morphism.

Shifting from addition to projection. The construction of Z and h uses a morphism $\sigma : \mathbf{P}^1 \times \mathbf{A}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$ compactifying the automorphism $\mathbf{A}^1 \times \mathbf{A}^1 \xrightarrow{\sim} \mathbf{A}^1 \times \mathbf{A}^1$ sending (x, y) to $(x, y - x)$, and therefore transforming the addition map $\mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$ into the second projection. The image of σ is $\mathbf{A}^1 \times \mathbf{A}^1 \cup \{(\infty, \infty)\}$, and $\sigma^{-1}(\infty, \infty) = \{\infty\} \times \mathbf{A}^1$ is denoted by E. Starting from compactifications $\overline{X} \xrightarrow{\overline{f}} \mathbf{P}^1$ and $\overline{Y} \xrightarrow{\overline{g}} \mathbf{P}^1$ of the given morphisms, obtained using proposition 6.4, one considers the pullback diagram

$$\begin{array}{ccc} T & \xrightarrow{\widehat{\sigma}} & \overline{X} \times \overline{Y} \\ {\scriptstyle \theta}\downarrow & & \downarrow{\scriptstyle \overline{f} \times \overline{g}} \\ \mathbf{P}^1 \times \mathbf{A}^1 & \xrightarrow{\sigma} & \mathbf{P}^1 \times \mathbf{P}^1 \end{array}$$

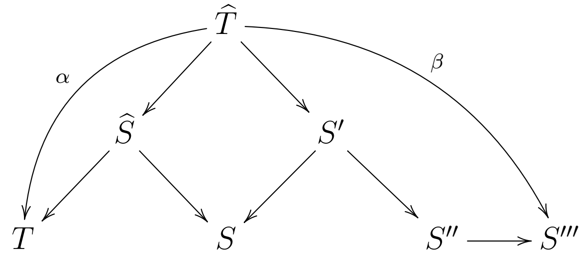
Lunts and Schnürer's proof then goes on with proving that $(T, \theta^{-1}(I_E)\mathcal{O}_T)$ satisfies condition (K), so that one may form the monomialisation $\gamma : Z \to T$, which provides a morphism

$$h : Z \xrightarrow{\gamma} T \xrightarrow{\theta} \mathbf{P}^1 \times \mathbf{A}^1 \xrightarrow{\mathrm{pr}_2} \mathbf{A}^1,$$

which they check satisfies all requested properties.

By what we said on monomialisation above, the construction of h and Z commutes with change of fields. Thus, by Lunts and Schnürer's result, $h_{\overline{k}}$ and $Z_{\overline{k}}$ do the job over $\overline{k}$, and it suffices to show that the properties they satisfy remain true over k. First of all, since $(T, \theta^{-1}(I_E)\mathcal{O}_T)$ satisfies condition (K) over $\overline{k}$, it does also satisfy it over k, so monomialisation may be performed. The fact that h is projective is clear since it is a composition of projective morphisms. Then points (i) and (iii) in the statement of the theorem are satisfied automatically. As for point (ii), Lunts and Schnürer's lemma says that the base change to $\overline{k}$ of the open immersion $\mathrm{Sing}(f \oplus g) \to \mathrm{Sing}(h)$ is in fact an isomorphism. By faithfully flat descent, we already have an isomorphism over k. Finally, it remains to check point (iv), or more precisely, the smoothness part of it, the projectivity being immediate. Again, this comes from faithfully flat descent. □

Remark 2.3.6.2. When reading carefully Lunts and Schnürer's proof, one notices that in fact it can be made to work without the assumption that k is algebraically closed. Indeed, they construct a diagram



where all arrows are étale and where S''' is of the form $\mathbf{A}^1_k \times L$ where

$$L = \operatorname{Spec} k[x_1, \ldots, x_s, y_1, \ldots, y_t]/(x^\mu - y^\nu)$$

with $x^\mu := x_1^{\mu_1} \ldots x_s^{\mu_s}$ and $y^\nu := y_1^{\nu_1} \ldots, y_t^{\nu_t}$ where the μ_i and ν_i are positive integers. This way one essentially reduces to verifying most properties directly for L or for S'''. The only point in the proof when one seems to use the algebraic closedness is when checking that $(L, (x^\mu))$ satisfies condition (K), but as we remarked above, since it is true geometrically it is already true over k.

Remark 2.3.6.3 (Compatibility of nearby cycles morphism with sums of proper morphisms). Let $a \in k$ and let $\mathscr{S}_{\mathrm{id}-a} : \mathscr{M}_{\mathbf{A}^1_k} \to \mathscr{M}_k^{\hat{\mu}}$ be the morphism from theorem 2.2.7.1 for $Y = \mathbf{A}^1$ and $g : \mathbf{A}^1 \to \mathbf{A}^1$ given by $x \mapsto x - a$. Let X and Y be smooth k-varieties and $f : X \to \mathbf{A}^1$, $g : Y \to \mathbf{A}^1$ projective morphisms. Proposition 2.3.6.1 shows that, though $f \oplus g : X \times Y \to \mathbf{A}^1$ is not necessarily proper, we nevertheless have

$$\mathscr{S}_{\mathrm{id}-a}([X \times Y \xrightarrow{f \oplus g} \mathbf{A}^1]) = (f \oplus g)_! \ \psi_{f \oplus g - a}.$$

Indeed, using notation from proposition 2.3.6.1, we may write

$$\mathscr{S}_{\mathrm{id}-a}([X \times Y \xrightarrow{f \oplus g} \mathbf{A}^1]) = \mathscr{S}_{\mathrm{id}-a}([Z \xrightarrow{h} \mathbf{A}^1_k]) + \sum_{I \neq \varnothing} (-1)^{|I|} \mathscr{S}_{\mathrm{id}-a}([D_I \xrightarrow{h_I} \mathbf{A}^1_k]).$$

Since all h_I are projective and smooth, we have

$$\mathscr{S}_{\mathrm{id}-a}[D_I \xrightarrow{h_I} \mathbf{A}^1_k] = (h_I)_! \psi_{h_I - a} = 0,$$

so that in fact, using that h is projective, we have

$$\mathscr{S}_{\mathrm{id}-a}([X \times Y \xrightarrow{f \oplus g} \mathbf{A}^1]) = \mathscr{S}_{\mathrm{id}-a}([Z \xrightarrow{h} \mathbf{A}^1_k]) = h_! \psi_{h-a}.$$

On the other hand, since $\operatorname{Sing}(h) = \operatorname{Sing}(f \oplus g)$, and ψ_{h-a} is supported on $\operatorname{Sing}(h)$, we have $\psi_{h-a} = \psi_{f \oplus g - a}$ (see corollary 3.6 in [LS16b]), whence the result.

2.4 The Thom-Sebastiani theorem: an explicit example

To illustrate the contents of section 2.3.2, we compute in this section both sides of equality (2.7) for $X_1 = X_2 = \mathbf{A}^1_{\mathbf{C}}$ over $k = \mathbf{C}$, with $f_1 = f_2 = (x \mapsto x^2)$.

2.4.1 Computation of left-hand side

Here we are dealing with the variety $X = \mathbf{A}^2$ together with the morphism $f : (x, y) \mapsto x^2 + y^2$. The only critical value is zero, so $\widetilde{\varphi}_f^{\mathrm{tot}}$ is just φ_f, seen as an element of $\mathscr{M}_X^{\hat{\mu}}$. Since in $\mathbf{C}$ we have the decomposition $f(x,y) = (x+iy)(x-iy)$, we may apply theorem 2.2.5.1 with $h = \mathrm{id}$, $E_1 = \{x+iy=0\}$, $E_2 = \{x-iy=0\}$, so that $a_1 = a_2 = 1$, $a_{12} = 1$ (in particular, $\hat{\mu}$-actions are trivial) and

$$\begin{aligned}\varphi_f &= [X_0(f) \xrightarrow{\mathrm{id}} X_0(f)] - [E_1^\circ \to X_0(f)] - [E_2^\circ \to X_0(f)] + (\mathbf{L}-1)[E_1 \cap E_2 \to X_0(f)] \\ &= \mathbf{L}[\{(0,0)\} \to X_0(f)] \in \mathscr{M}_{X_0(f)}^{\hat{\mu}}\end{aligned}$$

since $[X_0(f) \xrightarrow{\mathrm{id}} X_0(f)] = [E_1^\circ \to X_0(f)] + [E_2^\circ \to X_0(f)] + [E_1 \cap E_2 \to X_0(f)]$.

Thus,

$$\widetilde{\varphi}_f^{\,\mathrm{tot}} = \mathbf{L}[\{(0,0)\} \to \mathbf{A}^2] \in \mathscr{M}_{\mathbf{A}^2_{\mathbf{C}}}^{\hat{\mu}}.$$

2.4.2 Computation of the right-hand side

Here we are dealing with $Y = \mathbf{A}^1$ with $g : \mathbf{A}^1 \to \mathbf{A}^1$, $x \mapsto x^2$. Again, the only critical value is zero. We may again apply theorem 2.2.5.1 with $h = \mathrm{id}$, the only irreducible component of $Y_0(g)$ being $E = \{0\}$, with multiplicity $a = 2$, so that we consider a double cover $\tilde{E} \to E$ with μ_2-action. In other words, $\tilde{E}$ is just a pair of two points, exchanged by the action of the generator of μ_2. By the formula, we have

$$\varphi_f = [Y_0(g) \xrightarrow{\mathrm{id}} Y_0(g)] - [\tilde{E} \to Y_0(g)] \in \mathscr{M}_{Y_0(g)}^{\hat{\mu}},$$

so that $\widetilde{\varphi}_g^{\,\mathrm{tot}} = [\{0\} \to \mathbf{A}^1] - [\tilde{E} \to \mathbf{A}^1] \in \mathscr{M}_{\mathbf{A}^1}^{\hat{\mu}}$.

Remark 2.4.2.1. By example 2.7 in [LS16b], whenever we have two k-varieties S_1, S_2 and $p_i : Z_i \to S_i$ is a variety over S_i with $\hat{\mu}$-action, and the action of $\hat{\mu}$ on Z_2 is trivial, then

$$\Psi(Z_1 \times Z_2 \xrightarrow{p_1 \times p_2} S_1 \times S_2) = [Z_1 \times Z_2 \to S_1 \times S_2].$$

This shows that

$$\Psi(\widetilde{\varphi}_g^{\,\mathrm{tot}} \boxtimes \widetilde{\varphi}_g^{\,\mathrm{tot}}) = [\{(0,0)\} \to \mathbf{A}^2] - 2[\{0\} \times \tilde{E} \to \mathbf{A}^2] + \Psi([\tilde{E} \times \tilde{E} \to \mathbf{A}^2]).$$

Note that all the $\mathbf{A}^2$-varieties here are supported above the point $\{(0,0)\}$ of $\mathbf{A}^2$. We are therefore going to stop writing the morphisms to $\mathbf{A}^2$, which implicitly are all taken to be constant equal to $(0,0)$.

Now we are going to compute $\Psi([\tilde{E} \times \tilde{E}])$. By definition, this is

$$\Psi([\tilde{E} \times \tilde{E}]) = [(\tilde{E} \times \tilde{E}) \times^{\mu_2 \times \mu_2} F_0^2] - [(\tilde{E} \times \tilde{E}) \times^{\mu_2 \times \mu_2} F_1^2].$$

Denote the two points of $\tilde{E}$ by e_{-1} and e_1. Then the product $\tilde{E} \times \tilde{E} \times F_i^2$ is simply given by four copies of F_i^2, corresponding to each pair (e_i, e_j) for $i, j \in \{-1, 1\}$. Moreover, these copies are all identified via the $\mu_2 \times \mu_2$-action: indeed, any element $(\epsilon, \eta) \in \mu_2 \times \mu_2$ induces an isomorphism

$$\begin{array}{ccc}\{(e_1, e_1)\} \times F_i^2 & \to & \{(e_\epsilon, e_\eta)\} \times F_i^2 \\ (e_1, e_1, x, y) & \mapsto & (e_\epsilon, e_\eta, \epsilon x, \eta y)\end{array}$$

so that $(\tilde{E} \times \tilde{E}) \times^{\mu_2 \times \mu_2} F_i^2$ is in fact isomorphic to F_i^2, endowed with the diagonal μ_2-action.

Lemma 2.4.2.2. *1. The morphism*

$$\begin{array}{ccc} F_0^2 & \to & \mathbf{G}_m \times \{-1,1\} \\ (x,y) & \mapsto & (x, \frac{y}{ix}) \end{array}$$

is an isomorphism (over $\mathbf{C}$*), identifying* F_0^2 *with the disjoint union of two copies of* $\mathbf{G}_m$*. It is equivariant if one endows each copy of* $\mathbf{G}_m$ *with the obvious* μ_2*-action by translation.*

2. The morphism

$$\begin{array}{ccc} F_1^2 & \to & \mathbf{G}_m \setminus \{-1,1,i,-i\} \\ (x,y) & \mapsto & x+iy \end{array}$$

is an isomorphism (over $\mathbf{C}$*), identifying* F_1^2 *with* $\mathbf{G}_m \setminus \{-1,1,i,-i\}$*. It is equivariant if one endows* $\mathbf{G}_m \setminus \{-1,1,i,-i\}$ *with the action induced by the obvious* μ_2*-action by translation on* $\mathbf{G}_m$*.*

Proof. 1. A point (x,y) of $\mathbf{G}_m^2$ is an element of F_0^2 if and only if either $x = iy$ or $x = -iy$: an inverse to the map in the statement if therefore given by $(x,\epsilon) \mapsto (x, i\epsilon x)$. The statement about actions follows immediately.

2. Rewriting the equation of F_1^2 in the form $(x+iy)(x-iy) = 1$, we see that $x+iy$ is always non-zero, and that if $x+iy$ is equal to some $a \in \mathbf{G}_m$, then $x-iy$ is equal to a^{-1}. This remark allows us to construct an inverse

$$a \mapsto \left(\frac{a+a^{-1}}{2}, \frac{a-a^{-1}}{2i}\right),$$

which is well-defined and with image contained in F_1^2 whenever a is a complex number outside the set $\{0,1,-1,i,-i\}$. Again, the statement on actions is immediate.

□

Combining the results in the lemma, we have

$$\Psi([\tilde{E} \times \tilde{E}]) = 2[\mathbf{G}_m, \mu_2] - [\mathbf{G}_m \setminus \{-1,1,i,-i\}, \mu_2]$$

Denoting by $[\tilde{E}, \mu_2]$ the class of a union of two points exchanged by the generator of μ_2, as above, we have

$$[\mathbf{G}_m \setminus \{-1,1,i,-i\}, \mu_2] = [\mathbf{G}_m, \mu_2] - 2[\tilde{E}, \mu_2],$$

whence

$$\Psi([\tilde{E} \times \tilde{E}]) = [\mathbf{G}_m, \mu_2] + 2[\tilde{E}, \mu_2].$$

Thus, finally, we have, observing that $[\{0\} \times \tilde{E}, \mu_2] = [\tilde{E}, \mu_2]$,

$$\Psi({\widetilde{\varphi}_g}^{\mathrm{tot}} \boxtimes {\widetilde{\varphi}_g}^{\mathrm{tot}}) = 1 + [\mathbf{G}_m, \mu_2] = [\mathbf{A}^1, \mu_2],$$

that is, $\mathbf{A}^1$ with the generator of μ_2 acting through $x \mapsto -x$. By relation (2.2) in the Grothendieck ring $\mathscr{M}_{\mathbf{A}^2_{\mathbf{C}}}^{\hat{\mu}}$, this is equal to the left-hand side $\mathbf{L}$ (i.e. $\mathbf{A}^1$ with the trivial action) computed in section 2.4.1, whence the result.

Remark 2.4.2.3. Our computation shows that in $\mathscr{M}^{\hat{\mu}}_{\mathbf{A}^2_{\mathbf{C}}}$ we have the relation

$$\Psi(\widetilde{\varphi}_{x^2}^{\,\mathrm{tot}} \boxtimes \widetilde{\varphi}_{x^2}^{\,\mathrm{tot}}) = \widetilde{\varphi}_{x^2+y^2}^{\,\mathrm{tot}}$$

which in our calculation boils down to the equality

$$(1 - [\tilde{E}, \mu_2]) * (1 - [\tilde{E}, \mu_2]) = \mathbf{L}$$

in $\mathscr{M}^{\hat{\mu}}_{\mathbf{C}}$. Thus, the class $1 - [\tilde{E}, \mu_2]$ may be seen as a "square root" of $\mathbf{L}$ for the product $*$. For obvious dimensional reasons, such a square root does not exist in the ring $\mathscr{M}_{\mathbf{C}}$.

Chapter 3

Motivic Euler products

The possibility of writing a function as an Euler product, that is, an infinite product of "local factors", is a very important tool in number theory. In particular, the Hasse-Weil zeta function of a variety X over a finite field $\mathbf{F}_q$, defined by

$$\zeta_X(t) = \exp\left(\sum_{m\geq 1} \frac{|X(\mathbf{F}_{q^m})|}{m} t^m\right) \in \mathbf{Z}[[t]],$$

can be rewritten as a product over the closed points X_{cl} of X in the following manner:

$$\zeta_X(t) = \prod_{x\in X_{\mathrm{cl}}} \frac{1}{1-t^{\deg x}},$$

the expansion of which gives

$$\zeta_X(t) = \sum_{n\geq 0} |\{\text{effective zero-cycles of degree } n \text{ on } X\}| t^n. \tag{3.1}$$

The latter expression led Kapranov ([Kapr]) to define a motivic analogue of the Hasse-Weil zeta function: for a variety X over a field k, it is given by

$$Z_X(t) = \sum_{n\geq 0} [S^n X] t^n \ \in \mathrm{KVar}_k[[t]],$$

where S^nX is the n-th symmetric power of X, a variety over k which parametrises precisely effective zero-cycles of degree n on X, and $[S^nX]$ is its class in the Grothendieck ring of varieties KVar_k. When k is a finite field, the series $Z_X(t)$ specialises to $\zeta_X(t)$ via the counting measure

$$\begin{array}{ccc} \mathrm{KVar}_k & \to & \mathbf{Z} \\ [X] & \mapsto & |X(k)| \end{array}$$

as one can see from (3.1). Since then, several mathematicians have been studying the properties of this function and trying to measure the scope of the analogy with the Hasse-Weil zeta function. Kapranov himself showed for example that it was rational for a smooth projective curve having a zero-cycle of degree 1, whereas a result by Larsen and Lunts ([LL]) states that it is not rational for a surface. It is however an open question whether it is rational when regarded as a power series with coefficients in $\mathscr{M}_k = \mathrm{KVar}_k[\mathbf{L}^{-1}]$, the ring obtained from KVar_k by inverting the class of the affine line.

In this chapter we are going to broaden the parallel with the Hasse-Weil zeta function by showing that Kapranov's zeta function can be endowed with an Euler product decomposition. More precisely, we are going to give a way of making sense of expressions of the form

$$\prod_{x\in X}\left(1 + X_{1,x}t + X_{2,x}t^2 + \ldots\right)$$

where X is a variety over k, $(X_i)_{i\geq 1}$ is a family of varieties over X (or, more generally, a family of classes in KVar_X), and $X_{i,x}$ must be thought of as the class of the fibre of X above $x \in X$ in $\mathrm{KVar}_{k(x)}$, where $k(x)$ is the residue field at x. The decomposition of Kapranov's zeta function will in particular be covered by this definition, but so will many other power series, and in particular those occurring when studying motivic height zeta functions in chapter 6.

It is important to point out that our construction was inspired by [GZLM], where a first step towards infinite motivic products was made. Indeed, the authors define a notion of motivic "power", which is a special case of our construction, recovered when all X_i are of the form $X \times_k A_i$ for a family of varieties $(A_i)_{i\geq 1}$ over k, so that all factors are equal to

$$(1 + A_1t + A_2t^2 + \ldots) \in \mathrm{KVar}_k[[t]],$$

and the resulting product can be thought of as this series "raised to the power X".

Let us sketch the contents of this chapter. The overall idea is to generalise the notion of motivic zeta function in an appropriate way, to define the Euler product notation for all these generalised zeta functions, and then to show that this notation actually does behave like a product. Section 3.1 will be devoted to the definition of the coefficients of those zeta functions, which we will call symmetric products. They are a generalisation of the notion of symmetric power. The subsequent sections contain proofs of numerous properties of these products that are necessary to ensure subsequent good behaviour of our Euler products. In particular, in 3.2 we show how to iterate this construction, which will enable us to make sense of double products later. We explain there that the iteration makes it necessary to consider symmetric products of families indexed by more general sets than the set of positive integers. In 3.3 it is shown how the symmetric product of a family of varieties can be expressed in terms of symmetric products of constructible sets partitioning these varieties, which leads to multiplicative properties for zeta functions. Section 3.4 shows how symmetric products behave if the original varieties are multiplied by some affine spaces. Furthermore, we show that these definitions and properties may be extended to families of non-effective classes in a Grothendieck ring in 3.5, and to (classes of) varieties with exponentials in 3.6. Finally, in section 3.7, we define symmetric products of classes in localised Grothendieck rings.

Section 3.8 defines the Euler product notation and deduces all the properties following from the previous sections that show that we can think of it as a product and do calculations with it. Section 3.9 shows, thanks to a further slight generalisation of the notion of symmetric product, that inside the product, one can allow a finite number of constant terms not equal to 1.

Notation 3.0.0.1. We will denote by $\mathbf{N}$ the set of non-negative integers, and by $\mathbf{N}^*$ the set of positive integers. For any set I, we define

$$\mathbf{N}^{(I)} = \{(n_i)_{i\in I} \in \mathbf{N}^I,\ n_i = 0 \text{ for almost all } i\}$$

where "almost all" means "all but a finite number of". It is a monoid, if we endow it with addition coordinate by coordinate:

$$(n_i)_{i\in I} + (n'_i)_{i\in I} = (n_i + n'_i)_{i\in I}.$$

Moreover, it has a natural partial order defined by

$$(n_i)_{i\in I} \leq (n'_i)_{i\in I} \text{ if and only if } n_i \leq n'_i \text{ for all } i \in I.$$

For two elements $\pi = (n_i)_{i\in I}$ and $\pi' = (n'_i)_{i\in I}$ such that $\pi \leq \pi'$, we may also define their difference:

$$\pi' - \pi = (n'_i - n_i)_{i\in I} \in \mathbf{N}^{(I)}.$$

An element $\pi \in \mathbf{N}^{(I)}$ can be thought of as a finite collection of elements of I, each element i coming with a multiplicity n_i. That's why, especially in the case when $I = \mathbf{N}^*$, such an element will sometimes be written in the form $[a_1, \ldots, a_p]$ where $a_1, \ldots, a_p$ are elements of I, each appearing with the correct multiplicity, so that the integer p is equal to $\sum_{i\in I} n_i$. We denote by $|\pi|$ the integer $\sum_{i\in I} n_i$.

The special case $I = \mathbf{N}^*$ will be particularly important: in this case, an element $\pi \in \mathbf{N}^{(I)}$ is called a *partition*. Indeed, since $\pi = (n_i)_{i\geq 1} = [a_1, \ldots, a_p]$ is in this case a collection of positive integers with multiplicities, we may associate to it the number

$$n = \sum_{i\geq 1} i n_i = a_1 + \ldots + a_p$$

these integers sum to, and π is simply a partition of the integer n. The elements a_i are called the *parts* of the partition. Note that when denoting partitions of integers in the form $[a_1, \ldots, a_p]$, some authors require the sequence of the a_i to be non-decreasing. In order to simplify the statements of some results below, we prefer to say that the order of the a_i is of no importance: we consider the partitions $[a_1, \ldots, a_p]$ and $[a_{\sigma(1)}, \ldots, a_{\sigma(p)}]$ to be the same for any permutation $\sigma \in \mathfrak{S}_p$. However, when writing concrete partitions, we will often put the integers in increasing order for clarity.

In this chapter, R will be a variety over a perfect field k.

3.1 Symmetric products

3.1.1 Introduction: Symmetric powers of a variety

Let X be a quasi-projective variety over a perfect field k. For every non-negative integer n, there is a natural action of the symmetric group $\mathfrak{S}_n$ on the product X^n by permuting the coordinates, and it is a classical result that the quotient $S^nX = X^n/\mathfrak{S}_n$ exists as a variety (if X is quasi-projective, which we assumed). We will call this variety the n-th *symmetric power* of X. By convention S^0X will be $\operatorname{Spec} k$.

The rational points of the variety S^nX correspond to effective zero-cycles of degree n on X. Any such zero-cycle $\sum_x n_x x$ determines a partition

$$\sum_x \underbrace{(n_x + \ldots + n_x)}_{\deg x \text{ terms}} = n$$

of n, which we will denote by π. The subset $S^\pi X$ of S^nX consisting of the zero-cycles inducing this partition π is locally closed in S^nX, and can be constructed directly in the following way: for all $i \geq 1$, denote by n_i the number of times the integer i occurs in this partition. Every zero-cycle determining the partition π is of the form

$$\sum_{i\geq 1} i(x_{i,1} + \ldots + x_{i,n_i})$$

where the points $x_{i,j}$ are geometric points of X, all distinct. Consider therefore the product $X^{\sum_{i\geq 1} n_i}$, from which we remove the diagonal Δ, that is, the points having at least two equal coordinates. The product $\prod_{i\geq 1} \mathfrak{S}_{n_i}$ of symmetric groups has a left action on $\prod_{i\geq 1} X^{n_i} = X^{\sum_{i\geq 1} n_i}$ via

$$(\sigma_i)_{i\geq 1} \cdot (x_{i,1}, \ldots x_{i,n_i})_{i\geq 1} \mapsto (x_{i,\sigma_i^{-1}(1)}, \ldots, x_{i,\sigma_i^{-1}(n_i)})_{i\geq 1}$$

for any $\sigma = (\sigma_i)_{i\geq 1} \in \prod_{i\geq 1} \mathfrak{S}_{n_i}$. This action restricts to $X^{\sum_{i\geq 1} n_i} \setminus \Delta$, and the quotient will be naturally isomorphic to the above locally closed subset $S^\pi X$. This observation will be the starting point of the construction in the following paragraph.

The variety S^nX can thus be written as a disjoint union of locally closed sets $S^\pi X$ with π ranging over all partitions of n. In particular, we will denote by S^n_*X the open subset of S^nX corresponding to the partition $[1, \ldots, 1]$ of n, which parametrises étale zero-cycles of degree n on X.

This construction may be done with k replaced by a k-variety R, and products replaced by fibred products over R. The resulting objects will be varieties over R, denoted S^nX and $S^\pi X$ as well, or $S^n(X/R)$ and $S^\pi(X/R)$ if we want to keep track of the base variety. For any point $v \in R$, the fibre of $S^\pi(X/R)$ above v will be isomorphic to $S^\pi(X_v/\kappa(v))$ where X_v is the fibre of X over v.

Remark 3.1.1.1. Though we may define symmetric powers also over non-perfect fields, the above description of points will fail in this case. This justifies our condition on the base variety R.

3.1.2 Quotients of schemes by finite group actions

We gather here some facts about quotients by finite group actions. A detailed account may be found in Chapter 0 of [MFK]. Let X be a scheme endowed with an algebraic action of a finite group G.

Definition 3.1.2.1. A (categorical) quotient of X by G is a morphism of schemes $\pi : X \to Y$ with the following two properties:
— π is G-invariant;
— π is universal with this property: for every scheme Z over k and every G-invariant morphism $f : X \to Z$, there is a unique morphism $h : Y \to Z$ such that $h \circ \pi = f$.

Because of the universality, a quotient is unique up to canonical isomorphism if it exists. In this case, we write $Y = X/G$. Note that the universal property implies in particular that if X is an S-scheme, then so is X/G.

If $X = \operatorname{Spec} A$ is affine, of finite type over S (which may be assumed to be affine, equal to $\operatorname{Spec} C$ for some ring C), then A has a G-action, and we may define the subring A^G of A of all G-invariant elements of A. This induces a morphism

$$\pi : \operatorname{Spec} A \longrightarrow \operatorname{Spec} A^G.$$

One can show that this is the quotient of X by G.

Definition 3.1.2.2. We say that the action of G on X is good if any $x \in X$ has an open affine neighbourhood that is preserved by the G-action.

For example, if X is quasi-projective over a field k, then the action of G is good.

If the action of G on the variety X is good, then taking an affine cover $(U_i)_i$ of X by such affine subsets, one may construct a quotient X/G by glueing together the quotients U_i/G. It follows from [MFK], theorem 1.10, that this quotient is quasi-projective.

Proposition 3.1.2.3 ([Mus], Proposition A.8)**.** *Let G be a finite group acting by algebraic automorphisms on a quasi-projective variety X over k. Let H be a subgroup of G, and Y an open subset of X such that*

1. *Y is preserved by the action of H on X.*
2. *If $Hg_1, \ldots, Hg_r$ are the right equivalence classes of G modulo H, then*

$$X = \bigcup_{i=1}^{r} Yg_i$$

is a disjoint cover.

Then the natural morphism $Y/H \longrightarrow X/G$ is an isomorphism.

3.1.3 Symmetric products of a family of varieties

Let X be a variety over R, and let $\mathscr{X} = (X_i)_{i\geq 1}$ be a family of X-varieties with structural morphisms $\varphi_i : X_i \longrightarrow X$. All products in this section are fibred products over R.

Let $\pi = (n_i)_{i\geq 1}$ be a partition of the integer n. The variety $\prod_i X_i^{n_i}$ has a morphism $\prod_{i\geq 1} \varphi_i^{n_i}$ to $\prod_i X^{n_i}$. Denote by

$$\left(\prod_{i\geq 1} X^{n_i}\right)_*$$

the open subset of the latter obtained by removing the diagonal, that is, the points having at least two equal coordinates. By base change, we get the open subset

$$\prod_i X_i^{n_i} \times_{\prod_i X^{n_i}} \left(\prod_i X^{n_i}\right)_*$$

of elements mapping to $\sum_i n_i$-tuples of X which do not belong to the diagonal. This can be summarised by the following cartesian diagram:

$$\begin{array}{ccc} \left(\prod_{i\geq 1} X_i^{n_i}\right) \times_{\prod_{i\geq 1} X^{n_i}} \left(\prod_{i\geq 1} X^{n_i}\right)_* & \hookrightarrow & \prod_{i\geq 1} X_i^{n_i} \\ \downarrow & & \downarrow {\scriptstyle \prod_{i\geq 1} \varphi_i^{n_i}} \\ \left(\prod_{i\geq 1} X^{n_i}\right)_* & \hookrightarrow & \prod_{i\geq 1} X^{n_i} \end{array}$$

For simplicity, in what follows we will write $\left(\prod_{i\geq 1} X_i^{n_i}\right)_*$ for the variety at the top-left corner of this diagram (when we want to specify that points were removed with respect to coordinates in X, we may write $\left(\prod_{i\geq 1} X^{n_i}\right)_{*,X}$). Now the product $\prod_{i\geq 1} \mathfrak{S}_{n_i}$ of symmetric groups acts naturally on the varieties occurring in the right column of this diagram: each $\mathfrak{S}_{n_i}$ acts on the corresponding $X_i^{n_i}$ and X^{n_i} by permutation of coordinates. It restricts to the varieties in the left column, and is compatible with the vertical maps. Passing to the quotient, the left column gives us a variety which we will denote by $S^\pi(\mathscr{X}/R)$, or simply $S^\pi \mathscr{X}$, with a map to the variety $S^\pi X$ defined in the previous section.

Finally, taking the disjoint union $\cup_\pi S^\pi \mathscr{X}$ over all partitions of n, we get a variety $S^n \mathscr{X}$.

Remark 3.1.3.1. The horizontal inclusion maps of the cartesian square

$$\begin{array}{ccc} \left(\prod_{i\geq 1} X_i^{n_i}\right)_* & \longrightarrow & \prod_{i\geq 1} X_i^{n_i} \\ \downarrow & & \downarrow \\ \left(\prod_{i\geq 1} X^{n_i}\right)_* & \longrightarrow & \prod_{i\geq 1} X^{n_i} \end{array}$$

are compatible with taking the quotient, so we get well-defined maps

$$\begin{array}{ccc} S^\pi \mathscr{X} & \longrightarrow & \prod_{i\geq 1} S^{n_i} X_i \\ \downarrow & & \downarrow \\ S^\pi X & \longrightarrow & \prod_{i\geq 1} S^{n_i} X \end{array}$$

The diagonal is a closed subset Δ_X in $\prod_{i\geq 1} X^{n_i}$. It is stable by the action of $\prod_{i\geq 1} \mathfrak{S}_{n_i}$, and maps to a closed subset $\Delta_{X,\pi}$ inside $\prod_{i\geq 1} S^{n_i} X$ such that $S^\pi X \simeq \left(\prod_{i\geq 1} S^{n_i} X\right) \setminus \Delta_{X,\pi}$, and therefore $S^\pi \mathscr{X}$ is exactly the restriction of $\prod_{i\geq 1} S^{n_i} X_i$ to points mapping outside $\Delta_{X,\pi}$. In other words, the diagonal can be removed before or after passing to the quotient.

Notation 3.1.3.2. If the family $\mathscr{X}$ is constant, that is, all X_i are equal to some X-variety Y, then the resulting symmetric products will be denoted $S^\pi_X(Y)$ (resp. $S^n_X(Y)$). In particular, by definition, we have $S^\pi_X(X) = S^\pi X$.

Example 3.1.3.3. 1. If π is the partition $[1,\ldots,1]$, we are going to write $S^\pi\mathscr{X} = S^n_*\mathscr{X}$. It corresponds to the variety $S^n_{*,X} X_1$ parametrising effective zero-cycles on X_1 of degree n mapping to effective zero-cycles on X of degree n in which no point occurs with multiplicity strictly greater than one.

2. If π is the partition $[n]$, $S^\pi\mathscr{X} = X_n$.
3. If we take all X_i to be equal to some X-variety Y, then $S^\pi\mathscr{X} = S^\pi_X Y$ (see notation 3.1.3.2) corresponds exactly to effective zero-cycles of degree n on Y mapping to zero-cycles on X with partition π. In particular, if all X_i are equal to X, then we get the locally closed subset $S^\pi X$ of $S^n X$ described in section 3.1.1.
4. If $X = \operatorname{Spec} R$, then $\left(\prod_{i\geq 1} X^{n_i}\right)_*$ is empty whenever there is more than one factor, that is, except if $n_i = 0$ for all $i \geq 1$ but one (recall the product is over R). Since the n_i are subject to the relation $\sum_i i n_i = n$, this means that $n_i = 0$ for $i < n$, and $n_n = 1$. Thus, $S^\pi\mathscr{X}$ is empty for all π but $[n]$, and we have $S^n\mathscr{X} = X_n$.

3.2 Iteration of the symmetric product construction

If $\mathscr{X} = (X_i)_{i\geq 1}$ is a family of varieties over X, and X itself is a variety over some scheme R, then the symmetric product construction over R gives rise to a family of varieties

$$S^\bullet(\mathscr{X}/R) = (S^\pi(\mathscr{X}/R))_{\pi\in \mathbf{N}^{(\mathbf{N}^*)}}$$

over R, indexed by all partitions π. As it is now, our definition of symmetric products doesn't allow us to carry on and construct a symmetric product of this family. The aim of this section is to generalise our construction in a way that will make this possible. This generalisation is important in itself, as it gives the correct general setting in which symmetric products may be defined.

The idea is to replace families indexed by the set $\mathbf{N}^*$ of positive integers by families indexed by any set I. Then the family of their symmetric products will be indexed by the set

$$\mathbf{N}^{(I)} = \{(n_i)_{i\in I} \in \mathbf{N}^I,\ n_i = 0 \text{ for almost all } i\},$$

and the family of the symmetric products of those will be indexed by $\mathbf{N}^{\left(\mathbf{N}^{(I)}\right)}$.

3.2.1 Symmetric products of varieties indexed by any set

Let I be a set. The construction is completely analogous to the construction of symmetric products in the case where I is just the set of positive integers. Let X be a quasi-projective variety over R, and $\mathscr{X} = (X_i)_{i\in I}$ a family of quasi-projective X-varieties. Fix $\pi = (n_i)_{i\in I}$. The product

$$\prod_{i\in I} X_i^{n_i}$$

has a morphism to $\prod_{i\in I} X^{n_i}$. We consider the open subset

$$\left(\prod_{i\in I} X_i^{n_i}\right)_* \subset \prod_{i\in I} X_i^{n_i}$$

of points lying above the complement of the diagonal of $\prod_{i\in I} X^{n_i}$, that is mapping to points having pairwise distinct coordinates. We have a natural action of the product of symmetric groups $\prod_{i\in I} \mathfrak{S}_{n_i}$ by permutation of coordinates, and we define

$$S^\pi \mathscr{X} := \left(\prod_{i\in I} X_i^{n_i}\right)_* / \prod_{i\in I} \mathfrak{S}_{n_i},$$

which is a variety because the varieties we started with were quasi-projective. Note that in particular, in the case $\pi = 0$, we get $S^0\mathscr{X} = R$.

Remark 3.2.1.1. The construction is functorial in the sense that if we have two families of X-varieties $\mathscr{X} = (X_i)_{i\in I}$ and $\mathscr{Y} = (Y_i)_{i\in I}$, and if we are given, for every i, a morphism $f_i : X_i \to Y_i$, then the family of morphisms $f = (f_i)_{i\in I}$ induces, for every $\pi \in \mathbf{N}^{(I)}$, a morphism $S^\pi f : S^\pi\mathscr{X} \to S^\pi\mathscr{Y}$.

Example 3.2.1.2. If $X = R$ and $\pi \neq 0$, then as in Example 3.1.3.3, $S^\pi\mathscr{X}$ is empty except if there exists $i_0 \in I$ such that $\pi = (n_i)_{i\in I}$ satisfies $n_i = 0$ for all $i \neq i_0$, and $n_{i_0} = 1$, and in this case $S^\pi\mathscr{X} = X_{i_0}$.

Remark 3.2.1.3 (Case when I is a semigroup). Assume for a moment that I is of the form $I_0 \setminus \{0\}$ where I_0 is a commutative monoid, that is, I_0 is endowed with some associative and commutative composition law with zero-element 0. Then there is a well-defined map

$$\begin{array}{cccc} \lambda : & \mathbf{N}^{(I)} & \to & I_0 \\ & \pi = (n_i)_{i\geq 1} & \mapsto & \sum_{i\in I} i n_i \end{array}$$

and we may also define, for any $n \in I_0$, $S^n\mathscr{X}$ to be the disjoint union of all the $S^\pi\mathscr{X}$ for $\pi \in \lambda^{-1}(n)$.

Example 3.2.1.4. In the particular case where $I = \mathbf{N}^*$ all the X_i are equal to X, by definition, for any integer n, $S^n\mathscr{X}$ is the disjoint union of the locally closed subsets $S^\pi X$ described in section 3.1.1. In particular, we have the equality of classes $[S^n\mathscr{X}] = [S^n X]$ in KVar_R^+. Note that however, the natural scheme structure of the symmetric power $S^n X$ is not the same as the scheme structure of $S^n\mathscr{X}$. This won't have any importance for us because we will be mainly working in Grothendieck (semi)rings.

Example 3.2.1.5. If $X = R$ then by example 3.2.1.2, $S^n\mathscr{X} = X_n$.

The following definition comes as a natural generalisation of Kapranov's zeta function.

Definition 3.2.1.6. Let X be variety over R, $\mathscr{X} = (X_i)_{i\in I}$ a family of varieties over X. Consider also a family $\mathbf{t} = (t_i)_{i\in I}$ of indeterminates and denote by $\mathrm{KVar}_R^+[[\mathbf{t}]]$ the semi-ring of power series in those indeterminates over KVar_R^+. The zeta function associated to $\mathscr{X}$ is the formal power series given by

$$Z_{\mathscr{X}}(\mathbf{t}) = \sum_{\pi\in\mathbf{N}^{(I)}} [S^\pi\mathscr{X}]\mathbf{t}^\pi \quad \in \mathrm{KVar}_R^+[[\mathbf{t}]],$$

where $\mathbf{t}^\pi := \prod_{i\geq 1} t_i^{n_i}$. In particular, if one assumes $I = \mathbf{N}^*$ and specialises the t_i to $t_i = t^i$ for a single variable t, one gets a power series

$$Z_{\mathscr{X}}(t) = \sum_{n\geq 0}[S^n\mathscr{X}]t^n \quad \in \mathrm{KVar}_R^+[[t]].$$

More generally, if we assume $I = \mathbf{N}^p\backslash\{0\}$ for some integer $p \geq 1$, we get a multi-variate variant of the above zeta-function:

$$Z_{\mathscr{X}}(t_1,\ldots,t_p) = \sum_{\mathbf{n}\in\mathbf{N}^p} [S^{\mathbf{n}}\mathscr{X}]t_1^{n_1}\ldots t_p^{n_p} \quad \in \mathrm{KVar}_R^+[[t_1,\ldots,t_p]].$$

where for every $\mathbf{n} = (n_1,\ldots,n_p) \in \mathbf{N}^p$, the variety $S^{\mathbf{n}}\mathscr{X}$ is the disjoint union of the $S^\pi\mathscr{X}$ for all $\pi = (n_i)_{i\in I} \in \mathbf{N}^{(I)}$ such that $\sum_{i\in I} in_i = \mathbf{n}$.

When we want to specify R, we are going to write $Z_{\mathscr{X}/R}$ instead.

Example 3.2.1.7. Taking $X_i = X$ for all $i \geq 1$, and using the fact that by example 3.2.1.4 in this case $[S^n\mathscr{X}] = [S^n X]$, we recover Kapranov's zeta function

$$Z_X(t) = \sum_{n\geq 0}[S^n X]t^n \quad \in \mathrm{KVar}_R^+[[t]].$$

3.2.2 Describing points of symmetric products

Assume that I is a commutative semigroup. To describe the points of these symmetric products, it is convenient to use the term "effective zero-cycle" rather loosely, so that it applies to any finite formal sum of closed (or Galois orbits of geometric) points with coefficients in some semigroup.

Recall each X_i comes with a morphism $\varphi_i : X_i \to X$. Each point of $S^\pi \mathscr{X}$ has an image in $S^\pi X$ which, by construction, can be written as an effective zero-cycle on X of the form

$$\sum_{i\in I} i(v_{i,1} + \ldots + v_{i,n_i}) \in S^\pi X, \tag{3.2}$$

the $v_{i,j}$ being distinct (geometric) points of X. Moreover, for every $i \in I$, the degree i part

$$i(v_{i,1} + \ldots + v_{i,n_i})$$

comes from $(v_{i,1}, \ldots, v_{i,n_i}) \in X^{n_i}$, which in turn by definition comes from a point of $X_i^{n_i}$.

Thus, by analogy with the notation used in (3.2), we will write elements of $S^\pi \mathscr{X}$ as effective zero-cycles on the disjoint union of (a finite number of) the X_i, of the form

$$D' = \sum_{i\in I} i(x_{i,1} + \ldots + x_{i,n_i})$$

such that for all $i \in I$ and for all $j \in \{1, \ldots, n_i\}$, $x_{i,j}$ is a geometric point of X_i, and

$$\sum_{i\in I} i(\varphi_i(x_{i,1}) + \ldots + \varphi_i(x_{i,n_i})) \in S^\pi X,$$

that is, the $\varphi_i(x_{i,j})$ are distinct geometric points of X. One may also view an element of $S^\pi \mathscr{X}$ simply as a collection of effective zero-cycles $(D_i)_{i\in I}$ where for all $i \in I$, $D_i \in S^{n_i} X_i$, the support of the image of D_i in X is composed of n_i distinct geometric points, and the supports of the images of D_i and D_j for $i \neq j$ are disjoint.

If $D \in S^\pi X(\Omega)$ is a geometric point, for some algebraically closed field Ω, then for all $i \in I$ and $1 \leq j \leq n_i$, $v_{i,j} \in X(\Omega)$. Let $\Omega' \supset \Omega$ be an algebraically closed field. Then the Ω'-points of the fibre of $S^\pi X$ above D are exactly those where for all i, j, $x_{i,j} \in X_i(\Omega')$.

For $n \in I$, a geometric point of $S^n X$ is of the form $D = \sum_{v\in X} n_v v$, where the n_v are non-negative integers, almost all zero and such that $\sum_v n_v = n$, and the points v are distinct geometric points of X. The zero-cycle D is an element of $S^\pi X$ if and only if the partition of the integer n defined by the integers $(n_v)_v$ is exactly π. A geometric point of $S^\pi \mathscr{X}$ lying above D will be written in the form $\sum_{v\in X} n_v x_v$, where for every v, x_v is a geometric point of $X_{n_v,v}$. The fibre of $S^\pi \mathscr{X}$ above a geometric point $D \in S^\pi X$ is

$$(S^\pi \mathscr{X})_D = \prod_{v\in D} X_{n_v,v}. \tag{3.3}$$

3.2.3 Symmetric product of a family of symmetric products

Assume we are given a family of varieties $\mathscr{X} = (X_i)_{i\in I}$ over some variety X, which itself is defined over R, and assume R is itself a variety over some k-variety R'. For every $\pi \in \mathbf{N}^{(I)}$, this gives rise to a variety $S^\pi(\mathscr{X}/R)$ over R. We can now consider the family

$$S^\bullet(\mathscr{X}/R) = (S^\pi(\mathscr{X}/R))_{\pi\in\mathbf{N}^{(I)}},$$

of varieties over R and, replacing I by $\mathbf{N}^{(I)}$ in the previous paragraph, do the same construction again. We get a family of varieties indexed by the set

$$\mathbf{N}^{(\mathbf{N}^{(I)})} = \{(m_\pi)_\pi \in \mathbf{N}^{\mathbf{N}^{(I)}}, \quad m_\pi = 0 \text{ for all but finitely many } \pi\}.$$

For $\varpi \in \mathbf{N}^{(\mathbf{N}^{(I)})}$, we have

$$S^\varpi(S^\bullet(\mathscr{X}/R)/R') = \left(\prod_{\pi \in \mathbf{N}^{(I)}} (S^\pi(\mathscr{X}/R))^{m_\pi} \quad _{/R'}\right)_{*,R} / \prod_{\pi \in \mathbf{N}^{(I)}} \mathfrak{S}_{m_\pi},$$

where $/R'$ means the product is over R'.

A natural question arises now: What is the link between this family

$$\left(S^\varpi(S^\bullet(\mathscr{X}/R)/R')\right)_{\varpi \in \mathbf{N}^{(\mathbf{N}^I)}},$$

and the family $(S^\pi(\mathscr{X}/R'))_{\pi \in \mathbf{N}^{(I)}}$ obtained by doing the symmetric product construction for the family $\mathscr{X}$ but seeing X directly as an R'-variety?

3.2.4 Main result

Definition 3.2.4.1. The map $\mu : \mathbf{N}^{(\mathbf{N}^{(I)} \setminus \{0\})} \longrightarrow \mathbf{N}^{(I)}$ is defined to be the map that sends an element $(m_\pi)_{\pi \in \mathbf{N}^{(I)} \setminus \{0\}}$ to

$$\sum_{\pi \in \mathbf{N}^{(I)} \setminus \{0\}} m_\pi \pi \in \mathbf{N}^{(I)}.$$

In terms of the other notation, μ sends an element

$$[[a_{1,1}, \ldots, a_{1,m_1}], \ldots, [a_{r,1}, \ldots, a_{r,m_r}]] \in \mathbf{N}^{(\mathbf{N}^{(I)} \setminus \{0\})}$$

to

$$[a_{1,1}, \ldots, a_{1,m_1}, a_{2,1}, \ldots, a_{r,1}, \ldots, a_{r,m_r}] \in \mathbf{N}^{(I)}.$$

Before stating the main proposition, let us give a motivating example.

Example 3.2.4.2. Let $I = \mathbf{N}^*$ and $\pi = [1, 1, 2] \in \mathbf{N}^{(I)}$, so that

$$\mu^{-1}(\pi) = \{\ [[1], [1], [2]],\ [[1, 1], [2]],\ [[1], [1, 2]],\ [[1, 1, 2]]\ \}.$$

We keep the notation from section 3.2.3. The points of the variety

$$S^\pi(\mathscr{X}/R') = (X_1 \times_{R'} X_1 \times_{R'} X_2)_{*,X} / \mathfrak{S}_2 \times \mathfrak{S}_1 \tag{3.4}$$

are zero-cycles of the form $x + y + 2z$, with x, y, z having distinct images in X, but all mapping to the same $r \in R'$. We therefore may classify them depending on the relative positions of their images in R, which may be encoded by an element of $\mu^{-1}(\pi)$, by adding square brackets to gather integers corresponding to points having the same image in R. There are several cases to consider:

— The points x, y, z all have distinct images in R: this may be encoded by $\varpi_1 = [[1],[1],[2]]$.
— The points x and y have the same image in R, but not z: this corresponds to $\varpi_2 = [[1,1],[2]]$.
— The point z has the same image as one of the points x or y, but not the other: this is represented by $\varpi_3 = [[1],[1,2]]$.
— They all have the same image: this gives $\varpi_4 = [[1,1,2]]$.

Thus, we have a decomposition of $S^{\pi}(\mathscr{X}/R')$ into four locally closed subsets $S^{\pi}_{\varpi_i}(\mathscr{X}/R')$, $i = 1,2,3,4$ corresponding to these four cases. Proposition 3.2.4.3 gives a direct way of constructing varieties isomorphic to these locally closed subsets, in the flavour of what has been done in section 3.1.1, when we gave direct constructions for the locally closed subsets $S^{\pi}X$ of S^nX. For ϖ_2 for example, we may remark that giving an element of $S^{\pi}_{\varpi_2}(\mathscr{X}/R')$ is equivalent to giving a zero-cycle in $S^{[1,1]}(\mathscr{X}/R)$ and a zero-cycle in $S^{[2]}(\mathscr{X}/R)$, and making sure they have distinct images in R but the same image in R', which results in:

$$\left(S^{[1,1]}(\mathscr{X}/R) \times_{R'} S^{[2]}(\mathscr{X}/R)\right)_{*,R} = S^{\varpi_2}(S^{\bullet}(\mathscr{X}/R)/R') \tag{3.5}$$

The right-hand side means that this amounts exactly to applying the symmetric product construction for the element $\varpi_2 \in \mathbf{N}^{(N^{(I)}\setminus\{0\})}$, and the family $S^{\bullet}(\mathscr{X}/R)$ above the R'-variety R.

Proposition 3.2.4.3. *Let R' be a variety over k, R a variety over R', X a variety over R, and let $\mathscr{X} = (X_i)_{i\in I}$ be a family of varieties over X, indexed by a set I. Then for every $\pi \in \mathbf{N}^{(I)}$ and for every $\varpi \in \mu^{-1}(\pi)$, there is a piecewise isomorphism of the variety $S^{\varpi}(S^{\bullet}(\mathscr{X}/R)/R')$ onto a locally closed subset $S^{\pi}_{\varpi}(\mathscr{X}/R')$ of $S^{\pi}(\mathscr{X}/R')$, so that moreover $S^{\pi}(\mathscr{X}/R')$ is equal to the disjoint union of the sets $S^{\pi}_{\varpi}(\mathscr{X}/R')$. In particular, we have the equality*

$$\sum_{\varpi\in\mu^{-1}(\pi)} [S^{\varpi}(S^{\bullet}(\mathscr{X}/R)/R')] = [S^{\pi}(\mathscr{X}/R')]$$

in $\mathrm{KVar}^{+}_{R'}$.

3.2.5 Proof of proposition 3.2.4.3

To prove proposition 3.2.4.3, by a spreading-out argument, we may assume $R' = k$ is a field. We write $J = \mathbf{N}^{(I)}\setminus\{0\}$, and for any $j \in J$, $\pi_j = (n_i^j)_{i\in I} \in \mathbf{N}^{(I)}$, so that every element ϖ of $\mathbf{N}^{(J)}$ may be given as a family of multiplicities $(m_j)_{j\in J}$.

Put $\pi = (n_i)_{i\in I}$, and fix $\varpi \in \mu^{-1}(\pi)$. The condition that $\pi = \mu(\varpi)$ is equivalent to

$$n_i = \sum_{j\in J} m_j n_i^j$$

for all $i \in I$. Our proof decomposes in several steps. Both $S^{\varpi}(S^{\bullet}(\mathscr{X}/R))$ and $S^{\pi}(\mathscr{X})$ are constructed as some quotient of some product of the X_i. We are going to refrain from taking quotients first, and construct an immersion from the product giving the former to the product giving the latter.

The immersion before quotients For any two varieties V and W over R, there is an immersion $V \times_R W \hookrightarrow V \times W$. Thus, for every $j \in J$ there is an immersion

$$\prod_{i\in I} X_i^{n_i^j}{}_{/R} \hookrightarrow \prod_{i\in I} X_i^{n_i^j}.$$

Restricting to the complement of the diagonal in $\prod_{i\in I} X^{n_i^j}$, we get an immersion

$$\left(\prod_{i\in I} X_i^{n_i^j}{}_{/R}\right)_{*,X} \hookrightarrow \left(\prod_{i\in I} X_i^{n_i^j}\right)_{*,X}.$$

Taking the product over all $j \in J$ of the m_j-th powers of those varieties gives:

$$\prod_{j\in J}\left(\left(\prod_{i\in I} X_i^{n_i^j}{}_{/R}\right)_{*,X}\right)^{m_j} \hookrightarrow \prod_{j\in J}\left(\left(\prod_{i\in I} X_i^{n_i^j}\right)_{*,X}\right)^{m_j}.$$

For any two varieties V and W over X, we have the commutative diagram

$$\begin{array}{ccccc}
(V\times W)_{*,R} & \hookrightarrow & (V\times W)_{*,X} & \hookrightarrow & V\times W \\
\downarrow & & \downarrow & & \downarrow \\
(X\times X)_{*,R} & \hookrightarrow & (X\times X)_{*,X} & \hookrightarrow & X\times X \\
\downarrow & & & & \downarrow \\
(R\times R)_{*,R} & & \hookrightarrow & & R\times R
\end{array}$$

where the horizontal arrows are all open immersions: indeed, recall that we assumed R and X to be quasi-projective over k, and therefore separated, so that complements of diagonals are open (and therefore so are their inverse images by the structural morphisms). In particular, we have an open immersion $(V\times W)_{*,R} \to (V\times W)_{*,X}$. Thus, we can restrict to the complement of the diagonal of $\prod_{j\in J} R^{m_j}$ on the left, and to the complement of the diagonal of $\prod_{j\in J}\left(\prod_{i\in I} X^{n_i^j}\right)^{m_j}$ on the right, to get

$$\left(\prod_{j\in J}\left(\left(\prod_{i\in I} X_i^{n_i^j}{}_{/R}\right)_{*,X}\right)^{m_j}\right)_{*,R} \hookrightarrow \left(\prod_{j\in J}\left(\prod_{i\in I} X_i^{n_i^j}\right)^{m_j}\right)_{*,X}. \tag{3.6}$$

Note that using the assumption $\mu(\varpi) = \pi$, we may write

$$\left(\prod_{j\in J}\left(\prod_{i\in I} X_i^{n_i^j}\right)^{m_j}\right)_{*,X} = \left(\prod_{i\in I} X_i^{\sum_{j\in J} n_i^j m_j}\right)_{*,X} = \left(\prod_{i\in I} X_i^{n_i}\right)_{*,X}. \tag{3.7}$$

Composing (3.6) with this identification, we get an immersion

$$\left(\prod_{j\in J}\left(\left(\prod_{i\in I} X_i^{n_i^j}{}_{/R}\right)_{*,X}\right)^{m_j}\right)_{*,R} \hookrightarrow \left(\prod_{i\in I} X_i^{n_i}\right)_{*,X}. \tag{3.8}$$

Example 3.2.5.1. In example 3.2.4.2, the immersion (3.8) corresponding to ϖ_2 is written in the form

$$((X_1 \times_R X_1)_{*,X} \times X_2)_{*,R} \hookrightarrow (X_1 \times X_1 \times X_2)_{*,X},$$

where the variety on the left-hand side (resp. right-hand side) is exactly the one in (3.5) (resp. (3.4)), just without the permutation action quotients. (Recall we took $R' = k$.)

Description of the permutation actions Let V be the variety on the left-hand side, as well as its image through this morphism, and W the variety on the right-hand side. There is a natural action of $G = \prod_{i\in I} \mathfrak{S}_{n_i}$ on W. As for V, we can distinguish two groups acting on it. The first one is

$$\prod_{j\in J}\left(\prod_{i\in I}\mathfrak{S}_{n_i^j}\right)^{m_j}$$

which comes from the natural permutation action of $\mathfrak{S}_{n_i^j}$ on each product $\prod_{i\in I} X_i^{n_i^j}{}_{/R}$ for all $i \in I$ and all $j \in J$. Composing the morphisms $X_i \to X$ with the morphism $X \to R$, we get a map

$$\varphi : \left(\prod_{j\in J}\left(\left(\prod_{i\in I} X_i^{n_i^j}{}_{/R}\right)_{*,X}\right)^{m_j}\right)_{*,R} \longrightarrow \left(\prod_{j\in J} R^{m_j}\right)_{*} \tag{3.9}$$

the fibres of which are stable with respect to that action. On the other hand, there is also a permutation action of $\prod_{j\in J}\mathfrak{S}_{m_j}$ on $\left(\prod_{j\in J} R^{m_j}\right)_*$, which pulls back to an action on the variety on the left-hand side in the following manner: for $x \in V$, denoting for every $j \in J$ and every $\ell \in \{1,\ldots,m_j\}$ by $x_{j,\ell}$ the projection of x on the ℓ-th copy of $\left(\prod_{i\in I} X_i^{n_i^j}{}_{/R}\right)_{*,X}$ occurring in V, the element $\sigma = (\sigma_j)_{j\in J} \in \prod_{j\in J}\mathfrak{S}_{m_j}$ acts on $x = (x_{j,1},\ldots,x_{j,m_j})_{j\in I}$ via

$$\sigma \cdot \left((x_{j,1},\ldots,x_{j,m_j})_{j\in I}\right) = \left(\left(x_{j,\sigma_j^{-1}(1)},\ldots,x_{j,\sigma_j^{-1}(m_j)}\right)_{j\in J}\right).$$

Through immersion (3.8), these two actions give us two subgroups H_1 and H_2 of $G = \prod_{i\in I}\mathfrak{S}_{n_i}$.

Example 3.2.5.2.

1. In example 3.2.4.2, we have $W = (X_1^2 \times X_2)_{*,X}$. Let us examine the subgroups of $G := \mathfrak{S}_2 \times \mathfrak{S}_1$ corresponding to the different ϖ_i occurring in that example.
   — For $\varpi_1 = [[1],[1],[2]]$, we have $V = (X_1^2 \times X_2)_{*,R}$, $H_1 = \{1\}$ and $H_2 = G$.
   — For $\varpi_2 = [[1,1],[2]]$, we have $V = ((X_1 \times_R X_1)_{*,X} \times X_2)_{*,R}$, $H_1 = G$ and $H_2 = \{1\}$.
   — For $\varpi_3 = [[1,2],[1]]$, we have $V = ((X_1 \times_R X_2)_{*,X} \times X_1)_{*,R}$, and $H_1 = H_2 = \{1\}$.
   — For $\varpi_4 = [[1,1,2]]$, we have $V = ((X_1 \times_R X_1 \times_R X_2)_{*,X}$, $H_1 = G$ and $H_2 = \{1\}$.
2. Let us examine another example: $\pi = [1,1,1,1,1,1]$, $\varpi = [[1,1],[1,1],[1],[1]]$. Then $W = (X_1^6)_{*,X}$, $G = \mathfrak{S}_6$, and

$$V = ((X_1 \times_R X_1)_{*,X} \times (X_1 \times_R X_1)_{*,X} \times X_1 \times X_1)_{*,R},$$

so that H_1 is the subgroup of G generated by the permutations (12) and (34), whereas H_2 is generated by the permutations (13)(24) and (56).

Lemma 3.2.5.3. *The subgroup H_1 is normalised by H_2. The subgroup $H := H_1 H_2$ they generate is the largest subgroup of $\prod_{i\in I} \mathfrak{S}_{n_i}$ under the action of which V is invariant inside W.*

Proof. For all $\sigma \in H_1$ and $\tau \in H_2$, the element $\tau\sigma\tau^{-1} \in G$ stabilises the fibres of the morphism φ in (3.9), which means that it stabilises each factor $\prod_i X_i^{n_i^j}{}_{/R}$ of V. Thus, $\tau\sigma\tau^{-1}$ is an element of H_1.

Now, let $\sigma \in \prod_{i\in I} \mathfrak{S}_{n_i}$ be such that for all $x \in V$, $\sigma x \in V$. Let $x \in V$, and let $\tau \in H_2$ be the element such that $\tau(\varphi(x)) = \varphi(\sigma(x))$. Then $\sigma\tau^{-1}$ stabilises the fibres of the map φ in (3.9). This means that its action stabilises each factor $\prod_i X_i^{n_i^j}{}_{/R}$ of V, so $\sigma\tau^{-1}$ is an element of H_1. $\square$

Our aim now is to describe a locally closed subset $W(\varpi)$ of W containing V and stable under the natural $\prod_{i\in I} \mathfrak{S}_{n_i}$-action on W, and show that we can apply Proposition 3.1.2.3 to V and $W(\varpi)$ to get an isomorphism

$$V/H \simeq W(\varpi)/\prod_{i\in I} \mathfrak{S}_{n_i}$$

where the variety on the right-hand side will be called $S^{\pi}_{\varpi}(\mathscr{X})$.

Equivalence relations on coordinates of points of W Recall that W is the variety

$$\left(\prod_{i\in I} X_i^{n_i}\right)_{*,X}.$$

A point of this variety is of the form $x = (x_{i,p})_{\substack{i\in I\\ 1\leq p\leq n_i}}$ where for all $i \in I$ and for all $1 \leq p \leq n_i$, $x_{i,p} \in X_i$ and all coordinates $x_{i,p}$ have distinct images in X. Consider an equivalence relation ρ on the set of indices $\{(i,p)\}_{\substack{i\in I\\ 1\leq p\leq n_i}}$. Each equivalence class E is a subset of the latter, which we write in the form $E = \bigcup_{i\in I} E_i$ (disjoint union) where

$$E_i = E \cap X_i = \{(i,\alpha_{i,1}),\ldots,(i,\alpha_{i,\ell_i})\}$$

for some integers $(\ell_i)_{i\in I}$ (with $\ell_i \leq n_i$ for all i), and

$$1 \leq \alpha_{i,1} < \ldots < \alpha_{i,\ell_i} \leq n_i \quad \text{for all} \quad i \in I.$$

Note that the equivalence classes E of ρ form a partition of the set of indices of the coordinates of the point x, and that therefore for every $i \in I$, the sets $E_i = E \cap X_i$ form a partition of the set $\{(i,1),\ldots,(i,n_i)\}$. Thus, the sum of the ℓ_i over all equivalence classes E is equal to n_i.

To each such non-empty E we can associate the non-zero element $\pi(E) = (\ell_i)_{i\in I} \in \mathbf{N}^{(I)}$. The collection of all $\pi(E)$ for all equivalence classes E of ρ, counted with multiplicities, then gives an element $\varpi(\rho) \in \mathbf{N}^{(\mathbf{N}^{(I)}\setminus\{0\})}$ such that $\mu(\varpi(\rho)) = \pi$, since the sum of the ℓ_i over all equivalence classes E is n_i.

Definition of $W(\varpi)$ For every $x \in W$, we define an equivalence relation ρ_x on the set

$$\{(i,p)\}_{\substack{i\in I\\ 1\le p\le n_i}}$$

by: $(i,p) \sim (i',p')$ if and only if the coordinates $x_{i,p}$ and $x_{i',p'}$ have the same image in R.

Definition 3.2.5.4. For every equivalence relation ρ on $\{(i,p)\}_{\substack{i\in I\\ 1\le p\le n_i}}$ occurring in this way, define the locally closed subsets

$$W_\rho = \{x \in W,\ \rho_x = \rho\} \subset W$$

and

$$W(\varpi) = \bigcup_{\varpi(\rho)=\varpi} W_\rho \subset W,$$

(this is a finite and disjoint union).

Example 3.2.5.5. Let $I = \mathbf{N}^*$ and $\pi = [1,1,2]$, so that $W = (X_1^2 \times X_2)_{*,X}$. An element of W will be written $(x_{1,1}, x_{1,2}, x_{2,1})$. Let ρ be the equivalence relation on $\{(1,1),(1,2),(2,1)\}$ with equivalence classes $\{(1,1),(2,1)\}$ and $\{(1,2)\}$, giving rise respectively to the partitions $[1,2]$ and $[1]$, so that $\varpi := \varpi(\rho) = [[1,2],[1]]$. The only other equivalence relation giving this element of $\mu^{-1}(\pi)$ is the one with equivalence classes $\{(1,2),(2,1)\}$ and $\{(1,1)\}$, denoted by ρ'. Thus W_ρ is the locally closed subset of triples $(x_{1,1}, x_{1,2}, x_{2,1})$ such that in R, $x_{2,1}$ becomes equal to $x_{1,1}$ but not to $x_{1,2}$. In the same way, $W_{\rho'}$ corresponds to triples such that $x_{2,1}$ becomes equal to $x_{1,2}$ but not to $x_{1,1}$. Finally, $W(\varpi)$ is the union of these two sets, namely the set of triples such that in R, $x_{2,1}$ becomes equal either to $x_{1,1}$ or to $x_{1,2}$, where the "or" is exclusive.

Taking quotients

Lemma 3.2.5.6. *(a) $W(\varpi)$ is stable under the action of $\prod_{i\in I} \mathfrak{S}_{n_i}$ on W.*
(b) The group $\prod_{i\in I} \mathfrak{S}_{n_i}$ acts transitively on the set of the W_ρ with $\varpi(\rho) = \varpi$.

Proof. a) Let $\sigma = (\sigma_i)_{i\in I} \in \prod_{i\in I} \mathfrak{S}_{n_i}$, ρ some equivalence relation and $x \in W_\rho$. For every equivalence class E, the equivalence class $\sigma E = \bigcup_{i\in I} \sigma_i(E_i)$ gives the same numbers ℓ_i: therefore $\varpi(\sigma\rho) = \varpi$.

b) Let ρ and ρ' be two different equivalence relations such that $\varpi(\rho) = \varpi(\rho')$. Since they give rise to the same ϖ, they have the same number of equivalence classes, and moreover, to each equivalence class E of ρ we may associate an equivalence class E' of ρ' such that for every $i \in I$, we have

$$\#E_i = \#E'_i.$$

Denoting by ℓ_i this common value, write

$$E_i = \{(i,\alpha_{i,1}),\dots,(i,\alpha_{i,\ell_i})\} \quad \text{and} \quad E'_i = \{(i,\beta_{i,1}),\dots,(i,\beta_{i,\ell_i})\}$$

for all $i \in I$. Define the restriction of the element $\sigma = \prod_{i\in I} \sigma_i \in \prod_{i\in I} \mathfrak{S}_{n_i}$ to

$$\prod_{i\in I}\{\alpha_{i,1},\ldots,\alpha_{i,\ell_i}\}$$

by $\sigma_i(\alpha_{i,p}) = \beta_{i,p}$. Since the equivalence classes of ρ form a partition of $\{(i,p)\}_{\substack{i\in I\\ 1\leq p\leq n_i}}$, doing this for all equivalence classes completely defines an element $\sigma \in \prod_{i\in I}\mathfrak{S}_{n_i}$ such that for every $x \in W_\rho$, $\sigma x \in W_{\rho'}$.

□

Definition 3.2.5.7. For every $\varpi \in \mu^{-1}(\pi)$ we define $S^\pi_\varpi(\mathscr{X})$ to be the locally closed subset of $S^\pi\mathscr{X}$ given by taking the quotient of $W(\varpi) \subset W$ by $\prod_{i\in I}\mathfrak{S}_{n_i}$.

Lemma 3.2.5.8. *There is an equivalence relation ρ such that the image V of the immersion in (3.8) is equal to W_ρ.*

Proof. Fix $x = (x_{i,p})_{\substack{i\in I\\ 1\leq p\leq n_i}} \in W$ in the image of the morphism in (3.8), and for all $i \in I$, $j \in J$ and $1 \leq q \leq m_j$, denote by $E_{i,j,q}$ the set of indices of the coordinates of the projection of x to the q-th factor $X_i^{n_i^j}$. By definition, all coordinates $x_{i,p}$ of x with index $(i,p) \in E_{j,q} := \bigcup_{i\in I} E_{i,j,q}$ have the same image $r_{j,q}(x) \in R$, and the elements $r_{j,q}(x)$ for all $j \in J$ and $1 \leq q \leq m_j$ are distinct. Therefore, the sets $E_{j,q}$ in fact don't depend on x, so that the elements of V are exactly those subject to the equivalence relation ρ with classes $(E_{j,q})_{\substack{j\in J\\ 1\leq q\leq m_j}}$. □

Putting everything together and applying Proposition 3.1.2.3 to $V \subset W$ with actions of the groups $H \subset \prod_{i\in I}\mathfrak{S}_{n_i}$, we get the result, since $S^\varpi(S^\bullet(\mathscr{X}/R))$ is by definition equal to V/H.

3.3 Cutting into pieces

3.3.1 Introduction

The aim of this section is to state and prove an analogue in our setting of the following classical result about symmetric powers (see for example [CNS], chapter 6, proposition 1.1.7):

Proposition 3.3.1.1. *Let X be a quasi-projective variety over a field k, and Y a closed subvariety of X with open complement U. For any integers $n \geq 1$ and $r \in \{0,\ldots,n\}$, the variety*

$$S^rU \times S^{n-r}Y$$

can be identified with the locally closed subset of S^nX corresponding to effective zero-cycles of degree n the restriction of which to U has degree r. Moreover, S^nX is the disjoint union of these locally closed subsets. In particular, in terms of classes in KVar_k^+, we have

$$[S^nX] = \sum_{r=0}^{n}[S^rU][S^{n-r}Y] \in \mathrm{KVar}_k^+.$$

Thus, starting with a zero-cycle D of degree n in X, we get through restriction a pair (D_U, D_Y) of zero-cycles in $S^r U \times S^{n-r} Y$ for some r. Conversely, starting with a point of the latter variety, the sum of the two components yields an element in $S^n X$. More precisely, by the same argument we even have the following:

Proposition 3.3.1.2. *(Refinement of Proposition 3.3.1.1) Let k, X, U, Y be as in Proposition 3.3.1.1, and let $\pi \in \mathbf{N}^{(\mathbf{N}^*)}$ be a partition. Then $S^\pi X$ is the disjoint union of locally closed subsets isomorphic to $S^{\pi'} U \times S^{\pi-\pi'} Y$ where π' runs through all partitions such that $\pi' \leq \pi$. In particular, in terms of classes in* KVar_k^+*, we have*

$$[S^\pi X] = \sum_{\pi' \leq \pi} [S^{\pi'} U][S^{\pi-\pi'} Y] \in \mathrm{KVar}_k^+.$$

To motivate the construction in the following paragraph, let us examine what exactly we need to get a result of this flavour for symmetric products. Fix a set I, and let X be a variety over k, and $\mathscr{X} = (X_i)_{i \in I}$ a family of varieties over X. For all $i \in I$, let Y_i be a closed subvariety of X_i, and U_i its complement, so that we get families of X-varieties $\mathscr{Y} = (Y_i)_{i \in I}$ and $\mathscr{U} = (U_i)_{i \in I}$.

Let $\pi = (n_i)_{i \in I} \in \mathbf{N}^{(I)}$. Any point $D \in S^\pi \mathscr{X}$ is a zero-cycle contained in the disjoint union of (a finite number of) the X_i, and we can consider its restriction $D_{\mathscr{U}}$ to $\cup_{i \in I} U_i$. As in the discussion above, this clearly gives us a point in $S^{\pi'} \mathscr{U}$ for some $\pi' \leq \pi$, and the restriction $D_{\mathscr{Y}}$ to the elements of the family $\mathscr{Y}$ will then be a point in $S^{\pi-\pi'} \mathscr{Y}$. In other words, there is a well-defined immersion

$$\alpha : S^\pi \mathscr{X} \longrightarrow \bigcup_{\pi' \leq \pi} S^{\pi'} \mathscr{U} \times S^{\pi-\pi'} \mathscr{Y}.$$

On the other hand, this morphism α will in general not be an isomorphism. Indeed, the inverse mapping $(D_1, D_2) \mapsto D_1 + D_2$ that worked in the above cases is well-defined only when D_1 and D_2 have disjoint supports, since, by definition, points of $S^\pi \mathscr{X}$ are zero-cycles with supports mapping injectively to $S^\pi X$. Thus the image of α is the subset of $\bigcup_{\pi' \leq \pi} S^{\pi'} \mathscr{U} \times S^{\pi-\pi'} \mathscr{Y}$ mapping to pairs in $\bigcup_{\pi' \leq \pi} S^{\pi'} X \times S^{\pi-\pi'} X$ with disjoint supports. Thus, to generalise proposition 3.3.1.2, we are going to define more general symmetric products which are *mixed*, in the sense that we will combine different families of varieties before restricting to the complement of the diagonal in the base variety. In the case studied above, this construction will give us varieties $S^{\pi', \pi-\pi'}(\mathscr{U}, \mathscr{Y})$, the union of which for all $\pi' \leq \pi$ corresponds exactly to the image of α, so that the analogue of proposition 3.3.1.2 for symmetric products will take the form

$$[S^\pi \mathscr{X}] = \sum_{\pi' \leq \pi} [S^{\pi', \pi-\pi'}(\mathscr{U}, \mathscr{Y})]$$

in KVar_k^+.

3.3.2 Mixed symmetric products

Let X be a variety over R, $p \geq 1$ an integer, and $\mathscr{X}_1, \ldots, \mathscr{X}_p$ families of varieties over X, all indexed by the same set I. For all $j \in \{1, \ldots, p\}$ we write $\mathscr{X}_j = (X_{i,j})_{i \in I}$. We also fix for every $j \in \{1, \ldots, p\}$ an almost zero family of non-negative integers $\pi_j = (r_{i,j})_{i \in I}$ (that is, all $r_{i,j}$ but a finite number are zero). The product (over R)

$$\prod_{i \in I} X_{i,1}^{r_{i,1}} \times \ldots \times X_{i,p}^{r_{i,p}}$$

is a variety over

$$\prod_{i \in I} X^{r_{i,1}} \times \ldots \times X^{r_{i,p}}.$$

As before, we restrict it to the open subset $\left(\prod_{i \in I} X_{i,1}^{r_{i,1}} \times \ldots \times X_{i,p}^{r_{i,p}}\right)_*$ lying above the complement of the diagonal of $\prod_{i \in I} X^{r_{i,1}} \times \ldots \times X^{r_{i,p}} = X^{\sum_{i,j} r_{i,j}}$, that is, we remove points mapping to points having at least two equal coordinates. Then we take the quotient by the natural permutation action of $\prod_{i \in I} \mathfrak{S}_{r_{i,1}} \times \ldots \times \mathfrak{S}_{r_{i,j}}$. The resulting variety will be denoted by

$$S^{\pi_1, \ldots, \pi_p}(\mathscr{X}_1, \ldots, \mathscr{X}_p).$$

If for every $j \in \{1, \ldots, p\}$ all varieties $X_{i,j}$ are equal to some X_j, we may write this simply $S^{\pi_1, \ldots, \pi_p}(X_1, \ldots, X_p)$.

As it was the case in the construction of simple symmetric products, we could also first take the quotient, and then remove the closed set lying above the image of the diagonal by the quotient map.

The following properties are obvious consequences of the definition:

Fact 3.3.2.1.

1. In the case $p = 1$, we recover exactly the symmetric product $S^\pi \mathscr{X}$ from section 3.2.1.
2. For all $\sigma \in \mathfrak{S}_p$, there is an isomorphism
$$S^{\pi_1, \ldots, \pi_p}(\mathscr{X}_1, \ldots, \mathscr{X}_p) \simeq S^{\pi_{\sigma(1)}, \ldots, \pi_{\sigma(p)}}(\mathscr{X}_{\sigma(1)}, \ldots, \mathscr{X}_{\sigma(p)}).$$
3. If $\pi_p = 0$, then
$$S^{\pi_1, \ldots, \pi_p}(\mathscr{X}_1, \ldots, \mathscr{X}_p) = S^{\pi_1, \ldots, \pi_{p-1}}(\mathscr{X}_1, \ldots, \mathscr{X}_{p-1}).$$
4. If there exists $i \in I$ such that $r_{i,p} > 0$ and $X_{i,p} = \emptyset$, then
$$S^{\pi_1, \ldots, \pi_p}(\mathscr{X}_1, \ldots, \mathscr{X}_p) = \emptyset.$$

Remark 3.3.2.2. The open set $\left(\prod_{j=1}^p X^{\sum_{i \in I} r_{i,j}}\right)_*$ is a subset of $\prod_{j=1}^p \left(X^{\sum_{i \in I} r_{i,j}}\right)_*$ consisting of points with projection to $X^{\sum_{i \in I} r_{i,j}}$ having distinct coordinates for all j. Thus, there is an open immersion

$$S^{\pi_1, \ldots, \pi_p}(\mathscr{X}_1, \ldots, \mathscr{X}_p) \longrightarrow S^{\pi_1} \mathscr{X}_1 \times \ldots \times S^{\pi_p} \mathscr{X}_p. \tag{3.10}$$

It is an isomorphism if there is a partition of X into locally closed subsets $V_1, \ldots, V_p$ such that for all $i \in I$ and all $j \in \{1, \ldots, p\}$, the image of $X_{i,j}$ in X is contained in V_i. Indeed, then all elements in any p-tuple $(D_1, \ldots, D_p) \in S^{\pi_1}\mathscr{X}_1 \times \ldots \times S^{\pi_p}\mathscr{X}_p$ have disjoint supports, and the inverse mapping is given by $(D_1, \ldots, D_p) \mapsto D_1 + \ldots + D_p$.

We can now state our generalisation of Propositions 3.3.1.1 and 3.3.1.2.

Proposition 3.3.2.3. *Let $X, \mathscr{X}_1, \ldots, \mathscr{X}_p$ be as above, and consider moreover another family $\mathscr{X} = (X_i)_{i\in I}$ of quasi-projective varieties over X, together with families $\mathscr{Y} = (Y_i)_{i\in I}$ and $\mathscr{U} = (U_i)_{i\in I}$ such that for every $i \in I$, Y_i is a closed subvariety of X_i and U_i its complement. Let $\pi_1, \ldots, \pi_p, \pi$ be elements of $\mathbf{N}^{(I)}$. Then for every $\pi' \leq \pi$ the variety*

$$S^{\pi_1,\ldots,\pi_p,\pi',\pi-\pi'}(\mathscr{X}_1, \ldots, \mathscr{X}_p, \mathscr{U}, \mathscr{Y})$$

is isomorphic to the locally closed subset of $S^{\pi_1,\ldots,\pi_p,\pi}(\mathscr{X}_1, \ldots, \mathscr{X}_p, \mathscr{X})$ corresponding to zero-cycles with $\mathscr{X}$-component inducing partition π' on $\mathscr{U}$. Moreover, the variety

$$S^{\pi_1,\ldots,\pi_p,\pi}(\mathscr{X}_1, \ldots, \mathscr{X}_p, \mathscr{X})$$

is the disjoint union of these locally closed subsets, so that in terms of classes in KVar_R^+, *we have*

$$[S^{\pi_1,\ldots,\pi_p,\pi}(\mathscr{X}_1, \ldots, \mathscr{X}_p, \mathscr{X})] = \sum_{\pi' \leq \pi} \left[S^{\pi_1,\ldots,\pi_p,\pi',\pi-\pi'}(\mathscr{X}_1, \ldots, \mathscr{X}_p, \mathscr{U}, \mathscr{Y})\right].$$

Proof. We write $\pi = (n_i)_{i\in I}$, $\pi' = (m_i)_{i\in I}$ and for every $j \in \{1, \ldots, p\}$, $\pi_j = (r_{i,j})_{i\in I}$ According to proposition 3.3.1.1, the variety $\prod_{i\in I} S^{r_{i,1}}X_{i,1} \times \ldots \times S^{r_{i,p}}X_{i,p} \times S^{n_i}X_i$ is the union of locally closed subsets isomorphic to

$$\prod_{i\in I} S^{r_{i,1}}X_{i,1} \times \ldots \times S^{r_{i,p}}X_{i,p} \times S^{m_i}U_i \times S^{n_i-m_i}Y_i$$

for all $\pi \leq \pi_p$. Restricting to the images of points lying above the complements of the diagonal, we get the result. □

3.3.3 Applications

As an immediate consequence of Proposition 3.3.2.3 (for $p = 0$) and Remark 3.3.2.2, we get

Corollary 3.3.3.1. *Let X be a variety over R and $\mathscr{X} = (X_i)_{i\in I}$ a family of varieties over X. Let Y be a closed subvariety of X, U its open complement, and for all $i \in I$, define varieties $Y_i = X_i \times_X Y$, $U_i = X_i \times_X U$, and families $\mathscr{Y} = (Y_i)_{i\in I}$ and $\mathscr{U} = (U_i)_{i\in I}$ of varieties over Y and U, respectively. Then for all $\pi \in \mathbf{N}^{(I)}$ and for all $\pi' \leq \pi$, the variety $S^{\pi'}\mathscr{U} \times S^{\pi-\pi'}\mathscr{Y}$ is isomorphic to the locally closed subset of $S^{\pi}\mathscr{X}$ corresponding to* *families of zero-cycles inducing partition π' on $\mathscr{U}$. Moreover, $S^\pi\mathscr{X}$ is the disjoint union of these locally closed subsets. In particular, in terms of classes in* KVar_R^+, *we have*

$$[S^\pi\mathscr{X}] = \sum_{\pi'\leq\pi}[S^{\pi'}\mathscr{U}][S^{\pi-\pi'}\mathscr{Y}].$$

In terms of zeta-functions, this means that

$$Z_{\mathscr{X}}(\mathbf{t}) = Z_{\mathscr{U}}(\mathbf{t})Z_{\mathscr{Y}}(\mathbf{t})$$

in $\mathrm{KVar}_R^+[[\mathbf{t}]]$.

Proposition 3.3.3.2. *Let X be a variety over R and $\mathscr{X} = (X_i)_{i\in I}$ and $\mathscr{Y} = (Y_i)_{i\in I}$ families of varieties over X. Fix an element $\pi = (n_i)_{i\in I} \in \mathbf{N}^{(I)}$, and assume that for all $i \in I$ such that $n_i > 0$, X_i and Y_i are piecewise isomorphic. Then $S^\pi\mathscr{X}$ and $S^\pi\mathscr{Y}$ are piecewise isomorphic: in other words, we have the equality*

$$Z_{\mathscr{X}}(\mathbf{t}) = Z_{\mathscr{Y}}(\mathbf{t})$$

in $\mathrm{KVar}_R^+[[\mathbf{t}]]$.

Proof. By assumption, there is an integer p with the property that, for all $i \in I$ satisfying $n_i > 0$, we can partition X_i (resp. Y_i) into locally closed sets $X_{i,1},\ldots,X_{i,p}$ (resp. $Y_{i,1},\ldots,Y_{i,p}$) such that for all $j \in \{1,\ldots,p\}$, $X_{i,j}$ is isomorphic to $Y_{i,j}$. For every $j \in \{1,\ldots,p\}$, we denote by $\mathscr{X}_j$ (resp. $\mathscr{Y}_j$) the family $(X_{i,j})_{i\in I}$ (resp. $(Y_{i,j})_{i\in I}$). Proposition 3.3.2.3 together with an induction shows that $S^\pi\mathscr{X}$ is the disjoint union of locally closed subsets isomorphic to $S^{\pi_1,\ldots,\pi_p}(\mathscr{X}_1,\ldots,\mathscr{X}_p)$ for partitions $\pi_1,\ldots,\pi_p$ such that $\pi_1+\ldots+\pi_p = \pi$. Since the same is true for $\mathscr{Y},\mathscr{Y}_1,\ldots,\mathscr{Y}_p$, and since the isomorphisms between the $X_{i,j}$ and $Y_{i,j}$ give isomorphisms

$$S^{\pi_1,\ldots,\pi_p}(\mathscr{X}_1,\ldots,\mathscr{X}_p) \simeq S^{\pi_1,\ldots,\pi_p}(\mathscr{Y}_1,\ldots,\mathscr{Y}_p)$$

for all $\pi_1,\ldots,\pi_p$ such that $\pi_1+\ldots+\pi_p = \pi$, the result follows. □

3.4 Symmetric products and affine spaces

The following proposition is a direct generalisation of a result by Totaro (see [CNS], Proposition 5.1.5). In fact, it may be obtained directly by applying Totaro's statement to each factor in $\prod_{i\in I} S^{n_i}X_i$ and then restricting to the open subset $S^\pi X \subset \prod_{i\in I} S^n X$, but we prefer to give a direct proof in the flavour of Totaro's argument: it is more straightforward here since one does not need to go through the step of cutting up the symmetric power with respect to different partitions.

Proposition 3.4.0.1. *Let X be a variety over R and $\mathscr{X} = (X_i)_{i\in I}$ a family of varieties over X. Moreover, let $\mathbf{m} = (m_i)_{i\in I}$ be a family of non-negative integers, $\mathscr{L}^{\mathbf{m}}$ the family* *of affine spaces* $(\mathbf{A}_R^{m_i})_{i\in I}$ *and* $\mathscr{X}\times\mathscr{L}^{\mathbf{m}}$ *the family* $(X_i\times\mathbf{A}_R^{m_i})_{i\in I}$*, each* $X_i\times\mathbf{A}_R^{m_i}$ *being seen as an* X*-variety through the first projection. Then for all* $\pi=(n_i)_{i\in I}\in\mathbf{N}^{(I)}$*, the variety* $S^{\pi}(\mathscr{X}\times\mathscr{L}^{\mathbf{m}})$ *is endowed with the structure of a vector bundle of rank* $\sum_{i\in I} m_i n_i$ *over* $S^{\pi}(\mathscr{X})$*, so that in particular*

$$[S^{\pi}(\mathscr{X}\times\mathscr{L}^{\mathbf{m}})]=[S^{\pi}(\mathscr{X})]\mathbf{L}^{\sum_{i\in I} m_i n_i}$$

in KVar_R^+.

Proof. The first projections induce natural maps

$$\prod_{i\in I}(X_i\times\mathbf{A}_R^{m_i})^{n_i}\longrightarrow\prod_{i\in I}X_i^{n_i}$$

over $\prod_{i\in I}X^{n_i}$. Restricting to points mapping to the diagonal in $\prod_{i\in I}X^{n_i}$, we get a map

$$\left(\prod_{i\in I}(X_i\times\mathbf{A}_R^{m_i})^{n_i}\right)_*\longrightarrow\left(\prod_{i\in I}X_i^{n_i}\right)_*$$

that gives us in turn a map

$$p: S^{\pi}(\mathscr{X}\times\mathscr{L}^{\mathbf{m}})\longrightarrow S^{\pi}(\mathscr{X})$$

after taking the quotient by the natural action of the group $\prod_{i\in I}\mathfrak{S}_{n_i}$.

On the other hand, the variety $(\prod_{i\in I}(X_i\times\mathbf{A}_R^{m_i})^{n_i})_*$ is by definition isomorphic to

$$\prod_{i\in I}(X_i\times\mathbf{A}_R^{m_i})^{n_i}\times_{\prod_{i\in I}X^{n_i}}\left(\prod_{i\in I}X^{n_i}\right)_*\simeq\left(\prod_{i\in I}X_i^{n_i}\right)_*\times\mathbf{A}_R^{\sum_{i\in I}n_i m_i}.$$

Thus, we have a cartesian diagram

$$\begin{array}{ccc}
(\prod_{i\in I}X_i^{n_i})_*\times\mathbf{A}_R^{\sum_{i\in I}n_i m_i} & \longrightarrow & (\prod_{i\in I}X_i^{n_i})_* \\
\downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\
S^{\pi}(\mathscr{X}\times\mathscr{L}^{\mathbf{m}}) & \xrightarrow{\;p\;} & S^{\pi}(\mathscr{X})
\end{array}$$

with q' being the quotient by the permutation action of $\prod_{i\in I}\mathfrak{S}_{n_i}$. Through our identification, this permutation action becomes linear, endowing $S^{\pi}(\mathscr{X}\times\mathscr{L}^{\mathbf{m}})$ étale-locally with a structure of vector bundle of rank $\sum_{i\in I}n_i m_i$ over $S^{\pi}(\mathscr{X})$. By Hilbert's theorem 90, $S^{\pi}(\mathscr{X}\times\mathscr{L}^{\mathbf{m}})$ is a vector bundle of the same rank over $S^{\pi}(\mathscr{X})$, whence the result. □

This implies the following formula for zeta-functions:

Corollary 3.4.0.2. *Let* X, $\mathscr{X}$, $\mathscr{L}^{\mathbf{m}}$ *be as above. Then writing* $\mathbf{L}^{\mathbf{m}}\mathbf{t}=(\mathbf{L}^{m_i}t_i)_{i\in I}$ *we have*

$$Z_{\mathscr{X}}(\mathbf{L}^{\mathbf{m}}\mathbf{t})=Z_{\mathscr{X}\times\mathscr{L}^{\mathbf{m}}}(\mathbf{t})$$

in $\mathrm{KVar}_R^+[[\mathbf{t}]]$.

3.5 Symmetric products of non-effective classes

For the moment, we have only defined the symmetric products $S^{\pi}\mathscr{X}$ in the case where the family $\mathscr{X}$ is a family of quasi-projective varieties over X. The aim of this section is to generalise this definition in the case when $\mathscr{X}$ is merely a family of classes in KVar_X. For this, recall that in remark 3.1.3.1, we noted that one could define the symmetric product $S^{\pi}\mathscr{X}$ of a family of varieties $\mathscr{X} = (X_i)_{i\in I}$ over a variety X for $\pi = (n_i)_{i\in I}$ alternatively by first considering the product $\prod_{i\in I} S^{n_i}X_i$ and then restricting to the appropriate open subset, namely the one lying above the open subset $S^{\pi}X$ of $\prod_{i\in I} S^{n_i}X$. In this section, we are first going to define, for a class $Y \in \mathrm{KVar}_X$, its symmetric power S^nY as an element of KVar_{S^nX}. The symmetric product $S^{\pi}\mathscr{X}$ of a family of classes $(X_i)_{i\in I}$ in KVar_X will then be defined, by analogy with remark 3.1.3.1, as the pullback to $\mathrm{KVar}_{S^{\pi}X}$ of the element

$$\prod_{i\in I} S^{n_i}X_i \in \mathrm{KVar}_{\prod_{i\in I} S^{n_i}X}.$$

3.5.1 Relative symmetric powers

Let X be a quasi-projective variety over R. All (fibred and exterior) products in this section are over R, though we will not write it to simplify notation. The product $\prod_{n\geq 1}\mathrm{KVar}_{S^nX}$ is an additive group, but it may also be endowed with a commutative multiplicative group structure, in the following manner:

For $a = (a_n)_{n\geq 1}$ and $b = (b_n)_{n\geq 1} \in \prod_{n\geq 1}\mathrm{KVar}_{S^nX}$, we put

$$(ab)_n = \sum_{k=0}^{n} a_k \boxtimes b_{n-k}$$

where by convention $a_0 = b_0 = 1$, and $a_k \boxtimes b_{n-k}$ is the image of (a_k, b_{n-k}) through the composition

$$\mathrm{KVar}_{S^kX} \times \mathrm{KVar}_{S^{n-k}X} \xrightarrow{\boxtimes} \mathrm{KVar}_{S^kX\times S^{n-k}X} \to \mathrm{KVar}_{S^nX}$$

where the latter morphism is obtained from the natural map $X^n \to S^nX$ by passing to the quotient with respect to the natural action of the group $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$ on X^n. Observe that the neutral element for this law is the family $(e_n)_{n\geq 1}$ with $e_i = 0$ for all $i \geq 1$. Associativity is obtained by using, for all $n \geq 1$ and all i, j, k such that $i + j + k = n$ the commutativity of the diagram

$$\begin{array}{ccccc} (S^iX \times S^jX) \times S^kX & \longrightarrow & S^{i+j}X \times S^kX & \longrightarrow & S^nX \\ \downarrow \simeq & & & \nearrow & \\ S^iX \times (S^jX \times S^kX) & \longrightarrow & S^iX \times S^{j+k}X & & \end{array}$$

On the other hand, starting from $a = (a_n)_{n\geq 1} \in \prod_{n\geq 1}\mathrm{KVar}_{S^nX}$, we may define its inverse $b = (b_n)_{n\geq 1} \in \prod_{n\geq 1}\mathrm{KVar}_{S^nX}$ by induction: put $b_0 = 1$, $b_1 = -a_1$, and assume $b_0, \ldots, b_{n-1}$ to be constructed. Then b_n is the element of KVar_{S^nX} defined by

$$b_n = -\sum_{k=1}^{n} a_k \boxtimes b_{n-k}.$$

Notation 3.5.1.1. We denote by $\mathrm{KVar}_{S^\bullet X}$ the group $\prod_{n\geq 1}\mathrm{KVar}_{S^nX}$ with this law.

Lemma 3.5.1.2. *There is a unique group morphism*

$$S : \mathrm{KVar}_X \to \mathrm{KVar}_{S^\bullet X},$$

such that the image of the class of a quasi-projective variety Y over X is the family $(S^nY)_{n\geq 1}$.

Proof. The condition in the statement defines a morphism S' on the free abelian group with generators the isomorphism classes of quasi-projective varieties over X. Consider a quasi-projective variety Y over X, and a closed subscheme Z of Y, with open complement U. We know that S^nY is the disjoint union of locally closed subsets isomorphic to $S^kZ\times S^{n-k}U$ for $k=0,\dots,n$, and these isomorphisms are over S^nX, so $S'(Y)=S'(Z)S'(U)$, and S' descends to a well-defined group morphism as in the statement of the lemma, using the fact that KVar_X is generated by classes of quasi-projective varieties. □

Corollary 3.5.1.3. *For every $n\geq 1$ the class in KVar_{S^nX} of the symmetric power S^nY of a variety Y over X depends only on the class of Y in KVar_X.*

Remark 3.5.1.4. In the same manner, one may put a (commutative) monoid structure on $\mathrm{KVar}^+_{S^\bullet X} := \prod_{i\geq 1}\mathrm{KVar}^+_{S^iX}$, and there is a unique morphism of monoids

$$S^+ : \mathrm{KVar}^+_X \to \mathrm{KVar}^+_{S^\bullet X},$$

such that the image of the class of a quasi-projective variety Y over X is the family $(S^nY)_{n\geq 1}$. Moreover, S^+ induces S on KVar_X.

Definition 3.5.1.5. Let $\mathfrak{a}\in\mathrm{KVar}_X$ and $n\geq 1$. The n-th symmetric power of $\mathfrak{a}$, denoted by $S^n\mathfrak{a}$, is the image of $S(\mathfrak{a})$ through the projection

$$\prod_{i\geq 1}\mathrm{KVar}_{S^iX}\to\mathrm{KVar}_{S^nX}$$

onto the n-th component.

Remark 3.5.1.6. The lemma implies that for any $\mathfrak{a},\mathfrak{b}\in\mathrm{KVar}_X$ and for any $n\geq 1$,

$$S^n(\mathfrak{a}+\mathfrak{b})=\sum_{k=0}^{n} S^k\mathfrak{a}\boxtimes S^{n-k}\mathfrak{b}$$

in KVar_{S^nX}, where every term on the right-hand side is seen as an element of S^nX via the natural map $S^kX\times S^{n-k}X\to S^nX$. In particular, we have the equality of classes

$$[S^n(\mathfrak{a}+\mathfrak{b})]=\sum_{k=0}^{n}[S^k\mathfrak{a}][S^{n-k}\mathfrak{b}]$$

in KVar_R.

3.5.2 Definition of symmetric products of classes in KVar_X

Let X be a quasi-projective variety over R, $\mathscr{A} = (\mathfrak{a}_i)_{i\in I}$ a family of elements of KVar_X, $n \geq 1$ an integer and $\pi = (n_i)_{i\in I}$ a partition of n. Consider the natural projection morphism

$$p : \prod_{i\in I} X^{n_i} \to \prod_{i\in I} S^{n_i} X,$$

which is finite and surjective. Let $U := \left(\prod_{i\in I} X^{n_i}\right)_* \subset \prod_{i\in I} X^{n_i}$ be the complement of the diagonal, that is, the open subset of points with all coordinates being distinct. Then by construction, $S^\pi X$ is the open subset $p(U)$ of the variety $\prod_{i\in I} S^{n_i} X$.

Using definition 3.5.1.5, we may consider the element $\prod_{i\in I} S^{n_i}\mathfrak{a}_i \in \mathrm{KVar}_{\prod_{i\in I} S^{n_i}X}$.

Definition 3.5.2.1. 1. The element $S^\pi\mathscr{A} \in \mathrm{KVar}_{S^\pi X}$ is defined to be the image of $\prod_{i\in I} S^{n_i}\mathfrak{a}_i$ through the restriction morphism

$$\mathrm{KVar}_{\prod_{i\in I} S^{n_i}X} \to \mathrm{KVar}_{S^\pi X}.$$

2. More generally, let $p \geq 1$ be an integer, $X_1, \ldots, X_p$ quasi-projective varieties over R, and for every $j \in \{1, \ldots, p\}$, let $\mathscr{A}_j = (\mathfrak{a}_{ij})_{i\in I}$ be a family of elements of KVar_{X_i}. For all partitions $\pi_1, \ldots, \pi_p$ with $\pi_i = (r_{i,j})_{i\in I}$ we define the *mixed symmetric product* $S^{\pi_1,\ldots,\pi_p}(\mathscr{A}_1, \ldots, \mathscr{A}_p)$ as the image of $\prod_{i\in I} S^{r_{i,1}}\mathfrak{a}_{i,1}\times\ldots\times S^{r_{i,p}}\mathfrak{a}_{i,p}$ through the restriction morphism

$$\mathrm{KVar}_{\prod_{i\in I} S^{r_{i,1}}X_1\times\ldots\times S^{r_{i,p}}X_p} \to \mathrm{KVar}_{S^{\pi_1,\ldots,\pi_p}(X_1,\ldots,X_p)}.$$

Remark 3.5.2.2. The above restriction morphism factors through $\mathrm{KVar}_{S^{\pi_1}X_1\times\ldots\times S^{\pi_p}X_p}$ by remark 3.3.2.2. In the case where immersion (3.10) is an isomorphism, we have the equality

$$S^{\pi_1}\mathscr{A}_1 \boxtimes \ldots \boxtimes S^{\pi_p}\mathscr{A}_p = S^{\pi_1,\ldots,\pi_p}(\mathscr{A}_1, \ldots, \mathscr{A}_p)$$

in $\mathrm{KVar}_{S^{\pi_1}X_1\times\ldots\times S^{\pi_p}X_p} = \mathrm{KVar}_{S^{\pi_1,\ldots,\pi_p}(X_1,\ldots,X_p)}$.

Remark 3.5.2.3. Using the first part of the definition, we may extend definition 3.2.1.6 and get a notion of zeta-function $Z_{\mathscr{A}}(\mathbf{t}) \in \mathrm{KVar}_R[[\mathbf{t}]]$ of a family of elements of KVar_X.

Proposition 3.5.2.4. *Let X be a quasi-projective variety over R. Let $\mathscr{A} = (\mathfrak{a}_i)_{i\in I}$, $\mathscr{B} = (\mathfrak{b}_i)_{i\in I}$, $\mathscr{C} = (\mathfrak{c}_i)_{i\in I}$ be families of elements of* KVar_X *such that for every $i \in I$, $\mathfrak{a}_i = \mathfrak{b}_i + \mathfrak{c}_i$. For every partition $\pi = (n_i)_{i\in I}$ we have the equality*

$$S^\pi\mathscr{A} = \sum_{\pi'\leq\pi} S^{\pi',\pi-\pi'}(\mathscr{B}, \mathscr{C})$$

in $\mathrm{KVar}_{S^\pi X}$*, where each term on the right-hand side is considered as an element of* $\mathrm{KVar}_{S^\pi X}$ *via the natural morphism $S^{\pi'}X \times S^{\pi-\pi'}X \to S^\pi X$.*

Proof. According to remark 3.5.1.6, we may write

$$\prod_{i\in I} S^{n_i}\mathfrak{a}_i = \prod_{i\in I}\left(\sum_{0\leq m_i\leq n_i} S^{m_i}\mathfrak{b}_i \boxtimes S^{n_i-m_i}\mathfrak{c}_i\right) = \sum_{\pi'=(m_i)_{i\in I}\leq \pi}\prod_{i\in I} S^{m_i}\mathfrak{b}_i\boxtimes S^{n_i-m_i}\mathfrak{c}_i$$

in $\mathrm{KVar}_{\prod_{i\in I} S^{n_i}X}$, where each $S^{m_i}\mathfrak{b}_i \times S^{n_i-m_i}\mathfrak{c}_i$ is seen as an element of $\mathrm{KVar}_{S^{n_i}X}$ through the natural morphism $S^{m_i}X\times S^{n_i-m_i}X\to S^{n_i}X$. This gives the result by restriction to $S^\pi X$. □

Corollary 3.5.2.5. *Let X be a quasi-projective variety over R, Y a closed subscheme of X and U its open complement. Let $\mathscr{A}=(\mathfrak{a}_i)_{i\in I}$ be a family of elements of KVar_X, and define $\mathscr{B}=(\mathfrak{b}_i)_{i\in I}$ and $\mathscr{C}=(\mathfrak{c}_i)_{i\in I}$ the families of elements of KVar_U (resp. KVar_Y) obtained by restriction from $\mathscr{A}$. For every partition $\pi=(n_i)_{i\in I}$ we have the equality*

$$S^\pi\mathscr{A} = \sum_{\pi'\leq\pi} S^{\pi'}\mathscr{B}\boxtimes S^{\pi-\pi'}\mathscr{C}$$

in $\mathrm{KVar}_{S^\pi X}$, where each term on the right-hand side is considered as an element of $\mathrm{KVar}_{S^\pi X}$ via the natural immersion $S^{\pi'}U\times S^{\pi-\pi'}Y\to S^\pi X$. In particular, we have the equality $Z_{\mathscr{A}}(\mathbf{t}) = Z_{\mathscr{B}}(\mathbf{t})Z_{\mathscr{C}}(\mathbf{t})$ in $\mathrm{KVar}_R[[\mathbf{t}]]$.

3.6 Symmetric products of varieties with exponentials

3.6.1 The symmetric product of a family of varieties with exponentials

Fix a variety X over R, as well as $(\mathscr{X},f)=(X_i,f_i)_{i\in I}$ a family of varieties over X with exponentials. For any $\pi\in\mathbf{N}^{(I)}$, recall that the symmetric product $S^\pi\mathscr{X}$ is constructed as a quotient of an open subset of the product

$$\prod_{i\in I} X_i^{n_i}.$$

The latter is endowed with the map

$$\begin{array}{cccc} f^\pi: & \prod_{i\in I} X_i^{n_i} & \longrightarrow & \mathbf{A}^1 \\ & (x_{i,1},\ldots,x_{i,n_i})_{i\in I} & \mapsto & \sum_{i\in I}(f_i(x_{i,1})+\ldots+f_i(x_{i,n_i})). \end{array}$$

This map restricts to points lying above the complement of the diagonal, and is invariant modulo the action of $\prod_{i\in I}\mathfrak{S}_{n_i}$. Therefore it descends to a map

$$f^{(\pi)}: S^\pi\mathscr{X}\longrightarrow \mathbf{A}^1,$$

and we may define the symmetric product $S^\pi(\mathscr{X},f)$ to be the variety $S^\pi\mathscr{X}$ endowed with the map $f^{(\pi)}$.

Remark 3.6.1.1. Let X be a variety over R, $f : X \to \mathbf{A}^1$ a morphism and $n \geq 1$ an integer. Then there is a morphism

$$f^{(n)} : \begin{array}{ccc} S^n X & \to & \mathbf{A}^1 \\ x_1 + \ldots + x_n & \mapsto & f(x_1) + \ldots + f(x_n) \end{array}$$

induced by $f^{\oplus n} : X^n \to \mathbf{A}^1$. Its restriction to the locally closed subset $S^\pi X$ is given by

$$\sum_{i \geq 1} i(x_{i,1} + \ldots + x_{i,n_i}) \mapsto \sum_{i \geq 1} i(f(x_{i,1}) + \ldots + f(x_{i,n_i})).$$

Thus, the restriction $f^{(n)}_{|S^\pi X}$ to $S^\pi X$ is exactly the map $g^{(\pi)}$ where g is the family $(if)_{i\geq 1}$ of morphisms $X \to \mathbf{A}^1$.

3.6.2 Iterated symmetric products

Here we generalise proposition 3.2.4.3 to varieties with exponentials. We refer to the corresponding section for the definition of the map μ and other pieces of notation. Let X be a variety over a variety R, which itself lies above some k-variety R', and let $(\mathscr{X}, f) = (X_i, f_i)_{i \in I}$ be a family of varieties with exponentials over X. Let $\pi \in \mathbf{N}^{(I)}$ and let $\varpi = (m_j)_{j \in J} \in \mu^{-1}(\pi)$. Recall from (3.8) and the discussion that follows that we have an immersion

$$\left(\prod_{j \in J} \left(\left(\prod_{i \in I} X_i^{n_i^j} {}_{/R} \right)_{*,X} \right)^{m_j}_{/R'} \right)_{*,R} \hookrightarrow \left(\prod_{i \in I} X_i^{n_i} {}_{/R'} \right)_{*,X}. \qquad (3.11)$$

which after passing to the quotient by the appropriate group actions, induces the isomorphism

$$u_\varpi : S^\varpi(S^\bullet(\mathscr{X}/R)/R') \to S^\pi_\varpi(\mathscr{X}/R').$$

By the construction in section3.6.1 there is a morphism $f^{(\varpi)} : S^\varpi(S^\bullet(\mathscr{X}/R)/R') \to \mathbf{A}^1$ induced by the morphism

$$\prod_{j \in J} \left(f^{(\pi_j)} \right)^{\oplus m_j} : \prod_{j \in J} \left(S^{\pi_j}(\mathscr{X}/R) \right)^{m_j} {}_{/R'} \to \mathbf{A}^1$$

where each $f^{(\pi_j)}$ is itself induced by

$$f^\pi = \prod_{i \in I} f_i^{\oplus n_i^j} : \prod_{i \in I} X_i^{n_i^j} {}_{/R} \to \mathbf{A}^1.$$

Thus, $f^{(\varpi)}$ is induced, via (3.11) and passing to the quotient, by the morphism $f^\pi = \prod_{i \in I} f_i^{\oplus n_i} = \prod_{j \in J} \left(\prod_{i \in I} f_i^{\oplus n_i^j} \right)^{\oplus m_j}$ defined on $\prod_{i \in I} X_i^{n_i} {}_{/R}$. We may conclude that for all ϖ, we have $f^{(\pi)} \circ u_\varpi = f^{(\varpi)}$, which gives the following proposition:

Proposition 3.6.2.1. *Let X be a variety over a variety R, which itself lies above some k-variety R', and let $(\mathscr{X}, f) = (X_i, f_i)_{i\in I}$ be a family of varieties with exponentials over X. Then for every $\pi \in \mathbf{N}^{(I)}$ and for every $\varpi \in \mu^{-1}(\pi)$, there is an isomorphism u_ϖ of the constructible set $S^\varpi(S^\bullet(\mathscr{X}/R)/R')$ onto a locally closed subset $S^\pi_\varpi(\mathscr{X}/R')$ of $S^\pi(\mathscr{X}/R')$, so that moreover $S^\pi(\mathscr{X}/R')$ is equal to the disjoint union of the sets $S^\pi_\varpi(\mathscr{X}/R')$ and $f^{(\pi)} \circ u_\varpi = f^{(\varpi)}$. In particular, we have the equality*

$$\sum_{\varpi\in\mu^{-1}(\pi)} [S^\varpi(S^\bullet(\mathscr{X}/R)/R'), f^{(\varpi)}] = [S^\pi(\mathscr{X}/R'), f^{(\pi)}]$$

in $\mathrm{KExpVar}^+_{R'}$.

3.6.3 Cutting into pieces

More generally, starting from families $(\mathscr{X}_1, f_1), \ldots, (\mathscr{X}_p, f_p)$ of varieties with exponentials over X, for any $\pi_1, \ldots, \pi_p \in \mathbf{N}^{(I)}$ the mixed symmetric product

$$S^{\pi_1,\ldots,\pi_p}(\mathscr{X}_1, \ldots, \mathscr{X}_p)$$

from 3.3.2 may be endowed with a map $(f_1, \ldots, f_p)^{(\pi_1,\ldots,\pi_p)}$ to $\mathbf{A}^1$. Moreover, proposition 3.3.2.3 may be extended in the following way:

Proposition 3.6.3.1. *Let X be a variety over R, and $(\mathscr{X}_1, f_1), \ldots, (\mathscr{X}_p, f_p)$ families of varieties with exponentials over X, and consider moreover another family $(\mathscr{X}, g) = (X_i, g_i)_{i\in I}$ of varieties over X, together with families*

$$(\mathscr{U}, g_{|\mathscr{U}}) = (U_i, g_{i|U_i})_{i\in I} \quad \text{and} \quad (\mathscr{Y}, g_{|\mathscr{Y}}) = (Y_i, g_{i|Y_i})_{i\in I}$$

such that for every $i \in I$, Y_i is a closed subvariety of X_i and U_i its complement. Let $\pi_1, \ldots, \pi_p, \pi$ be elements of $\mathbf{N}^{(I)}$. Then for every $\pi' \in \mathbf{N}^{(I)}$ such that $\pi' \leq \pi$ there is an isomorphism $u_{\pi'}$ from the variety

$$S^{\pi_1,\ldots,\pi_p,\pi',\pi-\pi'}(\mathscr{X}_1, \ldots, \mathscr{X}_p, \mathscr{U}, \mathscr{Y})$$

to the locally closed subset of $S^{\pi_1,\ldots,\pi_p,\pi}(\mathscr{X}_1, \ldots, \mathscr{X}_p, \mathscr{X})$ corresponding to points with $\mathscr{X}$-component inducing partition π' on $\mathscr{U}$, such that

$$(f_1, \ldots, f_p, g)^{(\pi_1,\ldots,\pi_p,\pi)} \circ u_{\pi'} = (f_1, \ldots, f_p, g_{|\mathscr{U}}, g_{|\mathscr{Y}})^{(\pi_1,\ldots,\pi_p,\pi',\pi-\pi')}.$$

Moreover, $S^{\pi_1,\ldots,\pi_p,\pi}(\mathscr{X}_1, \ldots, \mathscr{X}_p, \mathscr{X})$ is the disjoint union of these locally closed subsets, so that in terms of classes in $\mathrm{KExpVar}^+_R$*, we have*

$$\left[S^{\pi_1,\ldots,\pi_p,\pi}(\mathscr{X}_1, \ldots, \mathscr{X}_p, \mathscr{X}), (f_1, \ldots, f_p, g)^{(\pi_1,\ldots,\pi_p)}\right]$$

$$= \sum_{\pi'\leq\pi} \left[S^{\pi_1,\ldots,\pi_p,\pi',\pi-\pi'}(\mathscr{X}_1, \ldots, \mathscr{X}_p, \mathscr{U}, \mathscr{Y}), (f_1, \ldots, f_p, g_{|\mathscr{U}}, g_{|\mathscr{Y}})^{(\pi_1,\ldots,\pi_p,\pi',\pi-\pi')}\right].$$

3.6.4 Compatibility with affine spaces

Lemma 3.6.4.1. *Let Y be a quasi-projective variety over a field k and $f : Y \to \mathbf{A}^1$ a morphism. Then for every $\lambda \in k$, and every integer $n \geq 0$ we have the equality:*

$$[S^n(Y \times \mathbf{A}^1, f \oplus \lambda \mathrm{id})] = \begin{cases} [S^nY]\mathbf{L}^n & \text{if } \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$

in $\mathrm{KExpVar}_{S^nY}$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} (Y \times \mathbf{A}^1)^n & \xrightarrow{p'} & Y^n \\ \downarrow q' & & \downarrow q \\ S^n(Y \times \mathbf{A}^1) & \xrightarrow{p} & S^nY \end{array}$$

where the vertical arrows are the quotient maps. By the proof of Totaro's lemma (see lemma 3.4, or [CNS] chapter 6, Proposition 3.1.5), for every partition π of n, the lower arrow endows $p^{-1}(S^\pi Y)$ with the structure of a vector bundle of rank n over $S^\pi Y$ which is locally trivial for the Zariski topology. On the other hand, by definition, the symmetric product $S^n(Y \times \mathbf{A}^1)$ is endowed with the morphism $(f \oplus \lambda\mathrm{id})^{(n)}$ induced by the morphism $(f \oplus \lambda\mathrm{id})^{\oplus n}$ given by

$$\begin{array}{ccc} (Y \times \mathbf{A}^1)^n = Y^n \times \mathbf{A}^n & \to & \mathbf{A}^1 \\ (y_1, \ldots, y_n, t_1, \ldots, t_n) & \mapsto & f(y_1) + \ldots + f(y_n) + \lambda(t_1 + \ldots + t_n) \end{array}$$

Consider a point $y \in S^\pi Y$. We know that the fibre of $p^{-1}(S^\pi Y)$ above this point is an affine space $\mathbf{A}^n_{\kappa(y)}$. The linear form $(t_1, \ldots, t_n) \mapsto \lambda(t_1 + \ldots + t_n)$ on the general fibre of the trivial vector bundle in the top row of the diagram descends (via the permutation action, which is linear) to some linear form ℓ on $\mathbf{A}^n_{\kappa(y)}$, which will be zero if and only if λ is zero. Thus, since by construction the morphism $(f \oplus \lambda\mathrm{id})^{(n)}$ coincides with $f^{(n)}$ on the zero-section of the vector bundle $p^{-1}(S^\pi Y) \to S^\pi Y$, we have, for any $(t_1, \ldots, t_n) \in \mathbf{A}^n_{\kappa(y)}$

$$(f \oplus \lambda\mathrm{id})^{(n)}(y, t_1, \ldots, t_n) = f^{(n)}(y) + \ell(t_1, \ldots, t_n),$$

with ℓ a linear form, which is zero if and only if $\lambda = 0$. Note that because of the last axiom defining $\mathrm{KExpVar}_k$, we have, using a linear change of basis, that

$$[\mathbf{A}^n, \ell] = \begin{cases} \mathbf{L}^n & \text{if } \ell = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Taking y to be a generic point of $S^\pi Y$ and spreading out to some trivialising open set U for the vector bundle $p^{-1}(S^\pi Y) \to S^\pi Y$, we have

$$[p^{-1}(S^\pi Y)_{|U}, (f \oplus \lambda\mathrm{id})^{(n)}_{|p^{-1}(S^\pi(Y))_{|U}}] = [U, f^{(n)}_{|U}][\mathbf{A}^n, \ell] = \begin{cases} [U, f^{(n)}_{|U}]\mathbf{L}^n & \text{if } \lambda = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Repeat the process for a generic point of $S^\pi Y \setminus U$. By Noetherian induction, the result follows by taking the sum over all partitions of n. ∎

3.6.5 Symmetric products of classes in Grothendieck rings with exponentials

Let X be a quasi-projective variety. The same procedure as in section 3.5.1 endows the product $\prod_{i\geq 1} \mathrm{KExpVar}_{S^iX}$ with a group law, and this group will be denoted $\mathrm{KExpVar}_{S^\bullet X}$.

Proposition 3.6.5.1. *Let X be a quasi-projective variety. There is a unique group morphism*

$$S^{\exp} : \mathrm{KExpVar}_X \to \mathrm{KExpVar}_{S^\bullet X}$$

sending the class $[Y, f]$ of a quasi-projective variety Y with a morphism $f : Y \to \mathbf{A}^1$, to the family of classes $([S^iY, f^{(i)}])_{i\geq 1}$. Moreover, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{KVar}_X & \xrightarrow{S} & \mathrm{KVar}_{S^\bullet X} \\ \downarrow & & \downarrow \\ \mathrm{KExpVar}_X & \xrightarrow{S^{\exp}} & \mathrm{KExpVar}_{S^\bullet X} \end{array}$$

where the vertical arrows are given by the injections KVar $\to$ KExpVar.

Proof. Define a morphism S' from the free abelian group on pairs (Y, f), where Y is a quasi-projective variety over X and $f : Y \to \mathbf{A}^1$ a morphism, to $\mathrm{KExpVar}_{S^\bullet X}$ by $S'((Y, f)) = ([S^iY, f^{(i)}])_{i\geq 1}$. It suffices now to check that S' passes to the quotient through the three relations defining $\mathrm{KExpVar}_X$. It is clear that S' is constant on isomorphic pairs. If Z is a closed subscheme of Y with open complement U, then for any $n \geq 1$, $[S^nY, f^{(n)}] = \sum_{i=0}^{n}[S^iU, f^{(i)}_{|U}][S^{n-i}Z, f^{(n-i)}_{|Z}]$, so that $S'(Y) = S'(U)S'(Z)$. Finally, it follows from lemma 3.6.4.1 that for any quasi-projective variety Y over X, the class $[Y \times_{\mathbf{Z}} \mathbf{A}^1, \mathrm{pr}_2]$ goes to zero. The commutativity of the diagram may be checked for classes of varieties $[Y \to X]$, where it is immediate. □

The notion of symmetric product of a class in KVar_X may therefore be extended to $\mathrm{KExpVar}_X$ in the following manner:

Definition 3.6.5.2. Let $\mathfrak{a} \in \mathrm{KExpVar}_X$ and $n \geq 1$. The n-th symmetric product of $\mathfrak{a}$, denoted by $S^n\mathfrak{a}$ is the image of $S^{\exp}(\mathfrak{a})$ through the projection

$$\prod_{i\geq 1} \mathrm{KExpVar}_{S^iX} \to \mathrm{KExpVar}_{S^nX}$$

onto the n-th component.

Remark 3.6.5.3. If X be a quasi-projective variety over R and $\mathscr{A} = (A_i)_{i\geq 1}$ a family of varieties with exponentials, definition 3.5.2.1 may be extended in a compatible way to define, for any partition π, the symmetric product $S^\pi\mathscr{A}$ as an element of $\mathrm{KExpVar}_{S^\pi X}$. We can also define mixed symmetric products, and proposition 3.5.2.4 and corollary 3.5.2.5 are true more generally with KVar replaced by KExpVar.

3.7 Symmetric products in localised Grothendieck rings

In this section, we are going to generalise the notion of symmetric product further, to include classes in the localised Grothendieck rings $\mathscr{M}_X$ and $\mathscr{E}xp\mathscr{M}_X$. Since the Grothendieck ring KVar injects into KExpVar, it does not restrict generality to work with KExpVar directly, which we will do in this section.

Lemma 3.7.0.1. *For every* $\mathfrak{a} \in \mathrm{KExpVar}_X$*, for any* $n \geq 1$ *and for any* $m \geq 1$*, one has*

$$S^n(\mathfrak{a}\mathbf{L}^m) = S^n(\mathfrak{a})\mathbf{L}^{nm}$$

in $\mathrm{KExpVar}_{S^nX}$.

Proof. Lemma 3.6.4.1 shows that this holds for effective elements of $\mathrm{KExpVar}_X$. It suffices to prove that it holds for a difference $Y - Z$ of two effective elements. Using the fact that S (resp. $S^{\exp}$) is a group morphism, one may write, by induction on n,

$$\begin{aligned} S^n((Y-Z)\mathbf{L}^m) &= S^n(Y\mathbf{L}^m) - \sum_{k=0}^{n-1} S^{n-1-k}((Y-Z)\mathbf{L}^m)S^k(Z\mathbf{L}^m) \\ &= S^n(Y)\mathbf{L}^{mn} - \mathbf{L}^{mn}\sum_{k=0}^{n-1} S^{n-1-k}(Y-Z)S^k(Z) \\ &= S^n(Y-Z)\mathbf{L}^{mn}. \end{aligned}$$

□

Let X be a quasi-projective variety. The same procedure as in section 3.5.1 endows the product $\prod_{i\geq 1}\mathscr{E}xp\mathscr{M}_{S^iX}$ with a group law, and this group will be denoted $\mathscr{E}xp\mathscr{M}_{S^\bullet X}$.

Lemma 3.7.0.2. *There is a unique group morphism*

$$S^{\mathrm{loc}} : \mathscr{E}xp\mathscr{M}_X \to \mathscr{E}xp\mathscr{M}_{S^\bullet X}$$

given by $\mathfrak{a}\mathbf{L}^{-m} \mapsto ((S^i\mathfrak{a})\mathbf{L}^{-mi})_{i\geq 1}$.

Proof. We already have a group morphism

$$\mathrm{KExpVar}_X \to \mathscr{E}xp\mathscr{M}_{S^\bullet X} \tag{3.12}$$

obtained from S by composing with the localisation morphism $\mathrm{KExpVar}_{S^\bullet X} \to \mathscr{E}xp\mathscr{M}_{S^\bullet X}$, and given by $\mathfrak{a} \mapsto (S^i\mathfrak{a})_{i\geq 1}$. Lemma 3.7.0.1 shows that an element in the kernel of the localisation morphism $\alpha_X : \mathrm{KExpVar}_X \to \mathscr{E}xp\mathscr{M}_X$ will go to 0, so (3.12) factors through α_X, inducing a morphism

$$\mathrm{im}(\alpha_X) \to \mathscr{E}xp\mathscr{M}_{S^\bullet X}.$$

Let $\mathfrak{a} \in \mathscr{E}xp\mathscr{M}_X$. Then there exists an integer $m \geq 1$ such that $\mathfrak{a}\mathbf{L}^m$ belong to $\mathrm{im}(\alpha_X)$. Put

$$S^{\mathrm{loc}}(\mathfrak{a}) = (S^i(\mathfrak{a}\mathbf{L}^m)\mathbf{L}^{-mi})_{i\geq 1}$$

By lemma 3.7.0.1 this does not depend on the choice of m, so S^{loc} is well-defined. □

Remark 3.7.0.3. We may now define, for any $\mathfrak{a} \in \mathscr{E}xp\mathscr{M}_X$, its symmetric powers $S^i(\mathfrak{a})$ to be the components of $S^{\mathrm{loc}}(\mathfrak{a})$. More generally, for any partition $\pi = (n_i)_{i\in I}$ we may define the symmetric products $S^\pi \mathscr{A}$ of any family $\mathscr{A}$ of elements of $\mathscr{E}xp\mathscr{M}_X$. Furthermore, we may define mixed symmetric products of a finite number of families of elements of $\mathscr{E}xp\mathscr{M}_X$, and as in remark 3.6.5.3, proposition 3.5.2.4 remains true with KVar replaced with $\mathscr{M}$, KExpVar or $\mathscr{E}xp\mathscr{M}$.

We also have, for $A \in \{\mathrm{KVar}, \mathscr{M}, \mathrm{KExpVar}, \mathscr{E}xp\mathscr{M}\}$:

Proposition 3.7.0.4. *Let X be a variety over R, and $(X_i)_{i\in I}$ a family of elements in A_X. Let $\mathbf{m} = (m_i)_{i\in I}$ be a family of non-negative integers, and denote by $\mathscr{X}\mathbf{L}^{\mathbf{m}}$ the family $(X_i\mathbf{L}^{m_i})_{i\in I}$. Then for all $\pi = (n_i)_{i\in I} \in \mathbf{N}^{(I)}$, we have*

$$S^\pi(\mathscr{X}\mathbf{L}^{\mathbf{m}}) = (S^\pi \mathscr{X})\mathbf{L}^{\sum_{i\in I} m_i n_i}$$

in $A_{S^\pi X}$.

Proof. By lemma 3.7.0.1, we have, for all $i \in I$,

$$S^{n_i}(X_i\mathbf{L}^{m_i}) = S^{n_i}(X_i)\mathbf{L}^{m_i n_i}$$

in $A_{S^{n_i}X}$. Taking exterior products over $i \in I$ and restricting to $S^\pi X$ we get the result. □

3.8 Euler products

Let $A \in \{\mathrm{KVar}, \mathscr{M}, \mathrm{KExpVar}, \mathscr{E}xp\mathscr{M}\}$. Let X be a variety over R and $\mathscr{X} = (X_i)_{i\in I}$ a family of elements of A_X. We define the Euler product notation to be

$$\prod_{u\in X/R}\left(1 + \sum_{i\in I} X_{i,u}t_i\right) := \sum_{\pi\in\mathbf{N}^{(I)}} [S^\pi(\mathscr{X}/R)]\mathbf{t}^\pi = Z_{\mathscr{X}/R}(\mathbf{t}) \quad \in A_R[[\mathbf{t}]],$$

where the $t_i,\ i \in I$ are variables, and $\mathbf{t}^\pi$ is defined to be the finite product $\prod_{i\in I} t_i^{n_i}$, where $\pi = (n_i)_{i\in\mathcal{I}}$. When $R = \operatorname{Spec} k$ for a field k, we will leave out the mention of R in the product, writing simply $\prod_{u\in X}$.

We are going to start by checking that our "product" actually behaves like a product, thereby justifying our notation. A summary of the properties proved below is given in theorem 4 of the introduction.

Properties: Let X be a variety over R and $\mathscr{X} = (X_i)_{i\in I}$ a family of classes in A_X.

1. (Product with one factor) When $X = R$, the last part of Example 3.1.3.3 gives

$$\prod_{u\in R/R}\left(1 + \sum_{i\in I} X_{i,u}t_i\right) = 1 + \sum_{i\in I} X_i t_i. \tag{3.13}$$

2. (Associativity) Let $X = U \cup Y$ be a partition of X into a closed subscheme Y and its complement U, and $\mathscr{U} = (U_i)_{i\in I}$ (resp. $\mathscr{Y} = (Y_i)_{i\in I}$) the restriction of $\mathscr{X}$ to U (resp. to Y). Then corollary 3.5.2.5 can be reformulated as

$$\prod_{u\in X/R}\left(1+\sum_{i\in I}X_{i,u}t_i\right) = \prod_{u\in U/R}\left(1+\sum_{i\in 1}U_{i,u}t_i\right)\prod_{u\in Y/R}\left(1+\sum_{i\in I}Y_{i,u}t_i\right). \tag{3.14}$$

Here we use remarks 3.6.5.3 and 3.7.0.3, which state that corollary 3.5.2.5 is true more generally with KVar replaced with KExpVar, $\mathscr{M}$ or $\mathscr{E}xp\mathscr{M}$.

3. (Finite products) Combining the previous two properties, we see that in the case where X is a disjoint union of m varieties $Y_1, \ldots, Y_m$ isomorphic to R,

$$\prod_{v\in X/R}\left(1+\sum_{i\in I}X_{i,v}t_i\right) = \prod_{j=1}^{m}\left(1+\sum_{i\in I}X_{i,j}t_i\right) \in A_R[[\mathbf{t}]] \tag{3.15}$$

where for all $i \in I$ $X_{i,j}$ is the restriction of X_i to $Y_j \simeq R$, and the product on the right-hand side is a finite product of power series (in the classical sense).

4. (Change of variables of the form $\mathbf{t} \mapsto \mathbf{L}^{\mathbf{m}}\mathbf{t}$) In terms of Euler products, Proposition 3.7.0.4 means that for any $(m_i)_{i\in I} \in \mathbf{N}^I$,

$$\prod_{u\in X/R}\left(1+\sum_{i\in I}X_{i,u}(\mathbf{L}^{m_i}t_i)\right) = \prod_{u\in X/R}\left(1+\sum_{i\in I}(X_{i,u}\mathbf{L}^{m_i})t_i\right),$$

where the right-hand-side is the Euler product associated to the family $(X_i\mathbf{L}^{m_i})_{i\in I}$, that is, the Euler product notation is compatible with respect to changes of variables of the form $(t_i \mapsto \mathbf{L}^{m_i}t_i)_{i\in I}$, and factors of the form $\mathbf{L}^{m_i}$ may pass from the variables to the coefficients. One must pay attention that this is specific to affine spaces because of their good behaviour with respect to Euler products.

5. (Double products) Let $R, R', X, \mathscr{X}$ be as in Proposition 3.2.4.3. Using our notation for the family of varieties $S^\bullet(\mathscr{X}/R)$ over R, we have

$$\prod_{v\in R/R'}\left(1+\sum_{\pi\in\mathbf{N}^{(I)}\setminus\{0\}}(S^\pi(\mathscr{X}/R))_v\mathbf{t}^\pi\right) = \sum_{\varpi\in\mathbf{N}^{\left(\mathbf{N}^{(I)}\setminus\{0\}\right)}}[S^\varpi(S^\bullet(\mathscr{X}/R)/R')]\mathbf{t}^\varpi,$$

where, writing $\varpi = (m_\pi)_{\pi\in\mathbf{N}^{(I)}\setminus\{0\}}$, and denoting by $\pi[i]$ the number of times i occurs in π, $\mathbf{t}^\varpi$ is by definition

$$\prod_{\pi\in\mathbf{N}^{(I)}\setminus\{0\}}(\mathbf{t}^\pi)^{m_\pi} = \prod_{\pi\in\mathbf{N}^{(I)}\setminus\{0\}}\left(\prod_{i\in I}t_i^{\pi[i]}\right)^{m_\pi} = \prod_{i\in I}t_i^{\sum_{\pi\in\mathbf{N}^{(I)}\setminus\{0\}}m_\pi\pi[i]} = \mathbf{t}^{\mu(\varpi)}.$$

In particular, $\mathbf{t}^\varpi$ and $\mathbf{t}^{\varpi'}$ are equal if and only if, in the notations of definition 3.2.4.1, $\mu(\varpi) = \mu(\varpi')$. Thus, the above product can be rewritten in the form

$$\prod_{v\in R/R'}\left(1+\sum_{\pi\in\mathbf{N}^{(I)}\setminus\{0\}}(S^\pi(\mathscr{X}/R))_v\mathbf{t}^\pi\right)$$

$$= 1 + \sum_{\pi \in \mathbf{N}^{(I)} \setminus \{0\}} \left(\sum_{\varpi \in \mu^{-1}(\pi)} [S^{\varpi}(S^{\bullet}(\mathscr{X}/R)/R')] \right) \mathbf{t}^{\pi}.$$

We may therefore conclude, using proposition 3.2.4.3 that the double product

$$\prod_{v \in R/R'} \left(\prod_{u \in X/R} \left(1 + \sum_{i \in I} X_{i,u} t_i \right) \right)_v$$

makes sense and satisfies

$$\prod_{v \in R/R'} \left(\prod_{u \in X/R} \left(1 + \sum_{i \in I} X_{i,u} t_i \right) \right)_v = \prod_{u \in X/R'} \left(1 + \sum_{i \in I} X_{i,u} t_i \right). \tag{3.16}$$

By proposition 3.6.2.1 this remains true with KVar replaced by KExpVar if $\mathscr{X}$ is a family of varieties with exponentials. We may also pass to the respective localisations $\mathscr{M}$ and $\mathscr{E}xp\mathscr{M}$.

6. (Compatibility with finite products) In the setting of 5, assume that X is the disjoint union of two copies of R, written respectively Y and Z. For all $i \in I$, we will write Y_i (resp. Z_i) for the restriction of X_i to Y (resp. to Z). Using (3.15), we get that

$$\prod_{u \in X/R} \left(1 + \sum_{i \in I} X_{i,u} t_i \right) = \left(1 + \sum_{i \in I} Y_i t_i \right) \left(1 + \sum_{i \in I} Z_i t_i \right).$$

Taking the product over R relatively to R', we get

$$\prod_{v \in R/R'} \left(1 + \sum_{i \in I} Y_{i,v} t_i \right) \left(1 + \sum_{i \in I} Z_{i,v} t_i \right)$$

On the other hand, by (3.14), we have

$$\prod_{v \in X/R'} \left(1 + \sum_{i \in I} X_{i,v} t_i \right) = \prod_{v \in Y/R'} \left(1 + \sum_{i \in I} Y_{i,v} t_i \right) \prod_{v \in Z/R'} \left(1 + \sum_{i \in I} Z_{i,v} t_i \right).$$

Using (3.16), we finally get

$$\prod_{v \in R/R'} \left(1 + \sum_{i \in I} Y_{i,v} t_i \right) \left(1 + \sum_{i \in I} Z_{i,v} t_i \right)$$

$$= \prod_{v \in R/R'} \left(1 + \sum_{i \in I} Y_{i,v} t_i \right) \prod_{v \in R/R'} \left(1 + \sum_{i \in I} Z_{i,v} t_i \right). \tag{3.17}$$

Remark 3.8.0.1. Let us now try to get a grip on what each factor of such an infinite product actually represents. For simplicity, we will consider $R = \operatorname{Spec} k$. If $X = \operatorname{Spec} k$, then according to (3.13),

$$\prod_{u \in \operatorname{Spec} k} \left(1 + \sum_{i \in I} X_{i,u} t_i \right) = 1 + \sum_{i \in I} X_i t^i \in \mathrm{KVar}_k[[t]].$$

In this case the left-hand side has "only one factor", and identifying coefficients on both sides suggests that we can think of those objects $X_{i,u}$ as being the classes in KVar_k of the fibres of the X_i above the single closed point $u \in X$.

Now let X be the disjoint union of a copy Y of $\operatorname{Spec} k$ (with a single closed point y) and of its open complement U, and let $\mathscr{Y}$ and $\mathscr{U}$ be the respective restrictions of $\mathscr{X}$ to Y and U. Then using what we just remarked together with (3.14),

$$\begin{aligned}\prod_{x\in X}\left(1+\sum_{i\in I}X_{i,x}t_i\right) &= \prod_{y\in Y}\left(1+\sum_{i\in I}Y_{i,y}t_i\right)\prod_{u\in U}\left(1+\sum_{i\in I}U_{i,u}t_i\right)\\ &= \left(1+\sum_{i\in I}Y_it_i\right)\prod_{u\in U}\left(1+\sum_{i\in I}U_{i,u}t_i\right)\end{aligned}$$

This means that adding a closed point $\operatorname{Spec} k$ to the variety consists exactly in adding a factor with coefficient of t_i representing the class in $\mathrm{KVar}_{k(y)}$ of the fibre $Y_i = X_{i,y}$ above the added point y.

Example 3.8.0.2. 1. Kapranov's zeta function is obtained when taking $I = \mathbf{N}^*$, $X_i = X$ and specialising t_i to t^i for some single indeterminate t, for all i. Thus, for all closed points $v \in X$, the class of $X_{i,v}$ in $\mathrm{KVar}^+_{k(v)}$ is 1 and the Euler product decomposition of Kapranov's zeta function can be written as

$$\prod_{v\in X}\left(\sum_{i\geq 0}t^i\right) = 1+\sum_{n\geq 1}[S^nX]t^n,$$

or even

$$\prod_{v\in X}(1-t)^{-1} = 1+\sum_{n\geq 1}[S^nX]t^n,$$

2. For $I = \mathbf{N}^*$, put $X_1 = X$ and $X_i = \varnothing$ for all $i \geq 2$. Then the class of $X_{i,v}$ in $\mathrm{KVar}^+_{k(v)}$ is 1 if $i = 1$ and 0 otherwise. Thus, writing $t_1 = t$, we get the following Euler product decomposition for the generating function of zero-cycles without repetitions:

$$\prod_{v\in X}(1+t) = 1+\sum_{n\geq 1}[S^n_*X]t^n.$$

Example 3.8.0.3. Let X be a variety over R and $n \geq 1$ an integer. As an application of double products, let us compute the Kapranov zeta function of a $\mathbf{P}^n$-bundle Y over X in terms of Z_X. By definition,

$$Z_Y = \prod_{y\in Y}\frac{1}{1-t}.$$

Using (3.16), we have

$$\begin{aligned} Z_Y &= \prod_{x\in X}\left(\prod_{y\in Y/X}\frac{1}{1-t}\right)_x \\ &= \prod_{x\in X}\prod_{y\in \mathbf{P}^n}\frac{1}{1-t} \\ &= \prod_{x\in X}\frac{1}{1-t}\frac{1}{1-\mathbf{L}t}\cdots\frac{1}{1-\mathbf{L}^n t} \end{aligned}$$

where the last line comes from the exact expression of $Z_{\mathbf{P}^n}(t)$. Using compatibility with finite products, we finally get

$$Z_Y = \prod_{i=0}^{n}\prod_{x\in X}\frac{1}{1-\mathbf{L}^i t} = Z_X(t)Z_X(\mathbf{L}t)\dots Z_X(\mathbf{L}^n t).$$

Note that this result can be obtained without the Euler product, by cutting the projective bundle into affine bundles and applying propositions 3.4.0.1 and 3.3.3.1 (see [LL], Corollary 3.6).

3.9 Allowing other constant terms

Until now, for simplicity we have only worked with series having constant terms equal to 1. Though for obvious reasons of convergence we cannot abandon this hypothesis completely, it is still possible to make sense of Euler products where a finite number of factors have arbitrary constant terms, by generalising our symmetric products a bit further. Let us motivate the construction first. In the simplest setting, we want to make sense of Euler products

$$\prod_{x\in X}(X_{0,x} + X_{1,x}t + X_{2,x}t^2 + \dots)$$

where X is a variety over an algebraically closed field k and $(X_i)_{i\geq 0}$ is a family of varieties over X, such that $X_{0,x} = 1$ for almost all closed points $x\in X$. In other words, our product looks like

$$\prod_{x\in U}(1 + X_{1,x}t + X_{2,x}t^2 + \dots)\prod_{x\in F}(X_{0,x} + X_{1,x}t + X_{2,x}t^2 + \dots)$$

for a finite set of closed points $F\subset X$ with open complement U. When one expands this product naively, one sees that the contribution for a zero-cycle $D\in S^nX(k)$ depends on the intersection of the support $|D|$ of D with the set F. Let us assume for simplicity that $F = \{x_0\}$ is a singleton. Then the expansion of this product may formally be written

$$\sum_{n\geq 0}\left(\sum_{\substack{D=\sum n_x x\in S^nX(k)\\ x_0\in|D|}}\prod_{x\in X}X_{n_x,x} + \sum_{D=\sum n_x x\in S^nU(k)}X_{0,x_0}\prod_{x\in U}X_{n_x,x}\right)t^n$$

In other words, the fibre X_{0,x_0} appears whenever the zero-cycle with respect to which we expand does not contain x_0 in its support, forcing us to choose the term of degree zero in the factor corresponding to x_0. Thus, denoting by $\mathscr{X}$ the family $(X_i)_{i\geq 1}$, the coefficient of degree n in this power series should be the class of the constructible set given by the product $X_{0,x} \times S^n\mathscr{X}_{|S^nU}$ above the open subset S^nU, and by $S^n\mathscr{X}_{|S^nX\setminus S^nU}$ above its complement.

This construction generalises for general F: we cut S^nX up into locally closed subsets $(S^nX)^E$ for all subsets $E \subset F$ such that $(S^nX)^E$ corresponds to zero-cycles D such that $|D| \cap F = E$, and modify the symmetric product $S^n\mathscr{X}$ above each $(S^nX)^E$ accordingly, multiplying it by the product of the fibres $X_{0,x}$ for all x in $F \setminus E$.

3.9.1 Another generalisation of symmetric products

Throughout this section, let $A \in \{\mathrm{KVar}, \mathscr{M}, \mathrm{KExpVar}, \mathscr{E}xp\mathscr{M}\}$. Let I_0 be an additive monoid, and $I = I_0 \setminus \{0\}$. Let X be a variety over a perfect field k and V an open set of X such that its complement F is a finite set of closed points. Let $\mathscr{X} = (X_i)_{i\in I_0}$ be a family of varieties over X (resp. of classes in the ring A_X), such that $X_0 \times_X V \simeq V$ (resp. the image of X_0 in A_V is the class $1 = [V \xrightarrow{\mathrm{id}} V]$). Thus, as a motivic function, X_0 is constant equal to 1 on the set V. In this section we are going to define a slight modification of the notion of symmetric product which takes into account X_0, or more precisely, the finite number of fibres $X_{0,v}$ for $v \in F$. This will allow us to consider a finite number of constant terms other than 1 in our Euler products.

Recall that for any $\pi = (n_i)_{i\in I} \in \mathbf{N}^{(I)}$, $S^\pi X$ parametrises collections $D = (D_i)_{i\in I}$ of effective zero-cycles on X, with disjoint supports, such that $\deg D_i = n_i$. We denote by $|D| = \cup_{i\in I}|D_i|$ the union of the supports of the D_i. For any subset $E \subseteq F$ denote by $(S^\pi X)^E$ the constructible subset of $S^\pi X$ parametrising families D of effective zero-cycles such that $|D| \cap F = E$. Also, denote by $\widetilde{\mathscr{X}}$ the family $(X_i)_{i\in I}$ (that is, $\mathscr{X}$ with X_0 removed), and by $\left(S^\pi \widetilde{\mathscr{X}}\right)^E$ the restriction of $S^\pi\widetilde{\mathscr{X}}$ to $(S^\pi X)^E$ (resp. the image of $S^\pi\widetilde{\mathscr{X}}$ in $A_{(S^\pi X)^E}$).

Definition 3.9.1.1. 1. The symmetric product $S^\pi\mathscr{X}$ of the family of varieties $\mathscr{X} = (X_i)_{i\in I_0}$ is defined as the constructible set over $S^\pi X$ with restriction to $(S^\pi X)^E$ given by

$$S^\pi\mathscr{X}_{|(S^\pi X)^E} = \left(S^\pi\widetilde{\mathscr{X}}\right)^E \times \prod_{v\in F\setminus E} X_{0,v}.$$

2. Denote by i_E the immersion $(S^\pi X)^E \to S^\pi X$. The symmetric product $S^\pi\mathscr{X}$ of the family of classes $\mathscr{X} = (X_i)_{i\in I_0}$ in A_X is defined as the element of $A_{S^\pi X}$ given by

$$S^\pi\mathscr{X} = \sum_{E\subseteq F} (i_E)_! \left(\left(S^\pi\widetilde{\mathscr{X}}\right)^E\right) \prod_{v\in F\setminus E} X_{0,v}.$$

Note that since $X_{0,v}$ is trivial (that is, a point) for every $v \in V$, it makes sense to write

$$\prod_{v\in F\setminus E} X_{0,v} = \prod_{v\notin E} X_{0,v}.$$

Notation 3.9.1.2. We denote by $(S^\pi \mathscr{X})^E$ the pullback of $S^\pi \mathscr{X}$ along i_E, given by

$$\left(S^\pi \widetilde{\mathscr{X}}\right)^E \prod_{v \notin E} X_{0,v}.$$

By definition, the partition $\left\{(S^\pi X)^E,\ E \subseteq F\right\}$ of $S^\pi X$ into constructible subsets is such that for each $E \subseteq F$, the restriction $S^\pi \mathscr{X} \times_{S^\pi X} (S^\pi X)^E$ is a trivial fibration above $S^\pi \widetilde{\mathscr{X}} \times_{S^\pi X} (S^\pi X)^E$ with fibre $\prod_{v \notin E} X_{0,v}$.

Remark 3.9.1.3. It is clear from the definition of symmetric products of families of classes in A_X that both parts of the definition are compatible, that is, the class in $A_{S^\pi X}$ of the constructible set $S^\pi \mathscr{X}$ from part 1 will be the element constructed in part 2 with the classes of the varieties X_i in A_X.

Remark 3.9.1.4. Definition 3.9.1.1 does not depend on the choice of the set F. Choosing a bigger set F' only amounts to refining the partition $\left\{(S^\pi X)^E,\ E \subseteq F'\right\}$. Indeed, assume $F' = F \cup \{v_0\}$. Then for all $E \subset F$,

$$\{D,\ |D| \cap F = E\} = \{D,\ |D| \cap F' = E\} \cup \{D,\ |D| \cap F' = E \cup \{v_0\}\},$$

and $X_{0,v_0} = 1$, so that $\prod_{v \in F' \setminus E} X_{0,v} = \prod_{v \in F \setminus E} X_{0,v}$.

Remark 3.9.1.5. Assume that $F = \varnothing$, that is, $X_0 = X$ (resp. $X_0 = 1 \in A_X$). Then we get $S^\pi \mathscr{X} = S^\pi \widetilde{\mathscr{X}}$, so our definition is an extension of the previous definition of symmetric products.

Remark 3.9.1.6. For $E = \varnothing$, $(S^\pi X)^E$ is an open subset of $S^\pi X$. Thus, the constructible set $S^\pi \mathscr{X}$ is birationally equivalent to $S^\pi \widetilde{\mathscr{X}} \times \prod_{v \in F} X_{0,v}$.

Example 3.9.1.7. Take $\pi = 0 \in \mathbf{N}^{(I)}$. Then the only non-empty piece of $S^\pi \mathscr{X}$ is the one corresponding to $E = \varnothing$, and therefore

$$S^0 \mathscr{X} = \prod_{v \in X} X_{0,v}.$$

The fibre of $S^\pi \widetilde{\mathscr{X}}$ above a family of effective zero-cycles $D = (D_i)_{i \in I} \in S^\pi X$ is given by

$$\prod_{i \in I} \prod_{v \in |D_i|} X_{i,v}.$$

By definition, replacing $\widetilde{\mathscr{X}}$ by $\mathscr{X}$ consists in replacing this fibre by its product with $\prod_{v \in F \setminus |D|} X_{0,v} = \prod_{v \notin |D|} X_{0,v}$. Thus, the fibre of $S^\pi \mathscr{X}$ above D can be written

$$\left(\prod_{v \notin |D|} X_{0,v}\right) \left(\prod_{i \in I} \prod_{v \in |D_i|} X_{i,v}\right),$$

which is indeed a finite product since $\cup_i |D_i|$ is a finite set, and and only a finite number of fibres $X_{0,v}$ are non-trivial.

3.9.2 Application to Euler products

Definition 3.9.2.1. Let X be a variety over a perfect field k and V an open subset of X such that its complement F is a finite set of closed points. Let $\mathscr{X} = (X_i)_{i\in I_0}$ be a family of classes in A_X such that X_0 maps to 1 in A_V. We define the zeta-function

$$Z_{\mathscr{X}}(\mathbf{t}) := \sum_{\pi\in\mathbf{N}^{(I)}} [S^\pi \mathscr{X}]\mathbf{t}^\pi \in A_k[[\mathbf{t}]],$$

where $S^\pi\mathscr{X}$ is understood to be the generalised symmetric product from the previous paragraph, as well as the Euler product notation for $Z_{\mathscr{X}}$:

$$\prod_{v\in X}\left(\sum_{i\in I_0} X_{i,v}t_i\right) := Z_{\mathscr{X}}(\mathbf{t}).$$

Lemma 3.9.2.2. *Let $X, V, \mathscr{X}$ be like in Definition 3.9.2.1. Let Y be a closed subscheme of X and U its open complement. Define $\mathscr{U} = (U_i)_{i\in I_0}$ and $\mathscr{Y} = (Y_i)_{i\in I_0}$ to be the families of elements of A_U (resp. A_Y) obtained by restriction from $\mathscr{X}$. For every $\pi \in \mathbf{N}^{(I)}$ we have the equality*

$$S^\pi\mathscr{X} = \sum_{\pi'\leq\pi} S^{\pi'}\mathscr{U} \boxtimes S^{\pi-\pi'}\mathscr{Y}$$

in $A_{S^\pi X}$, where each term on the right-hand side is considered as an element of $A_{S^\pi X}$ via the immersion $S^{\pi'}U \times S^{\pi-\pi'}Y \to S^\pi X$. In particular, we have the equality

$$Z_{\mathscr{X}}(\mathbf{t}) = Z_{\mathscr{U}}(\mathbf{t})Z_{\mathscr{Y}}(\mathbf{t})$$

in $A_k[[\mathbf{t}]]$.

Proof. Let E be a subset of F. Write E_U (resp. E_Y, F_U, F_Y) for the intersection $E\cap U$ (resp. $E\cap Y, F\cap U, F\cap Y$). Define the families $\mathscr{U} = (U_i)_{i\in I_0}$, $\widetilde{\mathscr{U}} = (U_i)_{i\in I}$, $\mathscr{Y} = (Y_i)_{i\in I_0}$, $\widetilde{\mathscr{Y}} = (Y_i)_{i\in I}$. Applying corollary 3.5.2.5 and pulling back along $i_E : (S^\pi X)^E \to S^\pi X$, we get

$$\left(S^\pi\widetilde{\mathscr{X}}\right)^E = \sum_{\pi'\leq\pi}\left(S^{\pi'}\widetilde{\mathscr{U}}\right)^{E_U} \boxtimes \left(S^{\pi-\pi'}\widetilde{\mathscr{Y}}\right)^{E_Y}$$

in $A_{(S^\pi X)^E}$. Therefore, writing

$$\prod_{v\in F\setminus E} X_{0,v} = \prod_{v\in F_U\setminus E_U} U_{0,v}\prod_{v\in F_Y\setminus E_Y} Y_{0,v},$$

we get that

$$\begin{aligned}(S^\pi\mathscr{X})^E &= \sum_{\pi'\leq\pi}\left((S^{\pi'}\widetilde{\mathscr{U}})^{E_U}\prod_{v\in F_U\setminus E_U}U_{0,v}\right)\boxtimes\left((S^{\pi-\pi'}\widetilde{\mathscr{Y}})^{E_Y}\prod_{v\in F_Y\setminus E_Y}Y_{0,v}\right)\\ &= \sum_{\pi'\leq\pi}\left(S^{\pi'}\mathscr{U}\right)^{E_U}\times\left(S^{\pi-\pi'}\mathscr{Y}\right)^{E_Y}\end{aligned}$$

in $A_{S^\pi X}$. On the other hand, denoting by i_{E_U} (resp. i_{E_Y}) the immersion $(S^{\pi'}U)^{E_U} \to S^{\pi'}U$ (resp. $(S^{\pi-\pi'}Y)^{E_Y} \to S^{\pi-\pi'}Y$:

$$\begin{aligned} S^{\pi'}\mathscr{U} \boxtimes S^{\pi-\pi'}\mathscr{Y} &= \left(\sum_{E_U\subset F_U} (i_{E_U})_! \left((S^{\pi'}\mathscr{U})^{E_U}\right)\right) \boxtimes \left(\sum_{E_Y\subset F_Y} (i_{E_Y})_! (S^{\pi-\pi'}\mathscr{Y})^{E_Y}\right) \\ &= \sum_{E\subset F} (i_{E_U} \boxtimes i_{E_Y})_! \left(\left(S^{\pi'}\mathscr{U}\right)^{E_U} \boxtimes \left(S^{\pi-\pi'}\mathscr{Y}\right)^{E_Y}\right) \end{aligned}$$

in $A_{S^{\pi'}U\times S^{\pi-\pi'}Y}$. We may conclude by commutativity of the diagram

$$\begin{array}{ccc} (S^{\pi'}U)^{E_U} \times (S^{\pi-\pi'}Y)^{E_Y} & \xrightarrow{i_{E_U}\boxtimes i_{E_Y}} & S^{\pi'}U \times S^{\pi-\pi'}Y \\ \downarrow & & \downarrow \\ (S^\pi X)^E & \xrightarrow{\quad i_E \quad} & S^\pi X \end{array}$$

where the vertical arrow on the right is the immersion from the statement of the lemma, and the vertical arrow on the left is the immersion it induces above $(S^\pi X)^E$. $\square$

This lemma immediately implies that the associativity property from Section 3.8 extends to this Euler product with constant terms.

Chapter 4

Mixed Hodge modules and convergence of Euler products

The aim of this chapter is to introduce a topology on the Grothendieck ring of varieties which is suitable for getting the expected convergence results in chapter 6. This topology will be defined using the theory of mixed Hodge modules of Morihiko Saito. For any complex variety X there is a morphism

$$\chi_X^{\mathrm{Hdg}} : \mathscr{M}_X^{\hat{\mu}} \to K_0(\mathrm{MHM}_X^{\mathrm{mon}}),$$

called the Hodge realisation, between the localised Grothendieck group of varieties over X with $\hat{\mu}$-action, and the Grothendieck group associated to the category $\mathrm{MHM}_X^{\mathrm{mon}}$ of mixed Hodge modules on X with monodromy action. This morphism becomes a ring morphism when $\mathscr{M}_X^{\hat{\mu}}$ is endowed with the convolution product $*$, and $K_0(\mathrm{MHM}_X^{\mathrm{mon}})$ with its Hodge-theoretic version. The notion of weight of a Hodge module defines an increasing sequence of subgroups $(W_{\leq n}K_0(\mathrm{MHM}_X^{\mathrm{mon}}))_{n\in\mathbf{Z}}$ of $K_0(\mathrm{MHM}_X^{\mathrm{mon}})$ which gives a filtration on the ring $K_0(\mathrm{MHM}_X^{\mathrm{mon}})$. We define the weight function $w_X : K_0(\mathrm{MHM}_X^{\mathrm{mon}}) \to \mathbf{Z}$, by

$$w_X(\mathfrak{a}) = \inf\{n \in \mathbf{Z},\ \mathfrak{a} \in W_{\leq n}K_0(\mathrm{MHM}_X^{\mathrm{mon}})\}.$$

We show it behaves well with respect to some natural operations in the derived category of mixed Hodge modules, like pushforwards, pullbacks, exterior products, but also with respect to a notion of *symmetric product* of mixed Hodge modules due to Maxim, Saito and Schürmann: for any element M of the bounded derived category $D^b(\mathrm{MHM}_X^{\mathrm{mon}})$, denoting by S^nM its n-th symmetric product for any integer $n \geq 1$, which is naturally an element $D^b(\mathrm{MHM}_{S^nX}^{\mathrm{mon}})$, we show that

$$w_{S^nX}(S^nM) \leq n\, w_X(M). \tag{4.1}$$

Composing w_X with χ_X^{Hdg} gives also a weight function on $\mathscr{M}_X^{\hat{\mu}}$. Finally, composing the latter with the total vanishing cycles measure

$$\Phi_X : \mathscr{E}xp\mathscr{M}_X \to \mathscr{M}_X^{\hat{\mu}}$$

gives us a weight function on the ring $\mathscr{E}xp\mathscr{M}_X$. Using this, we may define a notion of convergence for power series with coefficients in this ring. Our aim is to formulate a result which, given a sufficiently convergent series $1+\sum_{i\geq 1} X_i t^i \in \mathscr{E}xp\mathscr{M}_X[[t]]$, predicts the convergence of its Euler product

$$\prod_{x\in X}\left(1+X_{1,x}t+X_{2,x}t^2+\ldots\right).$$

Thus we need to be able to bound the weights of the symmetric products of the family $\mathscr{X}=(X_i)_{i\geq 1}$ in terms of the weights of the elements of the family. In other words, we need to relate the weights of the elements

$$\chi^{\mathrm{Hdg}}_{S^nX}\circ\Phi_{S^nX}(S^n(\mathscr{X}))\in K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^nX})$$

for all $i\geq 1$ to the weights of the elements

$$\chi^{\mathrm{Hdg}}_X\circ\Phi_X(X_i)$$

for al $i\geq 1$. Our strategy consists in introducing a version of the total vanishing cycles measure for mixed Hodge modules, that is, a morphism

$$\Phi^{\mathrm{Hdg}}_X : K_0(\mathrm{MHM}_{\mathbf{A}^1_X})\to K_0(\mathrm{MHM}^{\mathrm{mon}}_X)$$

fitting into a commutative diagram

$$\begin{array}{ccc}\mathscr{M}_{\mathbf{A}^1_X} & \xrightarrow{\Phi'_X} & \mathscr{M}^{\hat{\mu}}_X\\ \downarrow{\scriptstyle \chi^{\mathrm{Hdg}}_{\mathbf{A}^1_X}} & & \downarrow{\scriptstyle \chi^{\mathrm{Hdg}}_X}\\ K_0(\mathrm{MHM}_{\mathbf{A}^1_X}) & \xrightarrow{\Phi^{\mathrm{Hdg}}_X} & K_0(\mathrm{MHM}^{\mathrm{mon}}_X)\end{array}$$

(here we replaced Φ_X with its composition Φ'_X with the quotient morphism $\mathscr{M}_{\mathbf{A}^1_X}\to\mathscr{E}xp\mathscr{M}_X$). More precisely, we define a functor

$$\varphi^{\mathrm{tot}}_X : \mathrm{MHM}_{\mathbf{A}^1_X}\to\mathrm{MHM}^{\mathrm{mon}}_X$$

and take Φ^{Hdg}_X to be the morphism it induces on Grothendieck groups. Again, it is important to show that this morphism is a ring morphism when the Grothendieck groups are endowed with appropriate products. Having done this, in the above problem, the composition $\chi\circ\Phi$ may be replaced with $\Phi^{\mathrm{Hdg}}\circ\chi$, which is easier to deal with because most of the work can be done in categories of mixed Hodge modules, which are better behaved than categories of varieties (for example, they are abelian).

Via the Hodge realisation, symmetric powers in the Grothendieck ring of varieties correspond to Maxim, Saito and Schürmann's symmetric powers of mixed Hodge modules, that is, for any complex variety Z and any $\mathfrak{a}\in\mathscr{M}_Z$, we have

$$S^n(\chi^{\mathrm{Hdg}}_Z(\mathfrak{a}))=\chi^{\mathrm{Hdg}}_{S^nZ}(S^n\mathfrak{a}).$$

Using a Thom-Sebastiani theorem for Hodge modules due to Saito, we show that the functor φ_X^{tot} is, in a certain sense, compatible with these symmetric powers. Combining these two compatibilities with estimates of the form (4.1) enables us to conclude.

We now describe the structure of this chapter. The first three sections are purely Hodge-theoretic. We will start by recalling some definitions and useful facts about mixed Hodge modules, with or without monodromy, in section 4.1. In section 4.2 we define the *total vanishing cycles functor* between categories of mixed Hodge modules and prove the Thom-Sebastiani property. We recall the definition of symmetric products of Hodge modules in section 4.3, explain that it extends without trouble to Hodge modules with monodromy and describe the behaviour of the total vanishing cycles functor with respect to these symmetric products.

In section 4.4, we go on to relate all this to the framework of chapter 2, introducing the Hodge realisation and showing compatibilities of the motivic and Hodge-theoretic objects. Then we finally arrive to the definition of the weight filtrations, first on Grothendieck rings of Hodge modules in section 4.5, then on Grothendieck rings of varieties in section 4.6. In section 4.7, we conclude the chapter by stating and proving a convergence lemma for motivic Euler products of power series with coefficients in a Grothendieck ring of varieties with exponentials which will be used in chapter 6. We also include a result showing how, for a power series $Z(T) = \sum_{n\geq 0}[M_n]T^n \in \mathrm{KVar}^+_{\mathbf{C}}[[T]]$ (that is, having effective coefficients) for which we know the exact order of the first pole and which has a meromorphic continuation beyond this pole, one may recover information about the growth of the dimensions and number of irreducible components of the coefficients M_n.

4.1 Mixed Hodge modules

We are going to use freely the language of mixed Hodge modules introduced by Saito and are going to fix some notations and recall some facts to this end in this section. References are the original works [S88] and [S90], the summary [S89] by Saito himself, the axiomatic introduction by Peters and Steenbrink in [PS] section 14.1.1, and Beilinson, Bernstein and Deligne's paper [BBD] for definitions and properties of perverse sheaves. If S is a variety over $\mathbf{C}$, we denote by MHM_S the abelian category of mixed Hodge modules on S, by $D(\mathrm{MHM}_S)$ its derived category and by $D^b(\mathrm{MHM}_S)$ its bounded derived category. For any integer a, we also denote by $D^{\leq a}(\mathrm{MHM}_S)$ the full subcategory of $D(\mathrm{MHM}_S)$ of complexes only having cohomology in degree $\leq a$. We use square brackets to denote the shifting of complexes: for any complex M of mixed Hodge modules, for every $n \in \mathbf{Z}$ and any $i \in \mathbf{Z}$, $(M[n])^i = M^{i+n}$.

4.1.1 The rat functor

For any variety S over $\mathbf{C}$, there is an exact and faithful functor

$$\mathrm{rat}_S : \mathrm{MHM}_S \to \mathrm{Perv}_S$$

to the abelian category of perverse sheaves on S, extending to a functor

$$\mathrm{rat}_S : D^b(\mathrm{MHM}_S) \to D^b_{cs}(S)$$

where the category on the right is the full subcategory of cohomologically constructible complexes inside the bounded derived category of sheaves of $\mathbf{Q}$-vector spaces on S. Saito showed (Theorem 0.1 in [S90] or Theorem 1.3 in [S89]) that the usual operations $\boxtimes$, $\otimes$, as well as $f_*, f^*, f_!, f^!$ for any morphism f of varieties over $\mathbf{C}$ lift to the corresponding derived categories of mixed Hodge modules in a way compatible with the functor rat. In particular, there are adjunctions (f^*, f_*) and $(f_!, f^!)$. There is a morphism $f_! \to f_*$ which is an isomorphism when f is proper.

4.1.2 Twists

In the case where S is a point, the category $\mathrm{MHM}_{\mathrm{pt}}$ is exactly the category of polarisable mixed Hodge structures (see [S89], Theorem 1.4), and the functor rat becomes the forgetful functor associating to a mixed Hodge structure its underlying $\mathbf{Q}$-vector space. For any integer $d \in \mathbf{Z}$, we denote by $\mathbf{Q}^{\mathrm{Hdg}}_{\mathrm{pt}}(d) \in \mathrm{MHM}_{\mathrm{pt}}$ the Hodge structure of type $(-d, -d)$ with underlying vector space $\mathbf{Q}$. For $d = 0$, it will be denoted simply by $\mathbf{Q}^{\mathrm{Hdg}}_{\mathrm{pt}}$. For any complex variety S, tensoring with $\mathbf{Q}^{\mathrm{Hdg}}_{\mathrm{pt}}(1)$ defines a Tate twist on $D^b(\mathrm{MHM}_S)$.

When f is smooth of relative dimension d, then

$$f^! \simeq f^*[2d](d). \tag{4.2}$$

4.1.3 Weight filtration

Each $M \in \mathrm{MHM}_S$ has a finite increasing weight filtration $W_\bullet M$, the graded parts of which will be denoted $\mathrm{Gr}^W_\bullet$. For a bounded complex of mixed Hodge modules $M^\bullet$, we say $M^\bullet$ has weight $\leq n$ if $\mathrm{Gr}^W_i \mathcal{H}^j(M^\bullet) = 0$ for all integers i and j such that $i > j + n$.

For varieties X and Y over $\mathbf{C}$ we say that a functor $F : D^b(\mathrm{MHM}_X) \to D^b(\mathrm{MHM}_Y)$ does not increase weights if for every $n \in \mathbf{Z}$ and every $M^\bullet \in D^b(\mathrm{MHM}_X)$ with weight $\leq n$, the complex $F(M^\bullet)$ is also of weight $\leq n$. In particular, for any morphism of complex varieties f, the functors $f_!$ and f^* do not increase weights (see [S90], (4.5.2)).

4.1.4 Cohomological functors and cohomological amplitude

The usual truncation functor $\tau_{\leq}$ on $D^b(\mathrm{MHM}_S)$ corresponds to the perverse truncation ${}^p\tau_{\leq}$ on $D^b_{cs}(S)$, so that $\mathrm{rat}_S \circ \mathcal{H}^\bullet = {}^p\mathcal{H}^\bullet \circ \mathrm{rat}_S$, where $\mathcal{H}^\bullet$ is the usual cohomology on $D^b(\mathrm{MHM}_S)$ and ${}^p\mathcal{H}^\bullet$ is the perverse cohomology on $D^b_{cs}(S)$.

Definition 4.1.4.1. Let $T : D_1 \to D_2$ be an exact functor between triangulated categories endowed with t-structures, and let a be an integer. The functor T is said to be of *cohomological amplitude* $\leq a$ if $T(D_1^{\leq 0}) \subset D_2^{\leq a}$. Denoting by ${}^t\mathcal{H}^\bullet$ the corresponding cohomological functors, this means that for all $i > a$ and all $X \in D_1^{\leq 0}$, ${}^t\mathcal{H}^i(T(X)) = 0$.

Lemma 4.1.4.2. *For a morphism $f : Y \to X$ of varieties over $\mathbf{C}$ with fibres of dimension $\leq d$, the functors*

$$f_! : D^b(\mathrm{MHM}_Y) \to D^b(\mathrm{MHM}_X) \quad \textit{and} \quad f^* : D^b(\mathrm{MHM}_X) \to D^b(\mathrm{MHM}_Y)$$

are of cohomological amplitude $\leq d$.

Proof. According to [BBD] 4.2.4, this is true for the corresponding functors

$$f_! : D^b_{cs}(Y) \to D^b_{cs}(X) \quad \text{and} \quad f^* : D^b_{cs}(X) \to D^b_{cs}(Y)$$

with the perverse t-structure. Let $M^\bullet \in D^{\leq 0}(\mathrm{MHM}_Y)$. Then by compatibility of rat with pushforwards and t-structures as explained above, we have, for every $i \in \mathbf{Z}$

$$\mathrm{rat}_X(\mathcal{H}^i f_!(M^\bullet)) = {}^p\mathcal{H}^i(f_!(\mathrm{rat}_Y(M^\bullet)))$$

The right-hand side is zero for $i > d$ by [BBD] 4.2.4. The functor rat being faithful, we therefore have $\mathcal{H}^i f_!(M^\bullet) = 0$ for $i > d$ by lemma 4.1.4.3 below. The same argument holds for f^*. □

Lemma 4.1.4.3. *Let $F : A \to B$ be a faithful functor between additive categories. If $F(X) = 0$ for some object X of A, then $X = 0$.*

Proof. The assumption $F(X) = 0$ implies that $F(\mathrm{id}_X) = 0 = F(0_X)$ where 0_X is the constant zero map on X. By faithfulness we have $\mathrm{id}_X = 0_X$, which means that $X = 0$. □

4.1.5 The trace morphism for mixed Hodge modules

Though the theory of trace morphisms for Hodge modules is probably classical, we include this paragraph for lack of an appropriate reference. We only treat the case of smooth morphisms because it is sufficient for our purposes.

Notation 4.1.5.1. For any complex variety S, we denote by $a_S : S \to \operatorname{Spec} \mathbf{C}$ its structural morphism and by $\mathbf{Q}_S^{\mathrm{Hdg}}$ the complex of mixed Hodge modules $a_S^* \mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}$.

Remark 4.1.5.2. In the case where S is smooth and connected, the complex of mixed Hodge modules $\mathbf{Q}_S^{\mathrm{Hdg}}$ is concentrated in degree $\dim S$, and $\mathcal{H}^{\dim S}\mathbf{Q}_S^{\mathrm{Hdg}}$ is pure of weight $\dim S$, given by the pure Hodge module associated to the constant (rank one) variation of Hodge structures of weight 0 on S. When S is not smooth, by lemma 4.1.4.2 the complex $\mathbf{Q}_S^{\mathrm{Hdg}}$ is still an object of $D^{\leq \dim S}(\mathrm{MHM}_S)$, of weight ≤ 0 because the functor a_S^* does not increase weights, so that $\mathrm{Gr}_i^W \mathcal{H}^{\dim S}(\mathbf{Q}_S^{\mathrm{Hdg}}) = 0$ for $i > \dim S$. On the other hand, by [S90], (4.5.9), $\mathrm{Gr}^W_{\dim S} \mathcal{H}^{\dim S}(\mathbf{Q}_S^{\mathrm{Hdg}})$ is non-zero and simple, given by the intermediate extension of the constant weight 0 variation of Hodge structures on an open subset of S.

Remark 4.1.5.3 (Duality and the trace morphism for Hodge modules). Let X be a smooth variety of dimension d over $\mathbf{C}$. Then according to (4.2), we have $a_X^! \mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}} \simeq \mathbf{Q}_X^{\mathrm{Hdg}}(d)[2d]$, and by [S90] (4.4.2), there is a morphism of complexes of mixed Hodge structures

$$(a_X)_! \mathbf{Q}_X^{\mathrm{Hdg}} \to \mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}(-d)[-2d]$$

lifting the corresponding morphism in the derived category of sheaves of $\mathbf{Q}$-vector spaces on X. The cohomology of the complex $(a_X)_!\mathbf{Q}_X^{\mathrm{Hdg}}$ is exactly the cohomology with compact supports of X, and the morphism $(a_X)_!\mathbf{Q}_X^{\mathrm{Hdg}} \to \mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}(-d)[-2d]$ induces the trace morphism $H_c^{2d}(X, \mathbf{Q}) \to \mathbf{Q}(-d)$ on the top cohomology. The lift described above is compatible with Deligne's Hodge theory (see e.g. lemma 14.8, corollary 14.9 and remark 14.10 in [PS]), and turns the trace morphism into a morphism of Hodge structures $H_c^{2d}(X, \mathbf{Q}) \to \mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}(-d)$ (which is an isomorphism when X is irreducible).

The following proposition generalises this over a base:

Proposition 4.1.5.4. *Let S be a variety over $\mathbf{C}$ of dimension n and $p : X \to S$ a smooth morphism with fibres of constant dimension $d \geq 0$. Then there exists a morphism of complexes of Hodge modules*

$$f : p_!\mathbf{Q}_X^{\mathrm{Hdg}} \to \mathbf{Q}_S^{\mathrm{Hdg}}(-d)[-2d]$$

inducing a morphism of mixed Hodge modules

$$\mathcal{H}^{2d+n}(p_!\mathbf{Q}_X^{\mathrm{Hdg}}) \to \mathcal{H}^{2d+n}(\mathbf{Q}_S^{\mathrm{Hdg}}(-d)[-2d])$$

which above every closed point $s \in S$ corresponds to the classical trace morphism

$$H_c^{2d}(X_s, \mathbf{Q}) \to \mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}(-d)$$

of mixed Hodge structures.

Proof. The counit $p_!p^! \to \mathrm{id}$ associated to the adjunction $(p_!, p^!)$ induces a morphism of complexes of Hodge modules

$$p_!p^!\mathbf{Q}_S^{\mathrm{Hdg}} \to \mathbf{Q}_S^{\mathrm{Hdg}}.$$

Since p is smooth, we have $p^! = p^*(d)[2d]$, and this morphism induces a morphism

$$f : p_!p^*\mathbf{Q}_S^{\mathrm{Hdg}} \to \mathbf{Q}_S^{\mathrm{Hdg}}(-d)[-2d].$$

Note that by lemma 4.1.4.2, both these complexes are objects of $D^{\leq 2d+n}(\mathrm{MHM}_S)$. Moreover, by proper base change ([S90] 4.4.3), above every closed point $s \in S$, this induces a morphism of complexes of Hodge structures $f_s : (p_s)_!(p_s)^*\mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}} \to \mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}(-d)[-2d]$, where $p_s : X_s \to \mathbf{C}$ is the pullback of p by the inclusion $i_s : s \to S$, that is, $p_s = a_{X_s}$ using notation 4.1.5.1. This morphism induces the trace map on the top cohomology as explained in remark 4.1.5.3. □

Remark 4.1.5.5. Let $p : X \to S$ be as in the proposition, and assume moreover that X is irreducible. Then the morphism of mixed Hodge modules

$$\mathcal{H}^{2d+n}(p_!\mathbf{Q}_X^{\mathrm{Hdg}}) \to \mathcal{H}^{2d+n}(\mathbf{Q}_S^{\mathrm{Hdg}}(-d)[-2d])$$

given by the proposition induces an isomorphism

$$\mathrm{Gr}^W_{2d+n}\mathcal{H}^{2d+n}(p_!\mathbf{Q}_X^{\mathrm{Hdg}}) \simeq \mathrm{Gr}^W_{2d+n}\mathcal{H}^{2d+n}(\mathbf{Q}_S^{\mathrm{Hdg}}(-d)[-2d]).$$

Indeed, denoting by K_1 (resp. K_2) the kernel (resp. cokernel) of the above morphism, we have an exact sequence of mixed Hodge modules

$$0 \to K_1 \to \mathcal{H}^{2d+n}(p_! \mathbf{Q}_X^{\mathrm{Hdg}}) \to \mathcal{H}^{2d+n}(\mathbf{Q}_S^{\mathrm{Hdg}}(-d)[-2d]) \to K_2 \to 0,$$

which above $s \in S$ becomes

$$0 \to K_{1,s} \to H_c^{2d}(X_s, \mathbf{Q}) \to \mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}(-d) \to K_{2,s} \to 0. \tag{4.3}$$

The trace morphism being always surjective, we may conclude that $K_2 = 0$. Moreover, for every $s \in S$, since X is irreducible, there is an open dense subset U of S such that for all $s \in S$, X_s is irreducible, so that the trace morphism for X_s is an isomorphism. In particular, K_1 is supported inside some closed subset of S contained in the complement of U. Denote by S_1 the support of the pure Hodge module $\mathrm{Gr}^W_{2d+n} K_1$. If S_1 is non-empty, there is an open dense subset of S_1 over which $\mathrm{Gr}^W_{2d+n} K_1$ corresponds to a variation of pure Hodge structures, which must be of weight $2d$ because of the exact sequence (4.3). This would mean that the corresponding pure Hodge module is of weight $\dim S_1 + 2d < \dim S + 2d$, a contradiction.

4.1.6 Mixed Hodge modules with monodromy

A reference for this is Saito's unpublished paper about the Thom-Sebastiani theorem for mixed Hodge modules [S]. One can also consult the summary in section 2.9 of [BBDJS]. We denote by $\mathrm{MHM}_X^{\mathrm{mon}}$ the category of mixed Hodge modules M on a complex variety X endowed with commuting actions of a finite order operator $T_s : M \to M$ and a locally nilpotent operator $N : M \to M(-1)$. The category MHM_X can be identified with a full subcategory of $\mathrm{MHM}_X^{\mathrm{mon}}$ via the functor

$$\mathrm{MHM}_X \to \mathrm{MHM}_X^{\mathrm{mon}}$$

sending a mixed Hodge M to itself with $T_s = \mathrm{id}$ and $N = 0$. The Tate twist and the cohomological pullback and pushforward operations still exist in this setting (see [BBDJS], second paragraph of section 2.9). However, we need a more appropriate, twisted version of the external tensor product. Saito gives two equivalent ways of defining it, one of them being the following. Let $M_i = (D_i, F, L_i, W; T_s, N)$, $i = 1, 2$ be two mixed Hodge modules with monodromy, with underlying $\mathcal{D}_X-$modules D_i, underlying perverse complexes L_i with isomorphisms $\mathrm{DR}(D_i) \simeq L_i \otimes \mathbf{C}$ given by the Riemann-Hilbert correspondence, Hodge filtrations F, weight filtrations W and monodromy actions (T_s, N).

For each rational number $\alpha \in (-1, 0]$, let $D_i^\alpha = \mathrm{Ker}(T_s - \exp(-2i\pi\alpha)) \subset D_i$. We define

$$M_1 \overset{T}{\boxtimes} M_2 = (D, F, L, W; T_s, N)$$

by $D = D_1 \boxtimes D_2$, $L = L_1 \boxtimes L_2$, $T_s = T_s \boxtimes T_s$, $N = N \boxtimes \mathrm{id} + \mathrm{id} \boxtimes N$, the Hodge filtration being given by

$$F_p(D_1^\alpha \boxtimes D_2^\beta) = \begin{cases} \sum_{i+j=p+1} F_i D_1^\alpha \boxtimes F_j D_2^\beta & \text{if } \alpha + \beta \leq -1, \\ \sum_{i+j=p} F_i D_1^\alpha \boxtimes F_j D_2^\beta & \text{if } \alpha + \beta > -1. \end{cases}$$

and the weight filtration by

$$W_k(D_1^\alpha \boxtimes D_2^\beta) = \begin{cases} \sum_{i+j=k} W_i D_1^\alpha \boxtimes W_j D_2^\beta & \text{if } \alpha\beta = 0, \\ \sum_{i+j=k-1} W_i D_1^\alpha \boxtimes W_j D_2^\beta & \text{if } \alpha\beta \neq 0, \alpha+\beta \neq -1, \\ \sum_{i+j=k-2} W_i D_1^\alpha \boxtimes W_j D_2^\beta & \text{if } \alpha+\beta = -1 \end{cases} \tag{4.4}$$

for the underlying $\mathcal{D}$-modules. The weight filtration on the complex $(L_1 \otimes \mathbf{C}) \boxtimes (L_2 \otimes \mathbf{C})$ is defined in the same manner, and gives a weight filtration on $L_1 \boxtimes L_2$ via the action of the Galois group of $\mathbf{Q}$. We refer to [S] for the definition of the isomorphism $\mathrm{DR}(D_1 \boxtimes D_2) \simeq (L_1 \otimes \mathbf{C}) \boxtimes (L_2 \otimes \mathbf{C})$.

Example 4.1.6.1. Consider the Hodge structure with monodromy $H = (\mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}, -\mathrm{id}, 0) \in \mathrm{MHM}_{\mathbf{C}}^{\mathrm{mon}}$. Let us compute $H \overset{T}{\boxtimes} H$. The underlying $\mathbf{Q}$-vector space is of dimension one. Moreover, observe that $H = H^{-\frac{1}{2}}$, so that by definition, we have $W_1(H \overset{T}{\boxtimes} H) = 0$ and $W_2(H \overset{T}{\boxtimes} H) = W_0 H \boxtimes W_0 H$. Thus, $H \overset{T}{\boxtimes} H$ is pure of weight 2, with trivial monodromy, that is, it is equal to the pure Hodge structure $\mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}(-1)$.

There is also a twisted tensor product on the derived category $D^b(\mathrm{MHM}_X^{\mathrm{mon}})$, defined for $M_1, M_2 \in D^b(\mathrm{MHM}_X^{\mathrm{mon}})$ by

$$M_1 \overset{T}{\otimes} M_2 := \delta^*(M_1 \overset{T}{\boxtimes} M_2)$$

where $\delta : X \to X \times X$ is the diagonal map. It is clear from the definition that these twisted products $\overset{T}{\boxtimes}, \overset{T}{\otimes}$ coincide with the usual products $\boxtimes, \otimes$ for Hodge modules with trivial monodromy.

More generally, for any complex variety X and for varieties Y, Z above X, we have a relative twisted exterior product

$$\overset{T}{\boxtimes}_X : D^b(\mathrm{MHM}_Y^{\mathrm{mon}}) \times D^b(\mathrm{MHM}_Z^{\mathrm{mon}}) \to D^b(\mathrm{MHM}_{Y \times_X Z}^{\mathrm{mon}})$$

given for $M_1 \in D^b(\mathrm{MHM}_Y^{\mathrm{mon}})$, $M_2 \in D^b(\mathrm{MHM}_Z^{\mathrm{mon}})$ by

$$M_1 \overset{T}{\boxtimes}_X M_2 = i^*(M_1 \overset{T}{\boxtimes} M_2)$$

where i is the closed immersion $Y \times_X Z \to Y \times_{\mathbf{C}} Z$. We denote simply by $\boxtimes_X$ the relative exterior product it induced on mixed Hodge modules with trivial monodromy.

4.1.7 Grothendieck ring of mixed Hodge modules

A reference for this is [CNS], chapter 1 section 3.1, especially 3.1.4, 3.1.9 and 3.1.10. The Grothendieck group $K_0(\mathrm{MHM}_S)$ associated to the abelian category MHM_S is defined as the quotient of the free abelian group on isomorphism classes of mixed Hodge modules by relations of the form $[X]-[Y]+[Z]$ for all objects $X,Y,Z \in \mathrm{MHM}_S$ forming a short exact sequence

$$0 \to X \to Y \to Z \to 0.$$

The full subcategory of MHM_S of objects of pure weight, denoted by HM_S, is semi-simple (see [S88], Lemme 5). On the other hand, the natural group morphism

$$K_0(\mathrm{HM}_S) \to K_0(\mathrm{MHM}_S)$$

is an isomorphism, with inverse given by sending the class of a mixed Hodge module M to the sum of the classes of its graded parts. Thus, $K_0(\mathrm{MHM}_S)$ may also be seen as the quotient of the free abelian group on isomorphism classes of Hodge modules of pure weight by relations of the form $[X]-[Y]+[Z]$ for all objects $X,Y,Z \in \mathrm{HM}_S$ forming a *split* short exact sequence

$$0 \to X \to Y \to Z \to 0.$$

On the other hand, the triangulated category $D^b(\mathrm{MHM}_S)$ has a Grothendieck group $K_0^{\mathrm{tri}}(D^b(\mathrm{MHM}_S))$, which is defined as the quotient of the free abelian group on isomorphism classes of objects of $D^b(\mathrm{MHM}_S)$, by relations of the form $[X]-[Y]+[Z]$, for any objects X,Y,Z of the category $D^b(\mathrm{MHM}_S)$ fitting into a distinguished triangle

$$X \to Y \to Z \to X[1].$$

The $\otimes$ operation endows the ring $K_0^{\mathrm{tri}}(D^b(\mathrm{MHM}_S))$ with a ring structure. There is a natural group morphism

$$K_0(\mathrm{MHM}_S) \to K_0^{\mathrm{tri}}(D^b(\mathrm{MHM}_S))$$

sending the class of a mixed Hodge module M to the class of the complex defined by this Hodge module placed in degree zero. This morphism is an isomorphism, with inverse given by sending the class of any complex $M^\bullet$ of mixed Hodge modules to the alternating series of the classes of its cohomology groups $\sum_{i\in\mathbf{Z}}(-1)^i[\mathcal{H}^i(M^\bullet)]$. In what follows, we will denote this group always by $K_0(\mathrm{MHM}_S)$, and consider it as a ring via the ring structure on $K_0^{\mathrm{tri}}(D^b(\mathrm{MHM}_S))$.

As for Grothendieck rings of varieties, a morphism $f: T \to S$ of complex varieties induces a group morphism

$$f_!: K_0(\mathrm{MHM}_T) \to K_0(\mathrm{MHM}_S)$$

and a ring morphism

$$f^*: K_0(\mathrm{MHM}_S) \to K_0(\mathrm{MHM}_T).$$

In particular, for any complex variety S, $K_0(\mathrm{MHM}_S)$ is endowed with a $K_0(\mathrm{MHM}_{\mathrm{pt}})$-algebra structure.

One may also consider Grothendieck rings of mixed Hodge modules with monodromy $K_0(\mathrm{MHM}_S^{\mathrm{mon}})$, defined in the same manner, the product being induced by $\overset{T}{\otimes}$. We denote this product by $*$.

4.2 Vanishing cycles and mixed Hodge modules

4.2.1 The classical theory of vanishing cycles

Here we recall briefly how nearby and vanishing cycles are defined in the classical transcendental setting. The main reference for this is Deligne's article [Del73] in SGA 7. For a summary with a view towards mixed Hodge modules, see section 8 of Schnell's notes [Schn]. Let X be a complex manifold, $D \subset \mathbf{C}$ the open unit disc, and $f : X \to D$ a holomorphic function. We denote by $D^* = D \setminus \{0\}$ the punctured unit disc and by $\widetilde{D}^*$ its universal covering space, which may be viewed as the complex upper half plane via the covering map

$$p : \widetilde{D}^* = \{z \in \mathbf{C},\ \mathrm{Im}(z) > 0\} \begin{array}{c}\to \\ \mapsto\end{array} \begin{array}{c} D^* \\ \exp(2i\pi z)\end{array}.$$

with $z \mapsto \exp(2i\pi z)$.

We denote by X^* (resp X_0) the inverse image $f^{-1}(D^*)$ (resp. $f^{-1}(0)$), and by $\widetilde{X}^*$ the complex manifold making the rightmost square in the following diagram cartesian:

$$\begin{array}{ccccccc}
X_0 & \xhookrightarrow{i} & X & \xleftarrow{j} & X^* & \xleftarrow{p} & \widetilde{X}^* \\
\downarrow f & & \downarrow f & & \downarrow f & & \downarrow \\
\{0\} & \hookrightarrow & D & \hookleftarrow & D^* & \xleftarrow{p} & \widetilde{D}^*
\end{array}$$

The nearby cycle functor

$$\psi_f : D^b_c(X) \to D^b_c(X_0)$$

is defined in the following way: for $\mathcal{F}^\bullet \in D^b_c(X)$ a bounded constructible complex of sheaves on X, we put

$$\psi_f \mathcal{F}^\bullet = i^{-1} R(j \circ p)_* \, (j \circ p)^{-1} \mathcal{F}^\bullet.$$

The deck transformation $z \mapsto z + 1$ on $\widetilde{D}^*$ induces an automorphism of $\psi_f \mathcal{F}^\bullet$ called the *monodromy.* The adjunction morphism

$$\mathcal{F}^\bullet \to R(j \circ p)_* \, (j \circ p)^{-1} \mathcal{F}^\bullet$$

gives, after applying the functor i^{-1}, a morphism

$$i^{-1} \mathcal{F}^\bullet \to \psi_f \mathcal{F}^\bullet.$$

The vanishing cycles complex $\varphi_f \mathcal{F}^\bullet$ of $\mathcal{F}^\bullet$ at 0 is defined as the cone of this morphism, so that there is a distinguished triangle

$$i^{-1} \mathcal{F}^\bullet \to \psi_f \mathcal{F}^\bullet \to \varphi_f \mathcal{F}^\bullet \to i^{-1} \mathcal{F}^\bullet[1].$$

The functors ψ_f and φ_f take distinguished triangles to distinguished triangles, and commute with shifting of complexes.

A theorem of Gabber (see [Bry], Théorème 1.2) says that if $\mathcal{F}^\bullet$ is a perverse complex, then both ${}^p\psi_f \mathcal{F}^\bullet := \psi_f \mathcal{F}^\bullet[-1]$ and ${}^p\varphi_f \mathcal{F}^\bullet := \varphi_f \mathcal{F}^\bullet[-1]$ are perverse.

Lemma 4.2.1.1. *Let X be a complex algebraic variety, $f : X \to \mathbf{A}^1_{\mathbf{C}}$ a morphism and $\mathcal{F}^\bullet \in D^b_c(X)$. Then $\varphi_{f-a}\mathcal{F}^\bullet = 0$ for all but a finite number of $a \in \mathbf{C}$.*

Proof. The case when $\mathcal{F}^\bullet$ is a constructible sheaf in degree zero follows from theorem 2.13 in [Del77]. Indeed, though the latter is formulated in an ℓ-adic setting, the proof is formal and goes over to the complex setting. Another argument may be given using local triviality results (see e.g. Corollaire 5.1 in [Verdier]).

Since vanishing cycles commute with shifting of complexes, we may then proceed by induction on the amplitude of the complex $\mathcal{F}^\bullet$: there are complexes $\mathcal{G}^\bullet$ and $\mathcal{G}'^\bullet$ of strictly smaller amplitude fitting into a distinguished triangle

$$\mathcal{G}^\bullet \to \mathcal{F}^\bullet \to \mathcal{G}'^\bullet \to \mathcal{G}^\bullet[1].$$

Applying the functor φ_{f-a} and using the induction hypothesis, we get, for all a but a finite number, $\varphi_{f-a}\mathcal{F}^\bullet = 0$ □

Remark 4.2.1.2. There are several conventions and notations concerning nearby and vanishing cycles, which may seem quite confusing. In this section we use the most common one, coming from SGA, which is also the one used by Saito. Another convention is the one by Kashiwara and Shapira from [KS]. It has the same definition $\psi_f^{KS} = \psi_f$ for nearby cycles, but vanishing cycles are shifted by one:

$$\varphi_f^{KS} := \varphi_f[-1],$$

so that in Kashiwara and Shapira's theory, the above distinguished triangle is shifted and takes the form

$$\varphi_f^{KS}\mathcal{F}^\bullet \to i^{-1}\mathcal{F}^\bullet \to \psi_f^{KS} \to \varphi_f^{KS}\mathcal{F}^\bullet[1]. \tag{4.5}$$

This latter convention is used, e.g., in David Massey's paper [Mas] on the Thom-Sebastiani theorem in the derived category of constructible sheaves, because the shift chosen by Kashiwara and Shapira is the one that makes the theorem of Thom-Sebastiani work. For this reason, it is in fact also the convention used in the motivic setting of chapter 2: the distinguished triangle (4.5) induces, in the triangulated Grothendieck ring of the category $D^b_c(X_0)$, the identity

$$[\varphi_f^{KS}\mathcal{F}^\bullet] = [i^{-1}\mathcal{F}^\bullet] - [\psi_f^{KS}\mathcal{F}^\bullet],$$

which corresponds to the way we defined motivic vanishing cycles in section 2.2.5 of chapter 2. This definition, which we took from Lunts and Schnürer's work [LS16b] is the one which, simultaneously, enables us to define a group morphism using total motivic vanishing cycles and makes the motivic Thom-Sebastiani theorem work, so that this group morphism is actually a ring morphism, our motivic vanishing cycles measure. Denef and Loeser's motivic vanishing cycles $\mathscr{S}_f^\varphi$ which are equal to $(-1)^{\dim X}$ times our motivic vanishing cycles, satisfy Thom Sebastiani (each side of the Thom-Sebastiani equality being multiplied by the same power of -1), but can't be combined into a group morphism because of obvious sign issues.

4.2.2 Nearby and vanishing cycles for Hodge modules

For a morphism $f : X \to \mathbf{A}^1$ on a complex variety X, denoting $X_0(f) = f^{-1}(0)$, there are nearby and vanishing cycles functors

$$\psi_f^{\mathrm{Hdg}}, \varphi_f^{\mathrm{Hdg}} : \mathrm{MHM}_X \to \mathrm{MHM}^{\mathrm{mon}}_{X_0(f)}$$

lifting the corresponding functors ${}^p\psi_f$ and ${}^p\varphi_f$ on perverse sheaves. The monodromy operator is quasi-unipotent, so that its semisimple part is of finite order. The action of T_s is given by this semisimple part, whereas the action of N is given by the logarithm of the unipotent part of the monodromy. For any mixed Hodge module $M \in \mathrm{MHM}_X$, there are decompositions

$$\psi_f^{\mathrm{Hdg}}(M) = \psi_{f,1}^{\mathrm{Hdg}}(M) \oplus \psi_{f,\neq 1}^{\mathrm{Hdg}}(M)$$

and

$$\varphi_f^{\mathrm{Hdg}}(M) = \varphi_{f,1}^{\mathrm{Hdg}}(M) \oplus \varphi_{f,\neq 1}^{\mathrm{Hdg}}(M)$$

in the category $\mathrm{MHM}_{X_0(f)}$ where $\psi_{f,1}^{\mathrm{Hdg}}(M) = \mathrm{Ker}(T_s - \mathrm{id})$ and $\psi_{f,\neq 1}^{\mathrm{Hdg}}(M) = \mathrm{Ker}(T_s^{d-1} + \ldots + T_s + \mathrm{id})$ where d is the order of T_s (same definition for φ_f).

The following compatibility result with proper morphisms is classical in the theory of nearby and vanishing cycles, and, in the case of Hodge modules, follows from [S90] Theorem 2.14.

Lemma 4.2.2.1. *Let $p : X \to Y$ and $f : Y \to \mathbf{A}^1$ be morphisms of complex varieties. Assume that p is proper.*

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ & \searrow^{f\circ p} & \downarrow f \\ & & \mathbf{A}^1 \end{array}$$

Denoting by $\tilde{p}$ the restriction of p to $(f \circ p)^{-1}(0)$, for any $M \in \mathrm{MHM}_X$ there are isomorphisms

$$\tilde{p}_*(\psi_{f\circ p}(M)) \simeq \psi_f(p_*(M))$$

and

$$\tilde{p}_*(\varphi_{f\circ p}(M)) \simeq \varphi_f(p_*(M))$$

4.2.3 Total vanishing cycles

Let X be a complex variety, and denote by pr the projection $\mathbf{A}^1_X \to \mathbf{A}^1_{\mathbf{C}}$. By lemma 4.2.1.1 and faithfulness of the functor rat, there is a well-defined *total vanishing cycles functor*:

$$\begin{array}{cccc} \varphi_X^{\mathrm{tot}} : & \mathrm{MHM}_{\mathbf{A}^1_X} & \to & \mathrm{MHM}_X^{\mathrm{mon}} \\ & M & \mapsto & \bigoplus_{a\in\mathbf{C}} \varphi^{\mathrm{Hdg}}_{\mathrm{pr}-a} M \end{array}$$

It satisfies a Thom-Sebastiani property:

Proposition 4.2.3.1. *For all $M_1, M_2 \in D^b(\mathrm{MHM}_{\mathbf{A}^1_X})$, denoting by* add *the addition morphism $\mathbf{A}^1_X \times \mathbf{A}^1_X \to \mathbf{A}^1_{X^2}$, one has the isomorphism*

$$\varphi^{\mathrm{tot}}_{X^2}\left((\mathrm{add}_!)(M_1 \boxtimes M_2)\right) \simeq \varphi^{\mathrm{tot}}_X(M_1) \overset{T}{\boxtimes} \varphi^{\mathrm{tot}}_X(M_2)$$

in $D^b(\mathrm{MHM}^{\mathrm{mon}}_{X^2})$.

Proof. Saito's Thom-Sebastiani theorem for Hodge modules (see [S]) gives, for any $a_1, a_2 \in \mathbf{C}$, an isomorphism

$$i^*_{a_1,a_2}\varphi^{\mathrm{Hdg}}_{\mathrm{pr}\oplus\mathrm{pr}-a_1-a_2}(M_1 \boxtimes M_2) \simeq \varphi^{\mathrm{Hdg}}_{\mathrm{pr}-a_1}M_1 \overset{T}{\boxtimes} \varphi^{\mathrm{Hdg}}_{\mathrm{pr}-a_2}M_2,$$

in $D^b(\mathrm{MHM}^{\mathrm{mon}}_{\mathrm{pr}^{-1}(a_1)\times\mathrm{pr}^{-1}(a_2)})$, where

$$i_{a_1,a_2} : \mathrm{pr}^{-1}(a_1) \times \mathrm{pr}^{-1}(a_2) \to (\mathrm{pr}\oplus\mathrm{pr})^{-1}(a_1 + a_2)$$

is the natural inclusion. For any $a \in \mathbf{C}$, the products $\mathrm{pr}^{-1}(a_1) \times \mathrm{pr}^{-1}(a_2)$ for all a_1, a_2 such that $a_1 + a_2 = a$ form a partition of $(\mathrm{pr}\oplus\mathrm{pr})^{-1}(a)$. Therefore, taking the direct sum over all a_1, a_2, we get an isomorphism

$$\bigoplus_{a\in\mathbf{C}} \varphi^{\mathrm{Hdg}}_{\mathrm{pr}\oplus\mathrm{pr}-a}(M_1 \boxtimes M_2) \simeq \varphi^{\mathrm{tot}}_X(M_1) \overset{T}{\boxtimes} \varphi^{\mathrm{tot}}_X(M_2)$$

in $D^b(\mathrm{MHM}^{\mathrm{mon}}_{X^2})$. As in the motivic setting, a compactification argument allows us to write the left-hand side as

$$\bigoplus_{a\in\mathbf{C}} \varphi^{\mathrm{Hdg}}_{\mathrm{pr}\oplus\mathrm{pr}-a}(M_1 \boxtimes M_2) \simeq \bigoplus_{a\in\mathbf{C}} \varphi^{\mathrm{Hdg}}_{\mathrm{pr}-a}((\mathrm{add})_!(M_1 \boxtimes M_2)),$$

whence the result. □

4.3 Symmetric products and vanishing cycles

4.3.1 Symmetric products of mixed Hodge modules

From now on, we are going to take symmetric products, so we assume the varieties to be quasi-projective over $\mathbf{C}$. A notion of *symmetric power* of a complex of mixed Hodge modules was defined by Maxim, Saito and Schürmann in [MSS]. Here we are going to explain how it extends to the setting of mixed Hodge modules with monodromy.

Theorem 1.9 in [MSS] gives, for an integer $n \geq 1$, bounded complexes of Hodge modules $M_1, \ldots, M_n$ on a quasi-projective complex variety X and for all $\sigma \in \mathfrak{S}_n$, an isomorphism

$$\sigma^\sharp : M_1 \boxtimes \ldots \boxtimes M_n \xrightarrow{\sim} \sigma_*\left(M_{\sigma(1)} \boxtimes \ldots \boxtimes M_{\sigma(n)}\right)$$

in $D^b(\mathrm{MHM}_{X^n})$ compatible with the analogous isomorphism on the underlying complexes of constructible sheaves. These isomorphisms in fact induce a right action of $\mathfrak{S}_n$ on the exterior product $M_1 \boxtimes \ldots \boxtimes M_n$. When X is smooth, these isomorphisms are constructed from analogous isomorphisms on the underlying complexes of $\mathcal{D}$-modules and constructible sheaves, which are checked to be compatible via the De Rham functor (proposition 1.5 in [MSS]). The general non-smooth case is deduced from this by an embedding argument.

Note that the above generalises to complexes of mixed Hodge modules with monodromy, if we replace the ordinary exterior product $\boxtimes$ by its twisted counterpart $\overset{T}{\boxtimes}$. Indeed, the underlying complexes of $\mathcal{D}$-modules and constructible sheaves and the De Rham functor relating them remain the same, and the monodromy is compatible with the above action.

Let now M be a bounded complex of mixed Hodge modules with monodromy over a quasi-projective complex variety X. Let $n \geq 1$ be an integer, and denote by $\pi : X^n \to S^n X$ the canonical projection. The above construction leads to the definition of a right $\mathfrak{S}_n$-action on the complex $\pi_*(\overset{T}{\boxtimes}{}^n M)$. Finally, as in [MSS], the idempotent $e = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \sigma \in \mathbf{Q}[\mathfrak{S}_n]$ defines an idempotent of $\pi_*(\overset{T}{\boxtimes}{}^n M)$ in the category $D^b(\mathrm{MHM}^{\mathrm{mon}}_{S^n X})$, which splits by corollary 2.10 in [BS], meaning that we may write

$$\pi_*(\overset{T}{\boxtimes}{}^n M) = \mathrm{Ker}(e) \oplus \mathrm{Im}(e).$$

For any bounded complex of mixed Hodge modules M with monodromy on a complex variety X, the symmetric power $S^n(M)$ is then defined to be

$$S^n(M) = \left(\pi_*\left(\overset{T}{\boxtimes}{}^n M\right)\right)^{\mathfrak{S}_n} := \mathrm{Im}(e) \in D^b(\mathrm{MHM}^{\mathrm{mon}}_{S^n X}),$$

where $\pi : X^n \to S^n X$ is the canonical projection.

Remark 4.3.1.1. In the case where M has trivial monodromy, we recover the definition from [MSS]. In particular, in this case $S^n M$ is an element of $D^b(\mathrm{MHM}_{S^n X})$.

As in chapter 3, section 3.5.1, we may define a multiplicative group structure on the product $\prod_{i\geq 1} K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^i X})$, by using the twisted exterior product $\overset{T}{\boxtimes}$: for all $a = (a_i)_{i\geq 1}$ and $b = (b_i)_{i\geq 1}$ in the product $\prod_{i\geq 1} K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^i X})$, put, for every $n \geq 1$,

$$(ab)_n = \sum_{k=0}^{n} a_i \boxtimes b_{n-i}$$

where by convention $a_0 = b_0 = 1$, and $a_i \overset{T}{\boxtimes} b_{n-i}$ is the image of (a_i, b_{n-i}) through the composition

$$K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^i X}) \times K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^{n-i} X}) \xrightarrow{\overset{T}{\boxtimes}} K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^i X \times S^{n-i} X}) \to K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^n X})$$

where the latter morphism is obtained from the quotient map $X^n \to S^n X$ by passing to the quotient with respect to the natural permutation action of the group $\mathfrak{S}_i \times \mathfrak{S}_{n-i}$ on X^n. We denote this group by $K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^\bullet X})$.

Lemma 4.3.1.2. *There is a unique group morphism*

$$S_X^{\mathrm{Hdg}} : K_0(\mathrm{MHM}_X^{\mathrm{mon}}) \to K_0(\mathrm{MHM}_{S^\bullet X}^{\mathrm{mon}}),$$

such that the image of the class of a complex of mixed Hodge modules M over X is the family of classes $([S^n M])_{n\geq 1}$.

Proof. We define a map S_X^{Hdg} from the free abelian group on Hodge modules to the group $K_0(\mathrm{MHM}_{S^\bullet X}^{\mathrm{mon}})$ by putting $S_X^{\mathrm{Hdg}}(M) = ([S^n M])_{n\geq 1}$. By the discussion about the definition of $K_0(\mathrm{MHM}_X^{\mathrm{mon}})$, to show that S_X^{Hdg} passes to the quotient, it suffices to show that whenever M, M_0, M_1 are Hodge modules such that $M = M_0 \oplus M_1$, we have $S_X^{\mathrm{Hdg}}(M) = S_X^{\mathrm{Hdg}}(M_0) S_X^{\mathrm{Hdg}}(M_1)$. In fact, denoting for every $k \in \{0, \ldots, n\}$ by i_k the natural morphism $S^{n-k}X \times S^k X \to S^n X$, we will show that for every $n \geq 1$, we have, in $K_0(\mathrm{MHM}_{S^n X}^{\mathrm{mon}})$, the equality

$$S^n M = \sum_{k=0}^{n} i_{k,*}\left(S^{n-k} M_0 \overset{T}{\boxtimes} S^k M_1\right).$$

For this, note that in $D^b(\mathrm{MHM}_{X^n}^{\mathrm{mon}})$ we have the direct sum decomposition

$$\overset{T}{\boxtimes}{}^n M = \bigoplus_{(\epsilon_1,\ldots,\epsilon_n)\in\{0,1\}^n} M_{\epsilon_1} \overset{T}{\boxtimes} \ldots \overset{T}{\boxtimes} M_{\epsilon_n} = \bigoplus_{k=0}^{n} \bigoplus_{\substack{(\epsilon_1,\ldots,\epsilon_n)\in\{0,1\}^n \\ \epsilon_1+\ldots+\epsilon_n = k}} M_{\epsilon_1} \overset{T}{\boxtimes} \ldots \overset{T}{\boxtimes} M_{\epsilon_n}.$$

Let $M_{(k)} := \oplus_{\substack{(\epsilon_1,\ldots,\epsilon_n)\in\{0,1\}^n \\ \epsilon_1+\ldots+\epsilon_n = k}} M_{\epsilon_1} \overset{T}{\boxtimes} \ldots \overset{T}{\boxtimes} M_{\epsilon_n}$ and let $\pi : X^n \to S^n X$ be the quotient morphism. After applying the exact functor π_* to the above decomposition, we get

$$\pi_*\left(\overset{T}{\boxtimes}{}^n M\right) = \bigoplus_{k=0}^{n} \pi_*(M_{(k)})$$

in $D^b(\mathrm{MHM}_{S^n X}^{\mathrm{mon}})$. Observe that each of the factors $\pi_*(M_{(k)})$ is stable under the action of the symmetric group $\mathfrak{S}_n$, so that we have

$$\pi_*\left(\overset{T}{\boxtimes}{}^n M\right)^{\mathfrak{S}_n} = \bigoplus_{k=0}^{n} (\pi_* M_{(k)})^{\mathfrak{S}_n}.$$

It suffices to prove that for every k, $i_{k,*}\left(S^{n-k} M_0 \overset{T}{\boxtimes} S^k M_1\right)$ is isomorphic to $(\pi_* M_{(k)})^{\mathfrak{S}_n}$. We denote again by e the restriction of the idempotent e to $\pi_* M_{(k)}$.

Fix $k \in \{0, \ldots, n\}$, and denote by $\pi_k : X^k \to S^k X$ and $\pi_{n-k} : X^{n-k} \to S^{n-k} X$ the quotient maps. The corresponding idempotent on $(\pi_{n-k})_* \overset{T}{\boxtimes}{}^{n-k} M_0$ (resp. $(\pi_k)_* \overset{T}{\boxtimes}{}^k M_1$) will be denoted by e^0 (resp. e^1). By the commutativity of the diagram

$$\begin{array}{ccc} X^{n-k} \times X^k & \overset{\mathrm{id}}{\to} & X^n \\ \downarrow {\scriptstyle \pi_{n-k}\times\pi_k} & & \downarrow {\scriptstyle \pi} \\ S^{n-k}X \times S^k X & \overset{i_k}{\to} & S^n X \end{array}$$

and exactness of π_*, the inclusion of the direct factor $(\overset{T}{\boxtimes}{}^{n-k} M_0) \overset{T}{\boxtimes} (\overset{T}{\boxtimes}{}^{k} M_1) \to M_{(k)}$ induces a monomorphism

$$i_{k,*}\left((\pi_{n-k})_*\left(\overset{T}{\boxtimes}{}^{n-k} M_0\right) \overset{T}{\boxtimes} (\pi_k)_*\left(\overset{T}{\boxtimes}{}^{k} M_1\right)\right) \xrightarrow{f} \pi_* M_{(k)}.$$

The $\mathfrak{S}_{n-k} \times \mathfrak{S}_k$-action on the left-hand side is compatible with the $\mathfrak{S}_n$-action on the right-hand side, when $\mathfrak{S}_{n-k} \times \mathfrak{S}_k$ is seen in a natural way as a subgroup of $\mathfrak{S}_n$. Therefore, taking invariants, we have a monomorphism

$$i_{k,*}\left(S^{n-k} M_0 \overset{T}{\boxtimes} S^k M_1\right) \xrightarrow{\tilde{f}} (\pi_* M_{(k)})^{\mathfrak{S}_n},$$

fitting into the commutative diagram

$$\begin{array}{ccc}
i_{k,*}\left((\pi_{n-k})_*\left(\overset{T}{\boxtimes}{}^{n-k} M_0\right) \overset{T}{\boxtimes} (\pi_k)_*\left(\overset{T}{\boxtimes}{}^{k} M_1\right)\right) & \xrightarrow{f} & \pi_* M_{(k)} \\
\big\downarrow{\scriptstyle i_{k,*}(e^0 \overset{T}{\boxtimes} e^1)} & & \big\downarrow{\scriptstyle e} \\
i_{k,*}\left(S^{n-k} M_0 \overset{T}{\boxtimes} S^k M_1\right) & \xrightarrow{\tilde{f}} & (\pi_* M_{(k)})^{\mathfrak{S}_n}
\end{array}$$

Since e is surjective, $\tilde{f}$ is an isomorphism. □

Definition 4.3.1.3. For any $\mathfrak{a} \in K_0(\mathrm{MHM}_X^{\mathrm{mon}})$ and any $n \geq 1$, we define $S^n\mathfrak{a}$ to be the element of $K_0(\mathrm{MHM}_{S^nX}^{\mathrm{mon}})$ given by the n-th component of $S^{\mathrm{Hdg}}(\mathfrak{a})$.

Remark 4.3.1.4. If $\mathfrak{a}$ is an element of $K_0(\mathrm{MHM}_X)$ (i.e. has trivial monodromy), then $S^n\mathfrak{a}$ is an element of $K_0(\mathrm{MHM}_{S^nX})$.

4.3.2 Compatibility between symmetric products and total vanishing cycles

Denote by add_n the addition map $\mathbf{A}^n \to \mathbf{A}^1$ on the group scheme $\mathbf{A}^n$, and for any quasi-projective variety Y, by π_Y the quotient map $Y^n \to S^nY$. Since add_n is invariant via the permutation of the coordinates of $\mathbf{A}^1$, for any quasi-projective complex variety X, it induces a morphism $\overline{\mathrm{add}}_n$ fitting into a commutative diagram

$$\begin{array}{ccc}
(\mathbf{A}^1_X)^n & \xrightarrow{\mathrm{add}_n} & \mathbf{A}^1_{X^n} \\
{\scriptstyle \pi_{\mathbf{A}^1_X}}\big\downarrow & & \big\downarrow{\scriptstyle \pi'_X} \\
S^n(\mathbf{A}^1_X) & \xrightarrow{\overline{\mathrm{add}}_n} & \mathbf{A}^1_{S^nX}
\end{array}$$

where we denote by π'_X the morphism $\mathbf{A}^1_{X^n} \to \mathbf{A}^1_{S^nX}$ induced by $\pi_X : X^n \to S^nX$.

Let $M \in D^b(\mathrm{MHM}_{\mathbf{A}^1_X})$. From 4.3.1 we know that there is a $\mathfrak{S}_n$-action on the mixed Hodge module $\boxtimes^n M$ on $(\mathbf{A}^1_X)^n$, compatible with the permutation action of $\mathfrak{S}_n$ on $(\mathbf{A}^1_X)^n$.

Since $\pi'_X \circ \mathrm{add}_n$ is equivariant if one equips $\mathbf{A}^1_{S^nX}$ with the trivial $\mathfrak{S}_n$-action, this $\mathfrak{S}_n$-action induces a $\mathfrak{S}_n$-action on $(\pi'_X \circ \mathrm{add}_n)_! (\boxtimes^n M)$. The relation $\pi'_X \circ \mathrm{add}_n = \overline{\mathrm{add}}_n \circ \pi_{\mathbf{A}^1_X}$ shows that the functor $(\overline{\mathrm{add}}_n)_!$ sending $(\pi_{\mathbf{A}^1_X})_*(\boxtimes^n M)$ to $(\pi'_X \circ \mathrm{add}_n)_!(\boxtimes^n M)$ is compatible with $\mathfrak{S}_n$-actions, which, taking invariants, gives us the relation:

Lemma 4.3.2.1. *Let $M \in D^b(\mathrm{MHM}_{\mathbf{A}^1_X})$. One has*

$$(\overline{\mathrm{add}}_n)_! S^n M = \left((\pi'_X \circ \mathrm{add}_n)_! \left(\boxtimes^n M\right)\right)^{\mathfrak{S}_n} \tag{4.6}$$

in $D^b(\mathrm{MHM}_{\mathbf{A}^1_{S^nX}})$.

Proposition 4.3.2.2. *For any $M \in D^b(\mathrm{MHM}_{\mathbf{A}^1_X})$, we have*

$$\varphi^{\mathrm{tot}}_{S^nX}\left(\left(\overline{\mathrm{add}}_n\right)_! S^n M\right) = S^n\left(\varphi^{\mathrm{tot}}_X(M)\right)$$

in $D^b(\mathrm{MHM}^{\mathrm{mon}}_{S^nX})$.

Proof. For all $M_1, \dots M_n \in D^b(\mathrm{MHM}_{\mathbf{A}^1_X})$, the Thom-Sebastiani theorem for φ^{tot} says

$$\varphi^{\mathrm{tot}}_{X^n}\left((\mathrm{add}_n)_! \left(M_1 \boxtimes \dots \boxtimes M_n\right)\right) \simeq \varphi^{\mathrm{tot}}_X(M_1) \overset{T}{\boxtimes} \dots \overset{T}{\boxtimes} \varphi^{\mathrm{tot}}_X(M_n)$$

in $D^b(\mathrm{MHM}^{\mathrm{mon}}_{X^n})$. Composing with the functor $(\pi_X)_*$ where $\pi_X : X^n \to S^nX$ is the projection, and using lemma 4.2.2.1, we have

$$\varphi^{\mathrm{tot}}_{S^nX}((\pi'_X \circ \mathrm{add}_n)_!(M_1 \boxtimes \dots \boxtimes M_n)) \simeq (\pi_X)_*\left(\varphi^{\mathrm{tot}}_X(M_1) \overset{T}{\boxtimes} \dots \overset{T}{\boxtimes} \varphi^{\mathrm{tot}}_X(M_n)\right),$$

where π'_X is the morphism $\mathbf{A}^1_{X^n} \to \mathbf{A}^1_{S^nX}$ induced by π_X. In the same manner, for every $\sigma \in \mathfrak{S}_n$, we also have

$$\begin{aligned} &\varphi^{\mathrm{tot}}_{S^nX}((\pi'_X \circ \mathrm{add}_n)_!(M_{\sigma^{-1}(1)} \boxtimes \dots \boxtimes M_{\sigma^{-1}(n)})) \\ &\simeq (\pi_X)_*\left(\varphi^{\mathrm{tot}}_X(M_{\sigma^{-1}(1)}) \overset{T}{\boxtimes} \dots \overset{T}{\boxtimes} \varphi^{\mathrm{tot}}_X(M_{\sigma^{-1}(n)})\right), \end{aligned}$$

Thus, for any $M \in D^b(\mathrm{MHM}_{\mathbf{A}^1_X})$, the idempotent $\frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \sigma \in \mathbf{Z}[\mathfrak{S}_n]$ induces idempotents e and e' of

$$(\pi'_X \circ \mathrm{add}_n)_! \left(\boxtimes^n M\right) \in D^b(\mathrm{MHM}_{\mathbf{A}^1_{X^n}})$$

and

$$(\pi_X)_* \overset{T}{\boxtimes}{}^n (\varphi^{\mathrm{tot}}_X(M)) \in D^b(\mathrm{MHM}^{\mathrm{mon}}_{X^n})$$

such that $\varphi^{\mathrm{tot}}_{S^nX}(e) = e'$. Therefore, splittings being preserved by any additive functor, we have

$$\varphi^{\mathrm{tot}}_{S^nX}\left(\left((\pi'_X \circ \mathrm{add}_n)_! \boxtimes^n M\right)^{\mathfrak{S}_n}\right) \simeq \left(\overset{T}{\boxtimes}{}^n(\varphi^{\mathrm{tot}}_X(M))\right)^{\mathfrak{S}_n},$$

which, using the isomorphism (4.6) above, gives the result. $\square$

4.4 Compatibility with motivic vanishing cycles

4.4.1 The Hodge realisation

Let S be a complex variety, and let $X \xrightarrow{p} S$ be an S-variety, endowed with a μ_n-action σ for some $n \geq 1$. Then $\sigma(e^{\frac{2i\pi}{n}})$ induces an automorphism $T_s(\sigma)$ of finite order on each cohomology group of the complex of mixed Hodge modules $p_!\mathbf{Q}_X^{\mathrm{Hdg}}$ (see notation 4.1.5.1). Thus, as explained in [GLM] section 3.16, this defines a group morphism

$$\begin{array}{cccc} \chi_S^{\mathrm{Hdg}}: & \mathscr{M}_S^{\hat{\mu}} & \to & K_0(\mathrm{MHM}_S^{\mathrm{mon}}) \\ & [X \xrightarrow{p} S, \sigma] & \mapsto & \sum_{i\in\mathbf{Z}}(-1)^i[\mathcal{H}^i(f_!\mathbf{Q}_X^{\mathrm{Hdg}}), T_s(\sigma), 0] \end{array}$$

called the *Hodge realisation morphism*. Here are some properties of this Hodge realisation (for a proof, see the ideas in [GLM], section 6):

Proposition 4.4.1.1. *Let S, T be complex varieties.*

1. *The morphism χ_S^{Hdg} commutes with twisted exterior products, that is, for any $\mathfrak{a} \in \mathscr{M}_S^{\hat{\mu}}, \mathfrak{b} \in \mathscr{M}_T^{\hat{\mu}}$, we have*

$$\chi_{S\times T}^{\mathrm{Hdg}}(\Psi(\mathfrak{a} \boxtimes \mathfrak{b})) = \chi_S^{\mathrm{Hdg}}(\mathfrak{a}) \overset{T}{\boxtimes} \chi_T^{\mathrm{Hdg}}(\mathfrak{b})$$

where $\Psi : \mathscr{M}_{S\times T}^{\hat{\mu}\times\hat{\mu}} \to \mathscr{M}_{S\times T}^{\hat{\mu}}$ is the convolution morphism from chapter 2, section 2.2.1.

2. *For any morphism $f : T \to S$ between complex varieties, we have*

$$\chi_S^{\mathrm{Hdg}} \circ f_! = f_! \circ \chi_T^{\mathrm{Hdg}} \quad \textit{and} \quad \chi_T^{\mathrm{Hdg}} \circ f^* = f^* \circ \chi_S^{\mathrm{Hdg}}.$$

3. *The group morphism χ_S^{Hdg} is a ring morphism*

$$(\mathscr{M}_S^{\hat{\mu}}, *) \to (K_0(\mathrm{MHM}_S^{\mathrm{mon}}), *).$$

Example 4.4.1.2. For $S = \operatorname{Spec} \mathbf{C}$, we have, for any separated complex variety X with μ_n-action σ,

$$\chi_{\mathrm{pt}}^{\mathrm{Hdg}}([X, \sigma]) = \sum_{i=0}^{\dim X} (-1)^i [H_c^i(X(\mathbf{C}), \mathbf{Q}), T_s(\sigma)],$$

where $[H_c^i(X(\mathbf{C}), \mathbf{Q}), \sigma]$ is the class of the mixed Hodge structure defined by Deligne [Del] on the singular cohomology group $H_c^i(X(\mathbf{C}), \mathbf{Q})$, together with the automorphism of finite order induced by $\sigma(e^{\frac{2i\pi}{n}})$.

Example 4.4.1.3. In section 2.4 of chapter 2, we showed that the motivic vanishing cycles φ_{x^2} of the function $\mathbf{A}^1 \to \mathbf{A}^1$, $x \mapsto x^2$ are equal to the class $1 - [\tilde{E}, \mu_2] \in \mathscr{M}_{\mathbf{C}}^{\hat{\mu}}$, where $[\tilde{E}, \mu_2]$ is the class of the union of two points with permutation action by μ_2. The Hodge realisation of $[\tilde{E}, \mu_2]$ is the Hodge structure with monodromy

$$\left((\mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}})^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),$$

which may be decomposed as a direct sum

$$(\mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}, \mathrm{id}) \oplus (\mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}, -\mathrm{id}).$$

Thus, the class $1 - [\tilde{E}, \mu_2]$ maps to the class of the Hodge structure with monodromy $H = (\mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}, -\mathrm{id}, 0)$.

We may conclude that the equality

$$(1 - [\tilde{E}, \mu_2]) * (1 - [\tilde{E}, \mu_2]) = \mathbf{L}$$

from section 2.4 of chapter 2 becomes the equality

$$H \overset{T}{\boxtimes} H = \mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}(-1)$$

in $K_0(\mathrm{MHM}_{\mathrm{pt}}^{\mathrm{mon}})$. It is consistent with example 4.1.6.1, where we actually proved the two sides were isomorphic.

Remark 4.4.1.4. The above Hodge realisation induces the classical Hodge realisation $\chi_S^{\mathrm{Hdg}} : \mathscr{M}_S \to K_0(\mathrm{MHM}_S)$ (see e.g. [CNS], Chapter 1, paragraph 3.3) on the corresponding rings without monodromy. In what follows, we are also going to use this realisation.

4.4.2 Compatibility with symmetric products

Lemma 4.4.2.1. *Let $p : Y \to X$ be a quasi-projective variety over X. Then $\chi_{S^nX}^{\mathrm{Hdg}}(S^nY) = S^n(\chi_X^{\mathrm{Hdg}}Y)$ in $D^b(\mathrm{MHM}_{S^nX})$.*

Proof. The relation we want is

$$(S^n p)_! \mathbf{Q}_{S^nY}^{\mathrm{Hdg}} = S^n(p_! \mathbf{Q}_Y^{\mathrm{Hdg}}),$$

which is exactly equation (1.11) in [CMSSY]. □

Lemma 4.4.2.2. *The diagram*

$$\begin{array}{ccc} (\mathscr{M}_X, +) & \xrightarrow{S} & \left(\prod_{n\geq 1} \mathscr{M}_{S^nX}, \cdot\right) \\ \downarrow & & \downarrow \\ (K_0(\mathrm{MHM}_X), +) & \xrightarrow{S_X^{\mathrm{Hdg}}} & \left(\prod_{n\geq 1} K_0(\mathrm{MHM}_{S^nX}), \cdot\right) \end{array}$$

where the vertical arrows are given by the Hodge realisation morphisms, commutes.

Proof. We checked it is true for classes of quasi-projective varieties Y over X in lemma 4.4.2.1, and such classes generate KVar_X. Moreover, note that for any $M \in D^b(\mathrm{MHM}_X)$, any $n \geq 1$ and any $k \in \mathbf{Z}$, we have $S^n(M(k)) = (S^n M)(nk)$ (recall that $M(k) = M \otimes \mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}(k)$). Thus, for any $\mathfrak{a} \in \mathrm{KVar}_X$, any $k \in \mathbf{Z}$ and any $n \geq 1$, we have

$$\begin{aligned} \chi^{\mathrm{Hdg}}_{S^n X}(S^n(\mathbf{L}^{-k}\mathfrak{a})) &= \chi^{\mathrm{Hdg}}_{S^n X}(\mathbf{L}^{-kn} S^n \mathfrak{a}) \\ &= (\chi^{\mathrm{Hdg}}_{S^n X}(S^n \mathfrak{a}))(kn) \\ &= S^n(\chi^{\mathrm{Hdg}}_X(\mathfrak{a})(k)) \\ &= S^n(\chi^{\mathrm{Hdg}}_X(\mathbf{L}^{-k}\mathfrak{a})) \end{aligned}$$

which shows that the diagram commutes also on the localisation. □

4.4.3 Grothendieck rings of Hodge modules over the affine line

There are two natural $K_0(\mathrm{MHM}_X)-$algebra structures on the group $K_0(\mathrm{MHM}_{\mathbf{A}^1_X})$. The first one is given by the pullback morphism

$$(\epsilon_X)^* : K_0(\mathrm{MHM}_X) \to K_0(\mathrm{MHM}_{\mathbf{A}^1_X})$$

where $\epsilon_X : \mathbf{A}^1_X \to X$ is the structural morphism.

Denote by $\star$ the product induced by the addition morphism $\mathrm{add} : \mathbf{A}^2_X \to \mathbf{A}^1_X$:

$$\star : K_0(\mathrm{MHM}_{\mathbf{A}^1_X}) \times K_0(\mathrm{MHM}_{\mathbf{A}^1_X}) \xrightarrow{\boxtimes_X} K_0(\mathrm{MHM}_{\mathbf{A}^2_X}) \xrightarrow{\mathrm{add}_!} K_0(\mathrm{MHM}_{\mathbf{A}^1_X})$$

(see the end of section 4.1.6 for the definition of $\boxtimes_X$ in the context of mixed Hodge modules). Moreover, we define $i_X : X \to \mathbf{A}^1_X$ and $i^2_X : X \to \mathbf{A}^2_X$ to be the morphisms induced by the inclusions $\{0\} \to \mathbf{A}^1_{\mathbf{C}}$ and $\{(0,0)\} \to \mathbf{A}^2_{\mathbf{C}}$, respectively.

Lemma 4.4.3.1. *The functor $(i_X)_!$ induces a ring morphism*

$$(i_X)_! : K_0(\mathrm{MHM}_X) \to (K_0(\mathrm{MHM}_{\mathbf{A}^1_X}), \star),$$

endowing the ring $(K_0(\mathrm{MHM}_{\mathbf{A}^1_X}), \star)$ with a $K_0(\mathrm{MHM}_X)$-algebra structure.

Proof. There is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i^2_X} & \mathbf{A}^2_X \\ & \searrow^{i_X} & \downarrow \mathrm{add} \\ & & \mathbf{A}^1_X \end{array}$$

which gives us for $M, M' \in D^b(\mathrm{MHM}_X)$,

$$\begin{aligned} (i_X)_!(M) \star (i_X)_!(M') &= (\mathrm{add})_!((i_X)_!(M) \boxtimes_X (i_X)_!(M')) \\ &= (\mathrm{add})_!(i^2_X)_!(M \otimes_X M') \\ &= (i_X)_!(M \otimes_X M') \end{aligned}$$

□

Lemma 4.4.3.2. *The Hodge realisation* $\chi^{\mathrm{Hdg}}_{\mathbf{A}^1_X}$ *is a morphism*

$$\chi^{\mathrm{Hdg}}_{\mathbf{A}^1_X} : (\mathscr{M}_{\mathbf{A}^1_X}, \star) \to (K_0(\mathrm{MHM}_{\mathbf{A}^1_X}), \star)$$

of $\mathscr{M}_X$*-algebras.*

Proof. Let $\mathfrak{a}, \mathfrak{b} \in \mathscr{M}_{\mathbf{A}^1_X}$. Using the fact that the Hodge realisation commutes with push-forwards, pullbacks and exterior products (proposition 4.4.1.1), we have

$$\begin{aligned}
\chi^{\mathrm{Hdg}}_{\mathbf{A}^1_X}(\mathfrak{a} \star \mathfrak{b}) &= \chi^{\mathrm{Hdg}}_{\mathbf{A}^1_X}(\mathrm{add}_!(\mathfrak{a} \boxtimes_X \mathfrak{b})) \\
&= \mathrm{add}_! \left(\chi^{\mathrm{Hdg}}_{\mathbf{A}^2_X}(\mathfrak{a} \boxtimes_X \mathfrak{b}) \right) \\
&= \mathrm{add}_! \left(\chi^{\mathrm{Hdg}}_{\mathbf{A}^1_X}(\mathfrak{a}) \boxtimes_X \chi^{\mathrm{Hdg}}_{\mathbf{A}^1_X}(\mathfrak{b}) \right) \\
&= \chi^{\mathrm{Hdg}}_{\mathbf{A}^1_X}(\mathfrak{a}) \star \chi^{\mathrm{Hdg}}_{\mathbf{A}^1_X}(\mathfrak{b}).
\end{aligned}$$

□

4.4.4 Compatibility with motivic vanishing cycles

In section 4.2.2 we defined a total vanishing cycle functor:

$$\varphi^{\mathrm{tot}}_X : \mathrm{MHM}_{\mathbf{A}^1_X} \to \mathrm{MHM}^{\mathrm{tot}}_X.$$

It induces a group morphism

$$\Phi^{\mathrm{Hdg}}_X : K_0(\mathrm{MHM}_{\mathbf{A}^1_X}) \to K_0(\mathrm{MHM}^{\mathrm{mon}}_X)$$

between the corresponding Grothendieck rings.

Proposition 4.4.4.1. *The morphism* Φ^{Hdg}_X *is a morphism of* $K_0(\mathrm{MHM}_X)$*-algebras*

$$\Phi^{\mathrm{Hdg}}_X : (K_0(\mathrm{MHM}_{\mathbf{A}^1_X}), \star) \to (K_0(\mathrm{MHM}^{\mathrm{mon}}_X), *).$$

Proof. This is a direct consequence of the Thom-Sebastiani property for total vanishing cycles, proposition 4.2.3.1 . □

On the other hand, in chapter 2 we defined, for every variety X over a field k of characteristic zero, the motivic vanishing cycles measure

$$\Phi_X : \mathscr{E}xp\mathscr{M}_X \to (\mathscr{M}^{\hat{\mu}}_X, *).$$

Here, to be able to compare it with Φ^{Hdg}_X, we are going to consider rather its composition

$$\Phi'_X : (\mathscr{M}_{\mathbf{A}^1_X}, \star) \to (\mathscr{M}^{\hat{\mu}}_X, *)$$

with the quotient morphism

$$(\mathscr{M}_{\mathbf{A}^1_X}, \star) \to \mathscr{E}xp\mathscr{M}_X$$

(which is given, by definition, by sending to zero the elements $[\mathbf{A}^1_Y \to \mathbf{A}^1_X]$ for all morphisms $Y \to X$, the morphism $\mathbf{A}^1_Y \to \mathbf{A}^1_X$ being the identity on the $\mathbf{A}^1$–components, see the definition of Grothendieck rings with exponentials in chapter 2, section 2.1.2). Recall from property 3 of lemma 4.4.1.1 and lemma 4.4.3.2 the multiplicative properties of the Hodge realisation morphisms.

Proposition 4.4.4.2. *The diagram*

$$\begin{array}{ccc} (\mathscr{M}_{\mathbf{A}^1_X}, \star) & \xrightarrow{\Phi'_X} & (\mathscr{M}^{\hat{\mu}}_X, *) \\ \downarrow{\scriptstyle \chi^{\mathrm{Hdg}}} & & \downarrow{\scriptstyle \chi^{\mathrm{Hdg}}} \\ (K_0(\mathrm{MHM}_{\mathbf{A}^1_X}), \star) & \xrightarrow{\Phi^{\mathrm{Hdg}}_X} & (K_0(\mathrm{MHM}^{\mathrm{mon}}_X), *) \end{array}$$

commutes.

Proof. Proposition 3.17 in [GLM] shows compatibility between the motivic nearby fibre morphism and the nearby fibre functor on mixed Hodge modules. As for the motivic vanishing cycle morphism, as noted just after notation 3.9 in [DL01], the motivic vanishing cycles $\mathscr{S}^{\varphi}_f$ for $f : X \to \mathbf{A}^1$ as they were defined by Denef and Loeser should be seen as the motivic incarnation of $\varphi_f[d-1]$ where d is the dimension of X. With our notation, $\varphi_f = (-1)^d \mathscr{S}^{\varphi}_f$, so that our vanishing cycles should be the motivic incarnation of $\varphi_f[-1]$, which is exactly the perverse sheaf underlying φ^{Hdg}_f. □

Recall that in section 4.3.2 we defined a morphism

$$\overline{\mathrm{add}}_n : S^n(\mathbf{A}^1_X) \to \mathbf{A}^1_{S^n X}$$

for every integer $n \geq 1$.

Corollary 4.4.4.3. *(a) For any $\mathfrak{a} \in \mathscr{M}_{\mathbf{A}^1_X}$ and any integer $n \geq 1$, we have*

$$\chi^{\mathrm{Hdg}}_{S^n X} \circ \Phi'_{S^n X}((\overline{\mathrm{add}}_n)_!(S^n \mathfrak{a})) = S^n(\chi^{\mathrm{Hdg}}_X \circ \Phi'_X(\mathfrak{a})).$$

(b) For any $\mathfrak{a} \in \mathscr{E}xp\mathscr{M}_X$ and any integer $n \geq 1$, we have

$$\chi^{\mathrm{Hdg}}_{S^n X} \circ \Phi_{S^n X}(S^n \mathfrak{a}) = S^n(\chi^{\mathrm{Hdg}}_X \circ \Phi_X(\mathfrak{a})).$$

Proof. By proposition 4.4.4.2, to prove (a), it suffices to prove

$$\Phi^{\mathrm{Hdg}}_{S^n X} \circ \chi^{\mathrm{Hdg}}_{\mathbf{A}^1_{S^n X}}((\overline{\mathrm{add}}_n)_!(S^n \mathfrak{a})) = S^n(\Phi^{\mathrm{Hdg}}_X \circ \chi^{\mathrm{Hdg}}_{\mathbf{A}^1_X}(\mathfrak{a})).$$

We have

$$\begin{aligned}
\Phi^{\mathrm{Hdg}}_{S^nX} \circ \chi^{\mathrm{Hdg}}_{\mathbf{A}^1_{S^nX}}((\overline{\mathrm{add}}_n)_!(S^n\mathfrak{a})) &= \Phi^{\mathrm{Hdg}}_{S^nX} \circ (\overline{\mathrm{add}}_n)_! \circ \chi^{\mathrm{Hdg}}_{S^n(\mathbf{A}^1_X)}(S^n\mathfrak{a}) \quad \text{by proposition 4.4.1.1}\\
&= \Phi^{\mathrm{Hdg}}_{S^nX} \circ (\overline{\mathrm{add}}_n)_!(S^n\chi^{\mathrm{Hdg}}_{\mathbf{A}^1_X}(\mathfrak{a})) \qquad \text{by lemma 4.4.2.1}\\
&= S^n(\Phi^{\mathrm{Hdg}}_X \circ \chi^{\mathrm{Hdg}}_{\mathbf{A}^1_X}(\mathfrak{a})) \qquad \text{by proposition 4.3.2.2.}
\end{aligned}$$

To prove (b), take $\mathfrak{a} \in \mathscr{E}xp\mathscr{M}_X$, and, denoting by $q : \mathscr{M}_{\mathbf{A}^1_X} \to \mathscr{E}xp\mathscr{M}_X$ the quotient map, pick $\mathfrak{a}' \in \mathscr{M}_{\mathbf{A}^1_X}$ such that $\mathfrak{a} = q(\mathfrak{a}')$. Applying (a) to $\mathfrak{a}'$, we have (recall $\Phi' = \Phi \circ q$)

$$\chi^{\mathrm{Hdg}}_{S^nX} \circ \Phi_{S^nX} \circ q \circ (\overline{\mathrm{add}}_n)_!(S^n\mathfrak{a}') = S^n(\chi^{\mathrm{Hdg}}_X \circ \Phi_X(\mathfrak{a})).$$

It therefore remains to prove that $q \circ (\overline{\mathrm{add}}_n)_!(S^n\mathfrak{a}') = S^n\mathfrak{a}$. In other words, we want to show the commutativity of the diagram

$$\begin{array}{ccc}
\mathscr{M}_{\mathbf{A}^1_X} & \xrightarrow{S} & \mathscr{M}_{S^\bullet(\mathbf{A}^1_X)}\\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q\circ(\overline{\mathrm{add}})_!}\\
\mathscr{E}xp\mathscr{M}_X & \xrightarrow{S} & \mathscr{E}xp\mathscr{M}_{S^\bullet X}
\end{array} \tag{4.7}$$

(recall the group morphisms S have been defined in chapter 3, sections 3.5 and 3.7), where $\overline{\mathrm{add}}_! = \prod_{n\geq 1}(\overline{\mathrm{add}}_n)_!$. Let us start by checking that $q \circ (\overline{\mathrm{add}})_!$ is a group morphism. Let $\mathfrak{a} = (\mathfrak{a}_i)_{i\geq 1}$ and $\mathfrak{b} = (\mathfrak{b}_i)_{i\geq 1}$ be elements of $\mathscr{M}_{S^\bullet(\mathbf{A}^1_X)}$. We have

$$\mathfrak{a}\mathfrak{b} = \left(\sum_{i=0}^{n} \gamma_!(\mathfrak{a}_i \boxtimes \mathfrak{b}_{n-i})\right)_{n\geq 1}$$

where γ is the morphism $S^i(\mathbf{A}^1_X) \times S^{n-i}(\mathbf{A}^1_X) \to S^n(\mathbf{A}^1_X)$ induced by the identity $(\mathbf{A}^1_X)^i \times (\mathbf{A}^1_X)^{n-i} \to (\mathbf{A}^1_X)^n$. To prove that

$$q \circ (\overline{\mathrm{add}})_!(\mathfrak{a}\mathfrak{b}) = q \circ (\overline{\mathrm{add}})_!(\mathfrak{a}) q \circ (\overline{\mathrm{add}})_!(\mathfrak{b})$$

in $\mathscr{E}xp\mathscr{M}_{S^\bullet X}$, it suffices to prove that for all $n \geq 1$ and all $i \in \{0,\ldots,n\}$, we have

$$q \circ (\overline{\mathrm{add}}_n)_!\gamma_!(\mathfrak{a}_i \boxtimes \mathfrak{b}_{n-i}) = \beta_!(q \circ (\overline{\mathrm{add}})_i(\mathfrak{a}_i) \boxtimes q \circ (\overline{\mathrm{add}}_{n-i})_!(\mathfrak{b}_{n-i})) \tag{4.8}$$

in $\mathscr{E}xp\mathscr{M}_{S^nX}$, where $\beta : S^iX \times S^{n-i}X \to S^nX$ is the morphism induced by the identity $X^i \times X^{n-i} \to X^n$. For this, consider the following diagram:

$$\begin{array}{ccccc}
\mathscr{M}_{S^i(\mathbf{A}^1_X)} \times \mathscr{M}_{S^{n-i}(\mathbf{A}^1_X)} & \xrightarrow{\boxtimes} & \mathscr{M}_{S^i(\mathbf{A}^1_X)\times S^{n-i}(\mathbf{A}^1_X)} & \xrightarrow{\gamma_!} & \mathscr{M}_{S^n(\mathbf{A}^1_X)}\\
{\scriptstyle (\overline{\mathrm{add}}_i)_!\times(\overline{\mathrm{add}}_{n-i})_!}\downarrow & & {\scriptstyle (\overline{\mathrm{add}}_i\times\overline{\mathrm{add}}_{n-i})_!}\downarrow & & \downarrow{\scriptstyle (\overline{\mathrm{add}}_n)_!}\\
\mathscr{M}_{\mathbf{A}^1_{S^iX}} \times \mathscr{M}_{\mathbf{A}^1_{S^{n-i}X}} & \xrightarrow{\boxtimes} & \mathscr{M}_{\mathbf{A}^1_{S^iX}\times\mathbf{A}^1_{S^{n-i}X}} & \xrightarrow{(\beta\circ\mathrm{add})_!} & \mathscr{M}_{\mathbf{A}^1_{S^nX}}\\
{\scriptstyle q\times q}\downarrow & & {\scriptstyle q\circ\mathrm{add}_!}\downarrow & & \downarrow{\scriptstyle q}\\
\mathscr{E}xp\mathscr{M}_{S^iX} \times \mathscr{E}xp\mathscr{M}_{S^{n-i}X} & \xrightarrow{\boxtimes} & \mathscr{E}xp\mathscr{M}_{S^iX\times S^{n-i}X} & \xrightarrow{\beta_!} & \mathscr{E}xp\mathscr{M}_{S^nX}
\end{array} \tag{4.9}$$

Here β denotes the morphism $S^iX \times S^{n-i}X \to S^nX$, as well as the morphism $\mathbf{A}^1_{S^iX\times S^{n-i}X} \to \mathbf{A}^1_{S^nX}$ it induces, and add refers to the morphism

$$\mathbf{A}^1_{S^iX} \times \mathbf{A}^1_{S^{n-i}X} \to \mathbf{A}^1_{S^iX\times S^{n-i}X}$$

induced by the addition morphism on the $\mathbf{A}^1$-components.

To prove (4.8), it suffices to prove that this diagram is commutative. We do this square by square. The commutativity of the top left square comes from the fact that pushdowns commute with exterior products. The commutativity of the top right square comes from the commutativity of the square

$$\begin{array}{ccc} (\mathbf{A}^1_X)^i \times (\mathbf{A}^1_X)^{n-i} & \xrightarrow{\mathrm{id}} & (\mathbf{A}^1_X)^n \\ \downarrow{\scriptstyle \mathrm{add}_i \times \mathrm{add}_{n-i}} & & \downarrow{\scriptstyle \mathrm{add}_n} \\ \mathbf{A}^1_{X^i} \times \mathbf{A}^1_{X^{n-i}} & \xrightarrow{\mathrm{add}} & \mathbf{A}^1_{X^n} \end{array}$$

after taking quotients by the appropriate permutation actions. For the bottom left square, by bilinearity, it suffices to check commutativity for effective elements. For any morphisms $Y \xrightarrow{f} \mathbf{A}^1_{S^iX}$, and $Z \xrightarrow{g} \mathbf{A}^1_{S^{n-i}X}$, we have, by definition,

$$\begin{aligned} q \circ \mathrm{add}_!([Y \xrightarrow{f} \mathbf{A}^1_{S^iX}] \boxtimes [Z \xrightarrow{g} \mathbf{A}^1_{S^{n-i}X}]) &= q \circ \mathrm{add}_!([Y \times Z \xrightarrow{f\times g} \mathbf{A}^1_{S^iX} \times \mathbf{A}^1_{S^{n-i}X}]) \\ &= [Y \times Z, \mathrm{add} \circ (f \times g)] \\ &= [Y, f] \boxtimes [Z, g] \\ &= q([Y \xrightarrow{f} \mathbf{A}^1_{S^iX}]) \boxtimes q([Z \xrightarrow{g} \mathbf{A}^1_{S^{n-i}X}]) \end{aligned}$$

in $\mathscr{E}xp\mathscr{M}_{S^iX\times S^{n-i}X}$. The commutativity of the last square comes from the fact that q commutes with $\beta_!$.

We come back to the proof of the main statement. Since all maps involved are group morphisms, it suffices to prove commutativity of diagram (4.7) for effective elements. Let therefore $f : Y \to \mathbf{A}^1_X$ be a morphism. We have, for all $n \geq 1$,

$$\begin{aligned} q \circ (\overline{\mathrm{add}}_n)_!(S^n[Y \xrightarrow{f} \mathbf{A}^1_X]) &= q \circ (\overline{\mathrm{add}}_n)_!([S^nY \xrightarrow{S^nf} S^n(\mathbf{A}^1_X)]) \\ &= q([S^nY \xrightarrow{\overline{\mathrm{add}}_n \circ f} \mathbf{A}^1_{S^nX}]) \\ &= [S^nY, f^{(n)}] \\ &= S^n([Y, f]) \\ &= S^n(q([Y \xrightarrow{f} \mathbf{A}^1_X])) \end{aligned}$$

in $\mathscr{E}xp\mathscr{M}_{S^nX}$, whence the result. $\square$

4.5 Weight filtration on Grothendieck rings of mixed Hodge modules

4.5.1 The weight filtration

For any integer n, denote by $W_{\leq n}K_0(\mathrm{MHM}_S^{\mathrm{mon}})$ the subgroup of $K_0(\mathrm{MHM}_S^{\mathrm{mon}})$ generated by classes of pure Hodge modules (M, id, N) (i.e. with trivial semi-simple monodromy) of weight at most n and by classes of pure Hodge modules with monodromy (M, T_s, N) of weight at most $n-1$.

Remark 4.5.1.1. The monodromy T_s is an automorphism of finite order, so that a pure Hodge module (M, T_s, N) over S of weight m with monodromy decomposes into $M = M^0 \oplus M^{\neq 0}$ where $M^0 = \mathrm{Ker}(T_s - \mathrm{id})$ and $M^{\neq 0} = \mathrm{Ker}(T_s^{k-1} + \ldots + T_s + \mathrm{id})$, where k is minimal such that $T_s^k = 1$. This Hodge module is an element of $W_{\leq m}K_0(\mathrm{MHM}_S^{\mathrm{mon}})$ if $M^{\neq 0} = 0$, and of $W_{\leq m+1}K_0(\mathrm{MHM}_S^{\mathrm{mon}})$ otherwise.

We have the following compatibility with respect to pushdowns, pullbacks and exterior products:

Lemma 4.5.1.2. *1. Let $f : Y \to X$ be a morphism of complex varieties with fibres of dimension $\leq d$, then for all integers n, we have*

$$f_!\left(W_{\leq n}K_0(\mathrm{MHM}_Y^{\mathrm{mon}})\right) \subset W_{\leq n+d}K_0(\mathrm{MHM}_X^{\mathrm{mon}})$$

and

$$f^*\left(W_{\leq n}K_0(\mathrm{MHM}_X^{\mathrm{mon}})\right) \subset W_{\leq n+d}K_0(\mathrm{MHM}_Y^{\mathrm{mon}}).$$

2. Let X and Y be complex varieties. Then for all integers n, m we have

$$W_{\leq n}K_0(\mathrm{MHM}_X^{\mathrm{mon}}) \overset{T}{\boxtimes} W_{\leq m}K_0(\mathrm{MHM}_Y^{\mathrm{mon}}) \subset W_{\leq n+m}K_0(\mathrm{MHM}_{X\times Y}^{\mathrm{mon}}).$$

Proof. Let M be a pure Hodge module of weight at most n (resp. $n-1$). Then, since the functor $f_!$ does not increase weights, $f_!M$ is a complex of weight $\leq n$ (resp. $\leq n-1$) which belongs to $D^{\leq d}(\mathrm{MHM}_X)$ by lemma 4.1.4.2, so

$$[f_!M] = \sum_{i\leq d}(-1)^i[\mathcal{H}^i f_!M]$$

is a sum of Hodge modules of weight $\leq n+d$ (resp. $\leq n-1+d$). If the monodromy on M is trivial, then it is also trivial on all mixed Hodge modules $\mathcal{H}^i f_!M$, so in any case, we have $[f_!M] \in W_{\leq n+d}K_0(\mathrm{MHM}_S^{\mathrm{mon}})$. The proof is the same for f^*.

Let $(M_1, T_{s,1}, N_1) \in W_{\leq n}K_0(\mathrm{MHM}_X)$ and $(M_2, T_{s,2}, N_2) \in W_{\leq m}K_0(\mathrm{MHM}_X)$ be two pure Hodge modules with monodromy. By remark 4.5.1.1, it suffices to treat the following cases :

— $M_1 = M_1^0$ is of weight n, $M_2 = M_2^0$ is of weight m;
— $M_1 = M_1^0$ is of weight n, $M_2 = M_2^{\neq 0}$ is of weight $m-1$;

— $M_1 = M_1^{\neq 0}$ is of weight $n-1$ and $M_2 = M_2^{\neq 0}$ is of weight $m-1$.

By the definition of the weight filtration on $M_1 \overset{T}{\boxtimes} M_2$, in the first case $M_1 \overset{T}{\boxtimes} M_2$ is pure of weight $m+n$, with trivial monodromy, and in the second case, pure of weight $m+n-1$. Thus, in both cases, $M_1 \overset{T}{\boxtimes} M_2$ is an element of $W_{\leq m+n} K_0(\mathrm{MHM}_S^{\mathrm{mon}})$. This leaves us with the third case. For any $\alpha, \beta \in (-1, 0]$ such that $\exp(-2i\pi\alpha)$ (resp. $\exp(-2i\pi\beta)$) is an eigenvalue of $T_{s,1}$ (resp. $T_{s,2}$), the complex number $\exp(-2i\pi(\alpha+\beta))$ is an eigenvalue of monodromy on $M_1 \overset{T}{\boxtimes} M_2$, and $(M_1 \overset{T}{\boxtimes} M_2)^1$ is non-zero if and only if there exist such α, β with $\alpha + \beta = -1$. The weight filtration on $M_1 \overset{T}{\boxtimes} M_2$ is such that $(M_1 \overset{T}{\boxtimes} M_2)^{\neq 1}$ is of weight $m+n-2$, and $(M_1 \overset{T}{\boxtimes} M_2)^1$ is of weight $m+n$, so that the Hodge module $M_1 \overset{T}{\boxtimes} M_2$ is an element of $W_{\leq m+n} K_0(\mathrm{MHM}_S^{\mathrm{mon}})$. □

For any element $\mathfrak{a} \in K_0(\mathrm{MHM}_S^{\mathrm{mon}})$, we put

$$w_S(\mathfrak{a}) := \inf\{n,\ \mathfrak{a} \in W_{\leq n} K_0(\mathrm{MHM}_S^{\mathrm{mon}})\},$$

which defines a function $w_S : K_0(\mathrm{MHM}_S^{\mathrm{mon}}) \to \mathbf{Z} \cup \{-\infty\}$. Lemma 4.5.1.2 gives us the following:

Lemma 4.5.1.3. *For any complex varieties S and T, any $\mathfrak{a}, \mathfrak{a}' \in K_0(\mathrm{MHM}_S^{\mathrm{mon}})$ and $\mathfrak{b} \in K_0(\mathrm{MHM}_T^{\mathrm{mon}})$ the weight function satisfies the following properties:*

(a) $w_S(0) = -\infty$

(b) $w_S(\mathfrak{a} + \mathfrak{a}') \leq \max\{w_S(\mathfrak{a}), w_S(\mathfrak{a}')\}$, *with equality if* $w_S(\mathfrak{a}) \neq w_S(\mathfrak{a}')$.

(c) $w_{S\times T}(\mathfrak{a} \overset{T}{\boxtimes} \mathfrak{b}) \leq w_S(\mathfrak{a}) + w_T(\mathfrak{b})$.

(d) If $f : S \to T$ is a morphism with fibres of dimension $\leq d$, then

$$w_T(f_!(\mathfrak{a})) \leq w_S(\mathfrak{a}) + d.$$

(e) If $f : S \to T$ is a morphism with fibres of dimension $\leq d$, then

$$w_S(f^*(\mathfrak{b})) \leq w_T(\mathfrak{b}) + d.$$

(f) If $M^\bullet \in D^{\leq a}(\mathrm{MHM}_S)$ is a complex of mixed Hodge modules of weight $\leq n$, then $w_S([M^\bullet]) \leq a + n$. Equality is achieved if and only if $\mathrm{Gr}^W_{a+n} \mathcal{H}^a(M^\bullet) \neq 0$.

Remark 4.5.1.4. As a special case of (d), denoting by w the weight function w_{pt} on $K_0(\mathrm{MHM}_{\mathrm{pt}}^{\mathrm{mon}})$ and by $a_S : S \to \mathrm{pt}$ the structural morphism, we have

$$w((a_S)_!\mathfrak{a}) \leq w_S(\mathfrak{a}) + \dim S.$$

4.5.2 Weights of symmetric powers of mixed Hodge modules

Recall that in section 4.3.1 we defined a group $K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^\bullet X})$ and a morphism

$$S_X^{\mathrm{Hdg}} : K_0(\mathrm{MHM}^{\mathrm{mon}}_X) \to K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^\bullet X})$$

sending the class of a mixed Hodge module M to $(S^n M)_{n\geq 1}$. Our goal here is to show that S_X^{Hdg} behaves well with respect to the weight filtration from section 4.5.1. We define the following natural filtration on $K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^\bullet X})$:

$$W_{\leq d}K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^\bullet X}) := \prod_{n\geq 1} W_{\leq nd}K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^n X}) \subset \prod_{n\geq 1} K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^n X}).$$

We have the following properties:

Proposition 4.5.2.1. *1. For every d, $W_{\leq d}K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^\bullet X})$ is a subgroup of the group $K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^\bullet X})$.*

2. For any integer d, $S_X^{\mathrm{Hdg}}(W_{\leq d}K_0(\mathrm{MHM}^{\mathrm{mon}}_X)) \subset W_{\leq d}K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^\bullet X})$.

3. For every $\mathfrak{a} \in K_0(\mathrm{MHM}^{\mathrm{mon}}_X)$ and every integer $n \geq 0$, we have $w_{S^n X}(S^n\mathfrak{a}) \leq n w_X(\mathfrak{a})$.

Proof. 1. Let $a = (a_n)_{n\geq 1}$ and $b = (b_n)_{n\geq 1}$ be two elements of $W_{\leq d}K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^\bullet X})$. Then for all $n \geq 1$ and all $i \in \{0, \dots, n\}$, by property 2 of lemma 4.5.1.2 we have $a_i \overset{T}{\boxtimes} b_{n-i} \in K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^i X\times S^{n-i}X})$ of weight $\leq id + (n-i)d = nd$. The map $S^iX \times S^{n-i}X \to S^nX$ has fibre dimension zero, so by property 1 of lemma 4.5.1.2 the same estimate is valid for $a_i \overset{T}{\boxtimes} b_{n-i}$ seen as an element of $K_0(\mathrm{MHM}^{\mathrm{mon}}_{S^n X})$.

2. Let $M \in \mathrm{MHM}^{\mathrm{mon}}_X$ be a pure Hodge module of weight d. Then $\overset{T}{\boxtimes}{}^n M$ is a pure Hodge module of weight nd, and denoting by p the quotient morphism $X^n \to S^nX$, the complex $p_!(\overset{T}{\boxtimes}{}^n M)$ is of weight $\leq nd$ by property 1 of lemma 4.5.1.2, since p has fibres of dimension 0. Finally, S^nM is obtained as a subobject of $p_!(\overset{T}{\boxtimes}{}^n M)$, so its weight is $\leq nd$ again. Statement 3 is a direct consequence of 2. □

4.6 Weight filtration on Grothendieck rings of varieties

In this section, we are going to use the previously defined weight filtration to define a notion of weight on Grothendieck rings of varieties with exponentials. For this, we are going to use the motivic vanishing cycles measure from chapter 2, and the Hodge realisation from section 4.4.1.

4.6.1 The weight filtration and completion

Let S be a complex variety. Recall that $\Phi_S : \mathscr{E}xp\mathscr{M}_S \to (\mathscr{M}_S^{\hat{\mu}}, *)$ is the motivic vanishing cycles measure from theorem 2.3.5.1 of chapter 2.

Definition 4.6.1.1. 1. The weight filtration on the ring $\mathscr{M}_S^{\hat{\mu}}$ is given by

$$W_{\leq n}\mathscr{M}_S^{\hat{\mu}} := (\chi_S^{\mathrm{Hdg}})^{-1}(W_{\leq n}K_0(\mathrm{MHM}_S^{\mathrm{mon}}))$$

for every $n \in \mathbf{Z}$. The weight function on $\mathscr{M}_S^{\hat{\mu}}$, again denoted by w_S, is the composition

$$\mathscr{M}_S^{\hat{\mu}} \xrightarrow{\chi_S^{\mathrm{Hdg}}} K_0(\mathrm{MHM}_S^{\mathrm{mon}}) \xrightarrow{w_S} \mathbf{Z}.$$

2. The weight filtration on the ring $\mathscr{E}xp\mathscr{M}_S$ is given by

$$W_{\leq n}\mathscr{E}xp\mathscr{M}_S := (\chi_S^{\mathrm{Hdg}} \circ \Phi_S)^{-1}(W_{\leq n}K_0(\mathrm{MHM}_S^{\mathrm{mon}}))$$

for every $n \in \mathbf{Z}$. The weight function on $\mathscr{E}xp\mathscr{M}_S$, again denoted by w_S, is the composition

$$\mathscr{E}xp\mathscr{M}_S \xrightarrow{\Phi_S} \mathscr{M}_S^{\hat{\mu}} \xrightarrow{\chi_S^{\mathrm{Hdg}}} K_0(\mathrm{MHM}_S^{\mathrm{mon}}) \xrightarrow{w_S} \mathbf{Z}.$$

Remark 4.6.1.2. Properties $(a)-(e)$ of lemma 4.5.1.3 remain true for w_S on $\mathscr{M}_S^{\hat{\mu}}$ or $\mathscr{E}xp\mathscr{M}_S$. Indeed, this is obvious for (a), for (d) and (e) it follows from the fact that the Hodge realisation commutes with pushforwards and pullbacks, and for (b) it comes from the fact that χ_S^{Hdg} and Φ_S are group morphisms. Property (c) comes from the Thom-Sebastiani property for Φ_S, and from the fact that χ^{Hdg} is compatible with twisted exterior products.

Remark 4.6.1.3. Both these definitions induce the same weight filtration $(W_{\leq n}\mathscr{M}_S)_{n\in\mathbf{Z}}$ and the same weight map $w_S : \mathscr{M}_S \to \mathbf{Z}$ on the localised Grothendieck ring $\mathscr{M}_S$, because the restriction of Φ_S to $\mathscr{M}_S$ is the inclusion $\mathscr{M}_S \to \mathscr{M}_S^{\hat{\mu}}$.

Notation 4.6.1.4. For a variety X over S, we use $w_S(X)$ as a shorthand for $w_S([X])$. We denote by w the weight function for $S = \operatorname{Spec}\mathbf{C}$.

Definition 4.6.1.5. We define the completion of the ring $\mathscr{E}xp\mathscr{M}_S$ with respect to the weight topology as

$$\widehat{\mathscr{E}xp\mathscr{M}}_S = \varprojlim_n \mathscr{E}xp\mathscr{M}_S / W_{\leq n}\mathscr{E}xp\mathscr{M}_S.$$

4.6.2 Weights of symmetric products

Lemma 4.6.2.1. *Let I be a set and let $\pi = (n_i)_{i\in I} \in \mathbf{N}^{(I)}$. Let X be a complex variety, and let $\mathscr{A} = (\mathfrak{a}_i)_{i\in I}$ be a family of elements of $\mathscr{E}xp\mathscr{M}_X$. Then*

$$w_{S^\pi X}(S^\pi \mathscr{A}) \leq \sum_{i\in I} n_i w_X(\mathfrak{a}_i).$$

Proof. Recall that by definition, $S^\pi \mathscr{A}$ is the element of $\mathscr{E}xp\mathscr{M}_{S^\pi X}$ obtained by pulling back the product $\prod_{i\in I} S^{n_i}\mathfrak{a}_i \in \mathscr{E}xp\mathscr{M}_{\prod_{i\in I} S^{n_i}X}$ along the open immersion $j : S^\pi X \to \prod_{i\in I} S^{n_i}X$. By property (c) of lemma 4.5.1.3 and property 3 of proposition 4.5.2.1 we have

$$w_{\prod_{i\in I} S^{n_i}X}\left(\boxtimes_{i\in I} S^{n_i}\mathfrak{a}_i\right) \leq \sum_{i\in I} w_{S^{n_i}X}(S^{n_i}\mathfrak{a}_i) \leq \sum_{i\in I} n_i w_X(\mathfrak{a}_i).$$

Applying property (d) of lemma 4.5.1.3 to j, which has fibre dimension 0, we get the result. □

4.6.3 Weight and dimension

For effective classes in the Grothendieck ring of varieties, weight and dimension are closely linked, as shown in the following lemma:

Lemma 4.6.3.1. *Let S be a complex variety and X a variety over S. One has the equality*

$$w_S(X) = 2\dim_S X + \dim S.$$

Proof. We are going to denote by $f : X \to S$ the structural morphism of X, and by d the relative dimension $\dim_S X$, that is, the supremum of the dimensions of the fibres of f. Since the functors a_X^* and $f_!$ do not increase weights, the complex $f_!\mathbf{Q}_X^{\mathrm{Hdg}} = f_! a_X^* \mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}$ is of weight ≤ 0. Moreover, we see by lemma 4.1.4.2 that $f_!\mathbf{Q}_X^{\mathrm{Hdg}}$ is an object of $D^{\leq 2d+\dim S}(\mathrm{MHM}_S)$. By property (f), it suffices to prove that the top cohomology $\mathcal{H}^{2d+\dim S}(f_!\mathbf{Q}_X^{\mathrm{Hdg}})$ has non-zero graded part of weight $2d+\dim S$. By [S89] 1.20, we may write the Leray spectral sequence for $a_X = a_S \circ f$: for any $M^\bullet \in D^b(\mathrm{MHM}_X)$,

$$\mathcal{H}^p(a_S)_!\left(\mathcal{H}^q f_! M^\bullet\right) \Longrightarrow \mathcal{H}^{p+q}((a_X)_! M^\bullet).$$

Applying this with $M^\bullet = \mathbf{Q}_X^{\mathrm{Hdg}}$, $p = \dim S$ and $q = 2d + \dim S$, and recalling that the cohomology of the complex of mixed Hodge structures $(a_X)_!\mathbf{Q}_X^{\mathrm{Hdg}}$ is exactly the cohomology with compact supports of X with coefficients in $\mathbf{Q}$ with its standard Hodge structure, we have

$$\mathcal{H}^{\dim S}(a_S)_!\left(\mathcal{H}^{2d+\dim S} f_!\mathbf{Q}_X^{\mathrm{Hdg}}\right) \Longrightarrow H_c^{2\dim X}(X, \mathbf{Q}).$$

If $\mathrm{Gr}^W_{2d+\dim S}\mathcal{H}^{2d+\dim S} f_!\mathbf{Q}_X^{\mathrm{Hdg}} = 0$, then the graded part of weight $2\dim X$ of the left-hand side is zero. But the right-hand side is a sub-object of a quotient of the left-hand side, and therefore its graded part of weight $2\dim X$ should be zero as well, which it is not: it is classical that $H_c^{2\dim X}(X, \mathbf{Q})$ is pure of weight $2\dim X$, isomorphic to $\mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}(-\dim X)^r$ where r is the number of irreducible components of X. □

As a consequence, we have

Lemma 4.6.3.2. *Let S be a complex variety and $\mathfrak{a}$ an element of $\mathscr{M}_S^{\hat{\mu}}$. Then*

$$w_S(\mathfrak{a}) \leq 2\dim_S \mathfrak{a} + \dim S.$$

Proof. We may assume $\mathfrak{a}$ is of the form $\mathbf{L}^{-m}([X]-[Y])$ for some S-varieties X and Y with $\hat{\mu}$-actions. Assume moreover that $\max\{\dim_S X, \dim_S Y\}$ is minimal, so that

$$\dim_S \mathfrak{a} = \max\{\dim_S X, \dim_S Y\} - m.$$

Then, using lemma 4.5.1.3 and the fact that $\chi_{\mathrm{pt}}^{\mathrm{Hdg}}(\mathbf{L}^{-m}) = \mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}(m)$ is of weight $-2m$, we have

$$\begin{aligned} w_S(\mathfrak{a}) &\leq w_{\mathrm{pt}}(\mathbf{L}^{-m}) + w_S([X]-[Y]) \\ &\leq -2m + \max\{w_S(X), w_S(Y)\} \end{aligned}$$

By lemma 4.6.3.1, we therefore get

$$w_S(\mathfrak{a}) \leq -2m + 2\max\{\dim_S(X), \dim_S(Y)\} + \dim S$$

whence the result. □

We may therefore deduce the triangular inequality for the weight topology:

Lemma 4.6.3.3 (Triangular inequality for weights)**.** *Let S be a variety over $\mathbf{C}$, X a variety over S and $f : X \to \mathbf{A}^1_{\mathbf{C}}$ a morphism. Then*

$$w_S([X,f]) \leq w_S(X).$$

Proof. By lemmas 2.3.5.2 and 4.6.3.1 we have

$$w_S([X,f]) \leq 2\dim_S(\Phi_S([X,f])) + \dim S \leq 2\dim_S X + \dim S = w_S(X).$$

□

The following property, which follows from our discussion of the trace morphism (lemma 4.1.5.4 and remark 4.1.5.5) and states that there is a drop in weights for certain simple non-effective classes, will be very important to us:

Lemma 4.6.3.4 (Cancellation of maximal weights)**.** *Let S be a complex variety and $p : X \to S$, $q : Y \to S$ morphisms with fibres of constant dimension $d \geq 0$, with X and Y irreducible. Then*

$$w_S([X \xrightarrow{p} S] - [Y \xrightarrow{q} S]) \leq 2d + \dim S - 1.$$

Proof. The classes $[X]$ and $[Y]$ are of weights $\leq 2d + \dim S$, and according to remark 4.1.5.5 the graded parts of weight exactly $2d + \dim S$ of the corresponding complexes of mixed Hodge modules cancel out. □

4.7 Convergence of power series

4.7.1 Radius of convergence

Recall that for a classical series $\sum_{i\geq 0} a_i z^i$, the radius of convergence is given by

$$\left(\limsup\left(|a_i|\right)^{\frac{1}{i}}\right)^{-1}.$$

Analogously, in our setting, we have:

Definition 4.7.1.1. Let $F(T)=\sum_{i\geq 0} X_i T^i \in \mathscr{E}xp\mathscr{M}_X[[T]]$. The radius of convergence of F is defined by

$$\sigma_F = \limsup_{i\geq 1} \frac{w_X(X_i)}{2i}.$$

We say that F converges for $|T|<\mathbf{L}^{-r}$ if $r\geq \sigma_F$.

When F converges for $|T|<\mathbf{L}^{-r}$, it converges also for $|T|<\mathbf{L}^{-r'}$ for any $r'>r$. The subset of power series converging for $|T|<\mathbf{L}^{-r}$ is a subring of $\mathscr{E}xp\mathscr{M}_X[[T]]$.

Remark 4.7.1.2. If $r>\sigma_F$, there is some i_0 such that for all $i\geq i_0$, $\frac{w_X(X_i)}{2i}<r$, which means that the set of integers $\{w_X(X_i)-2ri,\ i\geq 0\}$ is bounded from above. Conversely, if this set is bounded from above for some r, then we may conclude that $r\geq\sigma_F$, that is, F converges for $|T|<\mathbf{L}^{-r}$. Thus, in general, we are going to prove that a series converges by finding a linear bound for $w_X(X_i)$.

However, one does not in general have $\{w_X(X_i)-2\sigma_F i,\ i\geq 0\}$ bounded from above: see for example the series $\sum_{i\geq 0}\mathbf{L}^{i+\lceil\sqrt{i}\rceil}T^i$.

If $F(T)$ converges for $|T|<\mathbf{L}^{-r}$, then for any element $\mathfrak{a}\in\mathscr{E}xp\mathscr{M}_{\mathbf{C}}$ such that $w(\mathfrak{a})<-2r$, $F(\mathfrak{a})$ exists as an element of $\widehat{\mathscr{E}xp\mathscr{M}}_X$. In particular, $F(\mathbf{L}^{-m})$ exists as an element of $\widehat{\mathscr{E}xp\mathscr{M}}_X$ if $m>r$.

Example 4.7.1.3. Let X be a complex quasi-projective variety, and consider $Z_X(T)=\sum_{i\geq 0}[S^iX]T^i\in\mathscr{E}xp\mathscr{M}_{\mathbf{C}}[[T]]$ its Kapranov zeta function. We have

$$w(S^iX)=2i\dim X$$

for all $i\geq 0$, so that the radius of convergence of $Z_X(T)$ is $\dim X$.

4.7.2 A convergence criterion

Proposition 4.7.2.1. *Assume $F(T)=1+\sum_{i\geq 1}X_iT^i\in\mathscr{E}xp\mathscr{M}_X[[T]]$ is such that there exists an integer $M\geq 0$ and real numbers $\epsilon>0$, $\alpha<1$ and β such that*

— *for all $i\in\{1,\ldots,M\}$, $w_X(X_i)\leq(i-\frac{1}{2}-\epsilon)w(X)$*

— *for all $i\geq M+1$, $w_X(X_i)\leq(\alpha i+\beta-\frac{1}{2})w(X)$.*

Then there exists $\delta>0$ such that the Euler product $\prod_{v\in X}F_v(T)\in\mathscr{E}xp\mathscr{M}_{\mathbf{C}}[[T]]$

— *converges for* $|T| < \mathbf{L}^{-\frac{w(X)}{2}\left(1-\delta+\frac{\beta}{M+1}\right)}$

— *for any* $0 \leq \eta < \delta$, *takes non-zero values for* $|T| \leq \mathbf{L}^{-\frac{w(X)}{2}\left(1-\eta+\frac{\beta}{M+1}\right)}$ *(that is, for every* $\mathfrak{a} \in \mathscr{E}xp\mathscr{M}_{\mathbf{C}}$ *such that* $w(\mathfrak{a}) < -w(X)\left(1-\eta+\frac{\beta}{M+1}\right)$*).*

Proof. Let $n \geq 1$ be an integer, and $\pi = (n_i)_{i\geq 1}$ a partition of n. Then we have

$$\begin{aligned}
w(S^\pi \mathscr{X}) &\leq w_{S^\pi X}(S^\pi \mathscr{X}) + \dim(S^\pi X) \qquad \text{by remark 4.5.1.4}\\
&\leq \sum_{i\geq 1} n_i w_X(X_i) + \frac{1}{2}\sum_{i\geq 1} n_i w(X) \qquad \text{by lemmas 4.6.2.1 and 4.6.3.1}\\
&\leq \sum_{i=1}^{M} n_i i w(X) - \epsilon \sum_{i=1}^{M} n_i w(X) + \sum_{i\geq M+1} \alpha i n_i w(X) + \beta w(X) \sum_{i\geq M+1} n_i\\
&\leq \sum_{i=1}^{M} n_i i w(X) - \frac{\epsilon}{M}\sum_{i=1}^{M} n_i i w(X) + \alpha \sum_{i\geq M+1} i n_i w(X) + \frac{\beta w(X)}{M+1}\sum_{i\geq M+1} i n_i\\
&= \left(1-\frac{\epsilon}{M}\right)\sum_{i=1}^{M} n_i i w(X) + \left(\alpha + \frac{\beta}{M+1}\right)\sum_{i\geq M+1} n_i i w(X)\\
&\leq \left(1-\delta+\frac{\beta}{M+1}\right) n w(X)
\end{aligned}$$

where $1-\delta = \max\{1-\frac{\epsilon}{M}, \alpha\} < 1$ (in the case when $M = 0$, we put $1-\delta = \alpha$). The desired convergence follows. Moreover, one sees that for $n \geq 1$, any $0 \leq \eta < \delta$ and any $\mathfrak{a} \in \mathscr{E}xp\mathscr{M}_{\mathbf{C}}$ such that $w(\mathfrak{a}) \leq -w(X)\left(1-\eta+\frac{\beta}{M+1}\right)$, we have

$$w(S^n \mathscr{X}\,\mathfrak{a}^n) \leq -(\delta-\eta) n w(X) < 0,$$

so the value of the product at $\mathfrak{a}$ is equal to 1 plus some terms of negative weight: it is therefore non-zero.

□

Example 4.7.2.2. Let $Z_X(T) = \sum_{i\geq 0}[S^i X]T^i$ be Kapranov's zeta function for some quasi-projective variety X. Then $Z_X(T) = \prod_{v\in X} F_v(T)$ where

$$F(T) = \sum_{i\geq 1} T^i \in \mathscr{M}_X[[T]],$$

that is, every coefficient is equal to $1 = [X] \in \mathscr{M}_X$. Take $M = 0$, $\alpha = 0$, $\beta = 1$ and $\eta = 1 - \frac{1}{2\dim X} < \delta = 1$. Then, since $w_X(X) = \dim X = \frac{1}{2}w(X)$, the condition in the lemma is satisfied, and we get that $Z_X(T)$ converges for $|T| < \mathbf{L}^{-\dim X}$ and takes non-zero values for $|T| \leq \mathbf{L}^{-\dim X - \frac{1}{2}}$.

Note that each factor $F(T) = \sum_{i\geq 0} T^i$ has radius of convergence 0, so taking the Euler product has the effect of shifting the radius of convergence by exactly the dimension of the base variety.

Example 4.7.2.3. Let X be a quasi-projective variety over $\mathbf{C}$, and let $\mathfrak{a} \in \mathscr{M}_X$ be an element such that $w_X(\mathfrak{a}) \leq \dim X + 1$. As an example of such an element, by lemma 4.6.3.4 we may take $\mathfrak{a} = Y - Z$ for two irreducible varieties Y, Z over X of relative dimension 1. Consider the polynomial $F(T) = 1 + \mathfrak{a}T^2$, so that

$$\prod_{v \in X} F_v(T) = \prod_{v \in X} (1 + \mathfrak{a}_v T^2) = \sum_{n \geq 0} S^n_{*,X}(\mathfrak{a}) T^{2n}.$$

Taking $M = 2, \epsilon = 1 - \frac{1}{w(X)}, \alpha = 0, \beta = 0$, we get convergence for $|T| < \mathbf{L}^{-\frac{1}{2}\dim X - \frac{1}{4}}$.

Let us check that we get the same convergence by estimating the radius of convergence directly: for this, note that

$$w(S^n_{*,X}(\mathfrak{a})) \leq w_{S^n_* X}(S^n_{*,X}\mathfrak{a}) + \dim S^n_* X \leq n(\dim X + 1) + n \dim X = 2n \dim X + n.$$

Thus, taking the lim sup over all even n, the radius of convergence is smaller than

$$\limsup \frac{n \dim X + \frac{1}{2}n}{2n} = \frac{1}{2}\dim X + \frac{1}{4}.$$

4.7.3 Growth of coefficients

We finish this section by a lemma that allows one to get information about growth of coefficients of a power series from the fact that it possesses a pole of some order at $T = \mathbf{L}^{-1}$. It shows that we can predict the behaviour of a positive proportion of the coefficients of the Hodge-Deligne polynomials of the coefficient of degree n for large n. For any constructible set M, denote by $\kappa(M)$ the number of irreducible components of maximal dimension of M.

Proposition 4.7.3.1. *Let $Z(T) = \sum_{n \geq 0} [M_n] T^n \in \mathrm{KVar}^+_{\mathbf{C}}[[T]]$ be a power series with effective coefficients such that there exist integers $a, r \geq 1$, a real number $\delta > 0$ and a power series $F(T) = \sum_{i \geq 0} f_i T^i \in \mathscr{M}_{\mathbf{C}}[[T]]$ converging for $|T| < \mathbf{L}^{-1+\delta}$ and taking a non-zero effective value at $T = \mathbf{L}^{-1}$, such that*

$$Z(T) = \frac{F(T)}{(1 - \mathbf{L}^a T^a)^r}.$$

Then for every $p \in \{0, \ldots, a-1\}$, one of the following cases occur when n tends to infinity in the congruence class of p modulo a:

(i) Either $\limsup \frac{\dim(M_n)}{n} < 1$.

(ii) Or $\dim(M_n) - n$ has finite limit $d_0 \in \mathbf{Z}$ and $\frac{\log(\kappa(M_n))}{\log n}$ converges to some integer in the set $\{0, \ldots, r-1\}$. More generally, for every real number η such that $0 < \eta < \delta$ and for sufficiently large n in the congruence class of p modulo a, the coefficients of the Hodge-Deligne polynomial $HD(M_n)$ of degrees contained in the interval

$$[2(1-\eta)n + 2d_0, 2n + 2d_0]$$

are polynomials in $\frac{n-p}{a}$ of degree at most $r - 1$.

Moreover, the second case happens for at least one value of p.

Proof. Note first that since $[M_n]$ is effective, we have $2\dim[M_n] = w(M_n)$, so it suffices to prove the statement with dimensions replaced by weights divided by two.

First of all, replace $Z(T)$ by $Z(\mathbf{L}^{-1}T)$ and $F(T)$ with $F(\mathbf{L}^{-1}T)$, so that F is now a power series converging for $|T| < \mathbf{L}^{\delta}$ and taking a non-zero effective value at $T = 1$, and $Z(T) = F(T)(1 - T^a)^r$ with $Z(T) = \sum_{n\in\mathbf{Z}}[M_n]\mathbf{L}^{-n}T^n$. We are going to do calculations in the case $a = 1$, and explain later how one can reduce to this case. Note first that if F converges for $|T| < \mathbf{L}^{\delta}$, then the same is true for all its derivatives. We may write its Taylor expansion at $T = 1$:

$$F(T) = \sum_{i\geq 0}\frac{F^{(i)}(1)}{i!}(T-1)^i = \sum_{i\geq 0}\frac{F^{(i)}(1)(-1)^i}{i!}(1-T)^i.$$

Put $G(T) = \sum_{i\geq r}\frac{F^{(i)}(1)(-1)^i}{i!}(1-T)^{i-r}$. Then

$$\begin{aligned} G(T) &= \sum_{i\geq r}\frac{F^{(i)}(1)(-1)^i}{i!}\sum_{j=0}^{i-r}\binom{i-r}{j}(-1)^jT^j. \\ &= \sum_{j\geq 0}\left(\sum_{i\geq r+j}\frac{F^{(i)}(1)(-1)^i}{i!}\binom{i-r}{j}\right)(-T)^j, \end{aligned}$$

so that the coefficient of degree j of $G(T)$ is exactly $g_j = (-1)^j\left(\sum_{i\geq r+j}\frac{F^{(i)}(1)(-1)^i}{i!}\binom{i-r}{j}\right)$. Writing $F(T) = \sum_{i\geq 0} f_jT^j$, we have $w(f_j) \to -\infty$ as $j \to +\infty$. More precisely, for any η such that $0 < \eta < \delta$ and for sufficiently large j, we have

$$w(f_j) < -2\eta j. \tag{4.10}$$

Thus, since

$$F^{(i)}(1) = \sum_{j\geq i} j(j-1)\dots(j-i+1)f_j,$$

we see that as i grows, $w(F^{(i)}(1)) \to -\infty$ linearly in i. In particular, $w(g_j) \to -\infty$ linearly in j, and, more precisely, the estimate

$$w(g_j) < -2\eta j \tag{4.11}$$

coming from (4.10) holds for all sufficiently large j. Write now

$$\begin{aligned} Z(T) &= \frac{F(T)}{(1-T)^r} \\ &= G(T) + \sum_{i=0}^{r-1}\frac{F^{(i)}(1)(-1)^i}{i!(1-T)^{r-i}} \\ &= G(T) + \sum_{i=0}^{r-1}\frac{F^{(i)}(1)(-1)^i}{i!}\sum_{n\geq 0}\binom{n+r-i-1}{r-i-1}T^n \\ &= G(T) + \sum_{n\geq 0}\left(\sum_{i=0}^{r-1}\frac{F^{(i)}(1)(-1)^i}{i!}\binom{n+r-i-1}{r-i-1}\right)T^n. \end{aligned}$$

Thus, identifying coefficients, we have

$$[M_n]\mathbf{L}^{-n} = g_n + \sum_{i=0}^{r-1} \frac{F^{(i)}(1)(-1)^i}{i!}\binom{n+r-i-1}{r-i-1}. \tag{4.12}$$

Since by assumption $[M_n]$ is an element of KVar_k^+, its Hodge-Deligne polynomial is of the form

$$\kappa(M_n)(uv)^{\dim(M_n)} + \text{ terms of lower degree.} \tag{4.13}$$

To get asymptotics for $\dim(M_n)$ and for the coefficients of high degree of $HD(M_n)$ when n goes to infinity, we therefore need to keep track of the dominant terms of the Hodge-Deligne series of (4.12).

We denote by $\{\mathfrak{a}\}_d$ the coefficient of $(uv)^d$ in the Hodge-Deligne series of $\mathfrak{a} \in \widehat{\mathscr{M}_{\mathbf{C}}}$. Let d_0 be the largest integer d such that there exists $i \in \{0, \ldots, r-1\}$ with $\{F^{(i)}(1)\}_d \neq 0$. Such a d_0 does exist since by assumption, we have $F(1)$ effective and non-zero, and therefore there exists some integer b such that $\{F(1)\}_b \neq 0$. Then for all sufficiently large n (namely, for n such that $w(g_n) < 2d_0$), and for all $d \geq d_0$, , we have

$$\{[M_n]\mathbf{L}^{-n}\}_d = \sum_{i=0}^{r-1} \frac{(-1)^i}{i!}\binom{n+r-i-1}{r-i-1}\{F^{(i)}(1)\}_d. \tag{4.14}$$

Then, for $d > d_0$, the right-hand side of (4.14) is zero, forcing the left-hand side to be zero as well, so that $w(M_n\mathbf{L}^{-n}) \leq 2d_0$. Put now $d = d_0$, and let i_0 be the smallest i such that $\{F^{(i)}(1)\}_d \neq 0$. Then we have

$$\{[M_n]\mathbf{L}^{-n}\}_{d_0} \sim_{n\to\infty} \{F^{(i_0)}(1)\}_{d_0} \frac{(-1)^{i_0}}{i_0!(r-i_0-1)!} n^{r-i_0-1},$$

so that for sufficiently large n, $w([M_n]\mathbf{L}^{-n}) = 2d_0$, and moreover

$$\frac{\log \kappa(M_n)}{\log n} \longrightarrow r - i_0 - 1 \in \{0, \ldots, r-1\}.$$

More generally, going back to equation (4.12), we see that for sufficiently large n, the effective element M_n is the sum of the element $\mathbf{L}^n g_n$ of $\mathscr{M}_k$ which is of weight strictly less than $2(1-\eta)n$ by estimate (4.11), and of the sum

$$\sum_{i=0}^{r-1} \frac{F^{(i)}(1)(-1)^i}{i!}\binom{n+r-i-1}{r-i-1},$$

which is a polynomial of degree at most $r-1$ in n with coefficients in $\mathscr{M}_k$ and of weight $2n$. The statement on the coefficients of the Hodge-Deligne polynomial follows.

It remains to show how to reduce to this when $a > 1$. We may decompose F in the following manner:

$$F(T) = \sum_{p=0}^{a-1}\sum_{j\geq 0} f_{aj+p}T^{aj+p} = \sum_{p=0}^{a-1} T^p F_p(T^a),$$

where $F_p(T) = \sum_{j\geq 0} f_{aj+p}T^j$, so that

$$Z(T) = \sum_{p=0}^{a-1} T^p \frac{F_p(T^a)}{(1-T^a)^r}.$$

Using the expansion we did above and putting $G_p(T) = \sum_{i\geq r} \frac{F_p^{(i)}(1)(-1)^i}{i!}(1-T)^{i-r} = \sum_{m\geq 0} g_{p,m}T^m$, we then have

$$Z(T) = \sum_{p=0}^{a-1} T^p \left(G_p(T^a) + \sum_{m\geq 0} T^{am} \sum_{i=0}^{r-1} \frac{F_p^{(i)}(1)(-1)^i}{i!} \binom{m+r-i-1}{r-i-1} \right)$$

Thus, for every $p \in \{0, \ldots, a-1\}$ and every $m \geq 0$, we have

$$[M_{am+p}]\mathbf{L}^{-(am+p)} = g_{p,m} + \sum_{i=0}^{r-1} \frac{F_p^{(i)}(1)(-1)^i}{i!} \binom{m+r-i-1}{r-i-1}.$$

Fix $p \in \{0, \ldots, a-1\}$, and assume first that there is some $d \in \mathbf{Z}$ and some $i \in \{0, \ldots, r-1\}$ such that $\{F_p^{(i)}(1)\}_d \neq 0$. Then we may conclude as above. If on the contrary such a d does not exist, this means that

$$w([M_{am+p}]\mathbf{L}^{-(am+p)}) \to -\infty$$

linearly in m (because $w(g_{p,m}) \to -\infty$ linearly in m), so that $\limsup \frac{\dim M_n}{n} < 1$ when n goes to infinity in the congruence class of p modulo a.

It remains to show that this last case does not occur for all p. For this, recall that $F(1) = \sum_{p=0}^{a-1} F_p(1)$, and, $F(1)$ being effective and non-zero, there exists d such that $\{F(1)\}_d \neq 0$. This means that $\{F_p(1)\}_d \neq 0$ for at least one p. $\square$

Chapter 5

The motivic Poisson formula

The aim of this chapter is to extend the scope of Hrushovski and Kazhdan's motivic Poisson formula. As explained in the introduction, it is an analogue of a weakened form of the classical Poisson formula for Schwartz-Bruhat functions $f : (\mathbb{A}_F)^n \to \mathbf{C}$ on (the n-th power, for some $n \geq 1$, of) the adeles of a global field F:

$$\sum_{x \in F^n} f(x) = c \sum_{y \in F^n} \mathscr{F} f(y) \tag{5.1}$$

for some multiplicative constant $c \in \mathbf{C}$, the Fourier transform being calculated with respect to a Haar measure on the locally compact group $\mathbb{A}_F^n$. The restriction f_v of a Schwartz-Bruhat function $f : \mathbb{A}_F \to C$ to the completion F_v of F at some non-archimedean place v is locally constant and compactly supported: by compactness, this means that there are integers $M \leq N$ such that, denoting by $\mathcal{O}_v, t$ respectively the ring of integers and a uniformiser of F_v, the function f_v is supported inside $t^M \mathcal{O}_v$ and invariant modulo $t^N \mathcal{O}_v$. Thus, f_v may be viewed as a function on the quotient $t^M \mathcal{O}_v / t^N \mathcal{O}_v$. Denoting by $\kappa(v)$ the residue field of F_v, this quotient group can naturally be identified with the $\kappa(v)$-points of the affine space of dimension $N - M$ over $\kappa(v)$, via

$$\begin{array}{ccc} t^M \mathcal{O}_v / t^N \mathcal{O}_v & \to & \mathbf{A}_{\kappa(v)}^{(M,N)} := \mathbf{A}_{\kappa(v)}^{N-M} \\ t^M x_M + \ldots + t^{N-1} x_{N-1} + t^N \mathcal{O}_v & \mapsto & (x_M, \ldots, x_{N-1}) \end{array} \tag{5.2}$$

Since the residue field $\kappa(v)$ is finite, the function f_v takes only a finite number of values, and its integral over F_v is given by the formula

$$\int_{F_v} f_v = q_v^{-N} \sum_{x \in t^M \mathcal{O}_v / t^N \mathcal{O}_v} f_v(x), \tag{5.3}$$

where q_v is the cardinality of $\kappa(v)$.

All the above definitions and equalities make heavy use of the local compactness of the adeles and of the local fields F_v. When the field F is the function field $k(C)$ of a smooth projective connected curve C over an algebraically closed field k, these local compactness properties fail. Hrushovski and Kazhdan's formalism from [HK09] allows nevertheless to define analogous objects in this setting, via a form of motivic integration. For example, as suggested by the identification in (5.2), one defines local Schwarz-Bruhat functions as elements of a relative Grothendieck ring with exponentials $\mathscr{E}xp\mathscr{M}_{\mathbf{A}_k^{(M,N)}}$ for some integers $M \leq N$. When M, N vary, these rings fit canonically into an inductive system, via maps that are interpreted respectively as extension by zero and pullback of functions. On the other hand, following formula (5.3), the *definition* of the integral of such a function $f \in \mathscr{E}xp\mathscr{M}_{\mathbf{A}_k^{(M,N)}}$ is

$$\int f = \mathbf{L}^{-N} \sum_{x \in \mathbf{A}_k^{(M,N)}} f(x),$$

where the sum in the right-hand side is a notation which stands for the image of f in the absolute Grothendieck ring $\mathscr{E}xp\mathscr{M}_k$ via the forgetful morphism $\mathscr{E}xp\mathscr{M}_{\mathbf{A}_k^{(M,N)}} \to \mathscr{E}xp\mathscr{M}_k$. More generally, one can define Fourier transforms of such functions by using the same kind of analogy.

In the same manner, an element f of the relative Grothendieck ring

$$\mathscr{E}xp\mathscr{M}_{\prod_{s\in S}\left(\mathbf{A}_k^{(M_s,N_s)}\right)^n},$$

for integers $M_s \leq N_s$, $s \in S$ may be seen as a motivic analogue of a Schwartz-Bruhat function on a finite product of powers of local fields $\prod_{s\in S} F_s^n$, for some finite set S of non-archimedean places of the global field F and some integer $n \geq 1$. The inclusion of the Riemann-Roch space

$$L(D) = k(C) \cap \prod_{s\in S} t_s^{M_s}\mathcal{O}_s = \{0\} \cup \{x \in k(C)^\times,\ \mathrm{div}(x) \geq -D\}$$

for the divisor $D = -\sum_{s\in S} M_s[s]$ on C into $\prod t_s^{M_s}\mathcal{O}_s$ induces a morphism

$$\theta : L(D)^n \to \prod_{s\in S} \left(\mathbf{A}_k^{(M_s,N_s)}\right)^n$$

via which we can pull back f. The class of $\theta^* f$ in $\mathscr{E}xp\mathscr{M}_k$ is then denoted $\sum_{x\in k(C)^n} f(x)$. Moreover, the same kind of construction can be done for the motivic Fourier transform $\mathscr{F}f$ of f. Denoting by g the genus of the curve C, the motivic Poisson formula

$$\sum_{x\in k(C)^n} f(x) = \mathbf{L}^{(1-g)n} \sum_{y\in k(C)^n} \mathscr{F}f(y)$$

proved by Hrushovski and Kazhdan is the motivic analogue of the above classical Poisson formula (5.1) for such functions.

This chapter focuses on building a framework in which this Poisson formula may be applied for families of such Schwartz-Bruhat functions, with varying set S, which will be crucial in chapter 6. The notion of symmetric product of a family of varieties from chapter 3, in the special case where the set of indices is $\mathbf{N}^p$ for some integer $p \geq 1$, will be central here, and therefore, we start by a review of those in this case in section 5.1. In chapter 6, the integer p will be the cardinality of the set $\mathscr{A}$ of irreducible components of the divisor at infinity in the equivariant compactification we will consider. Our families of functions will be defined as elements of the relative localised Grothendieck ring with exponentials over symmetric products

$$\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N) := S^{\mathbf{m}}((\mathbf{A}_C^{(\alpha-M_\iota,\beta+N_\iota)})_{\iota\in\mathbf{N}^p}), \tag{5.4}$$

where

- $\mathbf{m}\in\mathbf{N}^p$ is a p-tuple, which in chapter 6 will contain the degrees of the sections we count with respect to each of the irreducible components of the divisor at infinity.
- $(M_\iota)_{\iota\in\mathbf{N}^p}$ and $(N_\iota)_{\iota\in\mathbf{N}^p}$ will be families of non-negative integers, with $M_0=N_0=0$.
- $\alpha,\beta : C\to\mathbf{Z}$, $\alpha\leq 0\leq\beta$, are functions on the curve which are zero over some dense open subset U of C, which will enable us to take into account the irregular behaviour of the height function at a finite number of places, e.g. places of bad reduction.
- $\mathbf{A}_C^{(\alpha-M_\iota,\beta+N_\iota)}$ is the constructible set over the curve C given by $U\times\mathbf{A}_k^{(-M_\iota,N_\iota)}$ above U, and with fibres above $v\in C\setminus U$ given by the affine spaces $\mathbf{A}_k^{(\alpha_v-M_\iota,\beta_v+N_\iota)}$.

By construction, this symmetric product has a morphism to $S^{\mathbf{m}}C$. A point $D\in S^{\mathbf{m}}C(k)$ may be seen as an "effective zero-cycle" $\sum_{v\in C}\mathbf{m}_v v$ where $\mathbf{m}_v\in\mathbf{N}^p$ is such that $\sum_v\mathbf{m}_v=\mathbf{m}$. The fibre of the variety (5.4) above D will be

$$\prod_{v\in C}\mathbf{A}_k^{(\alpha_v-M_{\mathbf{m}_v},\beta_v+N_{\mathbf{m}_v})}, \tag{5.5}$$

that is, a product of affine spaces on which Schwartz-Bruhat functions in the sense of Hrushovski and Kazhdan may be considered. In the framework of chapter 6, this fibre will be the domain of definition of the characteristic function of the sections with poles of orders the coordinates $m_{v,1},\ldots,m_{v,p}$ of the vector $\mathbf{m}_v$ along the p irreducible components of the divisor at infinity.

The identification (5.2) involves the choice of a uniformiser at the place v, which in Hrushovski and Kazhdan's theory may be made arbitrarily, because only a finite number of places of the field F are involved. To be able to make such identifications for all places of F and keep all operations on families of Schwartz-Bruhat functions algebraic, we discuss in section 5.3.3 how we can choose uniformisers in a "uniform" way.

Though we define families of Schwartz-Bruhat functions (of level $\mathbf{m}$) in all generality to be elements of the ring $\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)}$, in fact we will use this definition in two important special cases, namely

- If all the integers N_ι are zero, the family is said to be *uniformly smooth*. By looking at the fibre (5.5), we see indeed that all the functions in such a family will be invariant modulo the same "compact open" subset $\prod_{v\in C}t_v^{\beta_v}\mathcal{O}_v$ of the adele ring of F. This will be the case for the family of characteristic functions of sections with given poles mentioned above. The arithmetic analogue of this is the fact that height functions are invariant modulo some compact open subset of the adeles.

— If all the integers M_ι are zero, the family is said to be *uniformly compactly supported.* Again, the terminology is clear from the fact that all functions in the family are zero outside $\prod_{v\in C} t_v^{\alpha_v}\mathcal{O}_v$.

In section 5.4, we go on to define Fourier transformation for such families of functions, so that it coincides with Hrushovski and Kazhdan's Fourier transform in each fibre above a rational point. This operation exchanges the above two types of families of functions. In section 5.5, we extend Hrushovski and Kazhdan's summation over rational points to families of uniformly compactly supported functions. Finally, in section 5.6, we formulate the Poisson formula for families of uniformly smooth functions.

5.1 Symmetric products

5.1.1 Multidimensional partitions

Fix an integer $p \geq 1$, and denote $\mathcal{I} = \mathbf{N}^p - \{0\}$ and $\mathcal{I}_0 = \mathbf{N}^p$. Consider the free abelian monoid over $\mathcal{I}$:

$$\mathbf{N}^{(\mathcal{I})} = \{(m_\iota)_{\iota\in\mathcal{I}} \in \mathbf{N}^{\mathcal{I}},\ m_\iota = 0 \text{ for almost all } \iota\}.$$

To an element $\pi = (m_\iota)_{\iota\in\mathcal{I}} \in \mathbf{N}^{(\mathcal{I})}$ we can associate canonically a p-tuple

$$\lambda(\pi) = \sum_{\iota\in\mathcal{I}} m_\iota \iota \in \mathcal{I}_0.$$

Thus, we have a well-defined map

$$\lambda : \mathbf{N}^{(\mathcal{I})} \longrightarrow \mathcal{I}_0.$$

We say π is a partition of $\mathbf{m} \in \mathcal{I}$ if $\lambda(\pi) = \mathbf{m}$.

Notation 5.1.1.1. Recall from notation 3.0.0.1 that another notation for partitions is as follows: a partition of $\mathbf{m}$ can be written in the form $[\mathfrak{a}_1, \ldots, \mathfrak{a}_r]$ where $\mathfrak{a}_1, \ldots, \mathfrak{a}_r \in \mathcal{I}$ are not necessarily distinct and such that $\mathfrak{a}_1 + \ldots + \mathfrak{a}_r = \mathbf{m}$. The order of the $\mathfrak{a}_i$ in this notation is not important: we consider $[\mathfrak{a}_1, \ldots, \mathfrak{a}_r]$ to be the same as $[\mathfrak{a}_{\sigma(1)}, \ldots, \mathfrak{a}_{\sigma(r)}]$ for all $\sigma \in \mathfrak{S}_r$.

Example 5.1.1.2. For $p = 1$, we recover partitions of integers: indeed, in this case an element π of $\mathbf{N}^{(\mathcal{I})}$ is a finite family $(m_i)_{i\geq 0}$ of non-negative integers, $\lambda(\pi) = \sum_{i\geq 1} m_i i$ is some integer m, and π determines a partition

$$\sum_{i\geq 1} \underbrace{(i + \ldots + i)}_{m_i \text{ times}} = m$$

of the integer m, the non-negative integer m_i being the number of occurrences of i in this partition.

For $p = 2$, consider for example

$$\pi = \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}\right]$$

It is a partition of

$$\begin{pmatrix} 4 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Note that this 2-dimensional partition gives in particular a one-dimensional partition for each coordinate: $[2,2]$ for the first coordinate, and $[1,1,3]$ for the second one. However, it carries more information than just the choice of these two partitions, since it also matches up their parts in some way. Thus, the partition

$$\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

is different from π, but yields the same partitions of its coordinates.

5.1.2 From partitions to symmetric products

Let k be a perfect field. Let $\pi = (n_\iota)_{\iota \in \mathcal{I}} \in \mathbf{N}^{(\mathcal{I})}$ and let $\mathscr{X} = (X_\iota)_{\iota \in \mathcal{I}_0}$ be a family of constructible subsets of projective varieties over a quasi-projective k-variety X. Assume moreover that there is an open subset U of X such that $X_0 \times_X U \simeq U$ and such that $X \setminus U$ is a finite union of closed points. In chapter 3, in particular in sections 3.1.3, 3.2.1 and 3.9.1, we defined a notion of *symmetric product* $S^\pi \mathscr{X}$. It follows from the construction that $S^\pi \mathscr{X}$ comes with a natural morphism to $S^\pi X$. Define also for any $\mathbf{m} \in \mathcal{I}$, the constructible set $S^{\mathbf{m}}(\mathscr{X})$ to be the disjoint union of the $S^\pi(\mathscr{X})$ for all partitions π of $\mathbf{m}$.

Remark 5.1.2.1. Recall that when $p = 1$, for any quasi-projective variety X, the variety $S^n X$ can also be obtained directly by taking the quotient of X^n by the natural permutation action of the symmetric group $\mathfrak{S}_n$. For $p \geq 2$ and $\mathbf{n} = (n_1, \ldots, n_p) \in \mathbf{N}^p$, note that giving an element $\sum_v \mathbf{n}_v v$ of $S^{\mathbf{n}} X$ is equivalent to giving its p components

$$\left(\sum_v n_{i,v} v\right)_{1 \leq i \leq p} \in S^{n_1} X \times \ldots \times S^{n_p} X.$$

Thus, we in fact have a piecewise isomorphism

$$S^{\mathbf{n}} X \simeq S^{n_1} X \times \ldots \times S^{n_p} X.$$

5.2 Motivic Schwartz-Bruhat functions and Poisson formula of Hrushovski and Kazhdan

We start with a review of Hrushovski and Kazhdan's motivic Poisson formula from [HK09], following the exposition in sections 1.2 and 1.3 of [ChL]. Let k be a perfect field.

5.2.1 Local Schwartz-Bruhat functions

Let $F = k((t))$ be the completion of a function field of a curve at a closed point, with uniformiser t, ring of integers $\mathcal{O}$ and residue field k. In [HK09], Hrushovski and Kazhdan considered local motivic exponential Schwartz-Bruhat functions on F: such functions are analogues of classical Schwartz-Bruhat functions on non-archimedean local fields, that is, locally constant and compactly supported functions. For each such function φ, there exist integers $M \leq N$ such that φ is zero outside $t^M\mathcal{O}$, and invariant modulo $t^N\mathcal{O}$, so that φ can be seen as a function on the quotient $t^M\mathcal{O}/t^N\mathcal{O}$. The latter can be endowed with the structure of a k-variety, and more precisely of an affine space over k, through the following identification:

$$\begin{array}{ccc} t^M\mathcal{O}/t^N\mathcal{O} & \longrightarrow & \mathbf{A}_k^{N-M}(k) \\ x_M t^M + \ldots + x_{N-1}t^{N-1} + t^N\mathcal{O} & \mapsto & (x_M, \ldots, x_{N-1}) \end{array} \tag{5.6}$$

This affine space is denoted by $\mathbf{A}_k^{(M,N)}$. More generally, for any $n \geq 1$ we denote by $\mathbf{A}_k^{n(M,N)}$ the affine space $(\mathbf{A}_k^{(M,N)})^n$, which is viewed as a motivic incarnation of $(t^M\mathcal{O}/t^N\mathcal{O})^n$. Thus, a Schwartz-Bruhat function of level (M,N) on F^n will by definition be an element of $\mathscr{S}(F^n;(M,N)) := \mathscr{E}xp\mathscr{M}_{\mathbf{A}_k^{n(M,N)}}$. An element E of this ring can indeed be interpreted as a function

$$\varphi : \mathbf{A}_k^{n(M,N)} \longrightarrow \mathscr{E}xp\mathscr{M}_{k(x)}$$

by sending a point $x \in \mathbf{A}_k^{n(M,N)}$ to the class of the fibre E_x, where $k(x)$ is the residue field of x. As M and N vary, the rings $\mathscr{S}(F^n;(M,N))$ fit into a directed system the direct limit of which is the total ring $\mathscr{S}(F^n)$ of Schwartz-Bruhat functions. More precisely, let us point out that the natural injection $t^M\mathcal{O}/t^N\mathcal{O} \to t^{M-1}\mathcal{O}/t^N\mathcal{O}$ gives rise to the closed immersion

$$\begin{array}{cccc} i : & \mathbf{A}_k^{(M,N)} & \longrightarrow & \mathbf{A}_k^{(M-1,N)} \\ & (x_M, \ldots, x_{N-1}) & \mapsto & (0, x_M, \ldots, x_{N-1}) \end{array} \tag{5.7}$$

whereas the natural projection $t^M\mathcal{O}/t^{N+1}\mathcal{O} \to t^M\mathcal{O}/t^N\mathcal{O}$ induces a morphism

$$\begin{array}{cccc} p : & \mathbf{A}_k^{(M,N+1)} & \longrightarrow & \mathbf{A}_k^{(M,N)} \\ & (x_M, \ldots, x_N) & \mapsto & (x_M, \ldots, x_{N-1}) \end{array} \tag{5.8}$$

which is a trivial fibration with fibre $\mathbf{A}^1$. They induce ring morphisms $i_! : \mathscr{S}(F^n;(M,N)) \to \mathscr{S}(F^n;(M-1,N))$ (extension by zero) and $p^* : \mathscr{S}(F^n;(M,N)) \to \mathscr{S}(F^n;(M,N+1))$.

5.2.2 Integration

For any Schwartz-Bruhat function $\varphi \in \mathscr{S}(F^n)$, choosing a pair (M,N) such that $\varphi \in \mathscr{S}(F^n;(M,N))$ one may define, using the exponential sum notation,

$$\int_{F^n} \varphi(x)\mathrm{d}x = \mathbf{L}^{-nN} \sum_{x \in \mathbf{A}_k^{n(M,N)}} \varphi(x) \in \mathscr{E}xp\mathscr{M}_k.$$

This does not depend on the choice of (M, N), and defines an $\mathscr{E}xp\mathscr{M}_k$-linear map

$$\int_{F^n} : \mathscr{S}(F^n) \to \mathscr{E}xp\mathscr{M}_k.$$

5.2.3 Fourier kernel

Fix a k-linear function $r : F \longrightarrow k$ such that there is an integer a with $r_{|t^a\mathcal{O}} = 0$. The least such integer a is called the *conductor* of r and denoted by ν. Note that because of its linearity, r is invariant modulo $r^\nu\mathcal{O}$, so that for any pair of integers (M, N) such that $M \leq N$ and $\nu \leq N$, it induces a well-defined morphism $r^{(M,N)} : \mathbf{A}_k^{(M,N)} \longrightarrow \mathbf{A}_k^1$.

On the other hand, for any two pairs of integers (M, N) and (M', N') satisfying $M \leq N$ and $M' \leq N'$, the product map $F \times F \longrightarrow F$ induces a well-defined map on classes

$$t^M\mathcal{O}/t^N\mathcal{O} \times t^{M'}\mathcal{O}/t^{N'}\mathcal{O} \longrightarrow t^{M+M'}\mathcal{O}/t^{N''}\mathcal{O}$$

where $N'' \geq \min(M' + N, M + N')$, that is, a well-defined morphism

$$\mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')} \longrightarrow \mathbf{A}_k^{(M+M',N'')}. \tag{5.9}$$

Whenever $N'' \geq \nu$, this map may be composed with $r^{(M+M',N'')}$ which yields a morphism

$$\mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')} \longrightarrow \mathbf{A}_k^1.$$

More generally, taking n-th powers and summing the corresponding maps, we get a morphism

$$\mathbf{A}_k^{n(M,N)} \times \mathbf{A}_k^{n(M',N')} \longrightarrow \mathbf{A}_k^1.$$

Note that when $M' = \nu - N$ and $N' = \nu - M$, the condition $N'' \geq \nu$ is satisfied. The morphism

$$r : \mathbf{A}_k^{n(M,N)} \times \mathbf{A}_k^{n(\nu-N,\nu-M)} \to \mathbf{A}_k^1 \tag{5.10}$$

defined in this setting is called the *Fourier kernel*.

5.2.4 Local Fourier transform

The Fourier transform of a function $\varphi \in \mathscr{S}(F^n; (M, N))$ is defined to be the element $\mathscr{F}\varphi \in \mathscr{S}(F^n; (\nu - N, \nu - M))$ given by

$$\mathscr{F}\varphi = \mathbf{L}^{-Nn}\varphi \cdot [\mathbf{A}_k^{n(M,N)} \times \mathbf{A}_k^{n(\nu-N,\nu-M)}, r],$$

where r is the morphism (5.10), and the product is taken in $\mathscr{E}xp\mathscr{M}_{\mathbf{A}_k^{n(M,N)}}$, and viewed in $\mathscr{E}xp\mathscr{M}_{\mathbf{A}_k^{n(\nu-N,\nu-M)}}$. For every $y \in \mathbf{A}_k^{n(\nu-N,\nu-M)}$, using the notation from section 2.1.4 of chapter 2, as well as the definition of the integral in section 5.2.2 we have

$$\mathscr{F}\varphi(y) = \int_F \varphi(x)\psi(r(xy))\mathrm{d}x.$$

5.2.5 Global Schwartz-Bruhat functions

One can extend the above definitions to finite products of fields. Consider a finite family $(F_v)_{v\in S}$ of such fields $F_v = k_v((t_v))$, with local parameters t_v and residue fields k_v (which are assumed to be finite extensions of k) and an integer $n \geq 1$. For any family of pairs of integers $(M_v, N_v)_{v\in S}$, with $M_v \leq N_v$ the space of Schwartz-Bruhat functions on $\prod_{v\in S} F_v^n$ of levels $(M_v, N_v)_{v\in S}$ is defined to be

$$\mathscr{S}\left(\prod_{v\in S} F_v^n; (M_v, N_v)_{v\in S}\right) := \mathscr{E}xp\mathscr{M}_{\prod_{v\in S}\mathrm{Res}_{k_v/k}\mathbf{A}_{k_v}^{n(M_v,N_v)}},$$

where $\mathrm{Res}_{k_v/k}$ denotes the functor of Weil restriction of scalars. In the case where k is algebraically closed, we have

$$\mathscr{S}\left(\prod_{v\in S} F_v^n; (M_v, N_v)_{v\in S}\right) := \mathscr{E}xp\mathscr{M}_{\prod_{v\in S}\mathbf{A}_{k}^{n(M_v,N_v)}}.$$

The ring $\mathscr{S}\left(\prod_{v\in S} F_v^n\right)$ is defined as a direct limit of these rings, with the appropriate compatibilities. The notions of integral, Fourier kernel and Fourier transform defined above extend easily to such functions (see [ChL], 1.2.10).

We are going to use this in the following setting: let k be a perfect field, C a smooth projective curve over k, $F = k(C)$ its function field. Denote by $\mathbb{A}_F$ the ring of adeles of the field F. The rings $\mathscr{S}(\prod_{v\in S} F_v^n)$, for finite sets S of closed points of C, form a directed system, and their direct limit is the ring $\mathscr{S}(\mathbb{A}_F^n)$ of global motivic Schwartz-Bruhat functions on $\mathbb{A}_F^n$.

5.2.6 Summation over rational points

For details on the contents of this paragraph, see [ChL], 1.3.5. Let φ be a global Schwartz-Bruhat function on $\mathbb{A}_F^n$, represented by a class in the ring

$$\mathscr{E}xp\mathscr{M}_{\prod_{v\in S}\mathrm{Res}_{k_v/k}\mathbf{A}_{k_v}^{n(M_v,N_v)}}$$

for some finite set S of closed points of C and some family $(M_v, N_v)_{v\in S}$ of pairs of integers such that $M_v \leq N_v$ for all $v \in S$.

Consider the divisor $D = -\sum_{v\in S} M_v v$ on C. For every $v \in C$, the natural embedding of the field $F = k(C)$ into its completion F_v maps the Riemann-Roch space

$$L(D) = \{0\} \cup \{f \in k(C)^\times, \mathrm{div}(f) \geq \sum_v M_v v\}$$

into $t^{M_v}\mathcal{O}_v$. This gives rise to a morphism of algebraic varieties

$$\theta : L(D)^n \longrightarrow \left(\prod_v \mathrm{Res}_{k_v/k}\mathbf{A}_{k_v}^{(M_v,N_v)}\right)^n.$$

The sum over rational points of $\varphi \in \mathscr{E}xp\mathscr{M}_{\left(\prod_v \mathrm{Res}_{k_v/k}\mathbf{A}_{k_v}^{(M_v,N_v)}\right)^n}$, denoted by $\sum_{x\in F^n}\varphi(x)$, is then defined to be the image in $\mathscr{E}xp\mathscr{M}_k$ of the pull-back $\theta^*\varphi \in \mathscr{E}xp\mathscr{M}_{L(D)^n}$. It does not depend on choices.

Remark 5.2.6.1. This definition is motivated by the fact that, when k is a finite field, for a Schwartz-Bruhat function $\varphi : (\mathbf{A}_F)^n \to \mathbf{C}$ which is supported inside $\left(\prod_v t^{M_v}\mathcal{O}_v\right)^n$, we have $\varphi(x) = 0$ for all $x \notin F^n \cap \left(\prod_v t^{M_v}\mathcal{O}_v\right)^n = L(D)^n$, so that we have the equality

$$\sum_{x\in F^n}\varphi(x) = \sum_{x\in L(D)^n}\varphi(x).$$

5.2.7 Motivic Poisson formula

We fix a non-zero meromorphic differential form $\omega \in \Omega^1_{F/k}$. For every $v \in C$, we choose the linear map $r_v : F_v \to k$ defined by $r_v : x \mapsto \mathrm{res}_v(x\omega)$ and we compute Fourier transforms with respect to those. Theorem 1.3.10 in [ChL] states that for $\varphi \in \mathscr{S}(\mathbb{A}_F^n)$, we have $\mathscr{F}\varphi \in \mathscr{S}(\mathbb{A}_F^n)$ and

$$\sum_{x\in F^n}\varphi(x) = \mathbf{L}^{(1-g)n}\sum_{y\in F^n}\mathscr{F}\varphi(y).$$

5.3 Families of Schwartz-Bruhat functions

5.3.1 Parametrising domains of definition

Let k be an algebraically closed field of characteristic zero, and C a smooth projective connected curve over k.

Definition 5.3.1.1. Let X be a variety over k. A function $\alpha : X \to \mathbf{Z}$ is said to be constructible if for every $n \in \mathbf{Z}$, $\alpha^{-1}(n)$ is a constructible subset of X.

Remark 5.3.1.2. When $X = C$, $\alpha : C \to \mathbf{Z}$ is constructible if and only if it is constant on some dense open subset of C. If it is zero on some dense open subset of C, we say it is *almost zero*.

The value of a constructible function α at a point $v \in C$ will be denoted α_v.

Definition 5.3.1.3. Let $M, N : C \longrightarrow \mathbf{Z}$ be constructible functions such that $M \leq N$. Let $U \subset C$ be a dense open set over which they are constant, equal respectively to $M_0 \in \mathbf{Z}$ and $N_0 \in \mathbf{Z}$. We will denote by $\mathbf{A}_C^{(M,N)}$ the variety over C isomorphic to $U \times \mathbf{A}_k^{(M_0,N_0)}$ over U, and with fibre above $u \notin U$ given by $\mathbf{A}_k^{(M_u,N_u)}$. Furthermore, we will denote by $\left(\mathbf{A}_C^{(M,N)}\right)^n$, or $\mathbf{A}_C^{n(M,N)}$, the variety over C defined by

$$\mathbf{A}_C^{(M,N)} \times_C \ldots \times_C \mathbf{A}_C^{(M,N)},$$

where the product contains n factors.

Recall that we denote by $\mathcal{I}_0$ the additive monoid $\mathbf{Z}^p_{\geq 0}$ and $\mathcal{I} = \mathcal{I}_0 \setminus \{0\}$. Fix two almost zero functions $\alpha, \beta : C \longrightarrow \mathbf{Z}$ such that $\alpha \leq 0 \leq \beta$. Denote by U a dense open subset of C over which α and β are zero. Fix also two families of non-negative integers $M = (M_\iota)_{\iota\in\mathcal{I}_0}$ and $N = (N_\iota)_{\iota\in\mathcal{I}_0}$, with $M_0 = 0$ and $N_0 = 0$.

We have a family of varieties $\left(\mathbf{A}_C^{n(\alpha - M_\iota, \beta + N_\iota)}\right)_{\iota\in\mathcal{I}_0}$ over C, giving rise to symmetric products

$$\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N) := S^{\mathbf{m}}\left(\left(\mathbf{A}_C^{n(\alpha - M_\iota, \beta + N_\iota)}\right)_{\iota\in\mathcal{I}_0}\right) \tag{5.11}$$

for all $\mathbf{m} \in \mathcal{I}_0$.

By the definition of symmetric products, all these objects are varieties endowed with natural morphisms to $S^{\mathbf{m}}C$, which we denote $\varpi_{\mathbf{m}} : \mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N) \longrightarrow S^{\mathbf{m}}C$ for every $\mathbf{m} \in \mathcal{I}_0$.

Remark 5.3.1.4. For clarity, let us point out that, where in definition 5.3.1.3 objects denoted M, N were *constructible functions*, from now on, except when explicitly stated (that is, except in section 5.10), they will denote integers. The possible variation above a finite number of places will be taken care of by the almost zero functions α and β.

Remark 5.3.1.5. Denote by $\Sigma = \{x_1, \ldots, x_s\}$ the complement $C \setminus U$, which is a finite union of closed points. By corollary 3.3.3.1 from section 3.3.3 of chapter 3, $\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)$ is the disjoint union of locally closed subsets isomorphic to products

$$S^{\mathbf{m}_0}\left(\left(\mathbf{A}_U^{n(-M_\iota,N_\iota)}\right)_{\iota\in\mathcal{I}}\right) \times \prod_{j=1}^{s} S^{\mathbf{m}_j}\left(\left(\mathbf{A}_{\{x_j\}}^{n(\alpha_{x_j} - M_\iota, \beta_{x_j} + N_\iota)}\right)_{\iota\in\mathcal{I}_0}\right) \tag{5.12}$$

for all $\mathbf{m}_0, \ldots, \mathbf{m}_s \in \mathcal{I}_0$ such that $\mathbf{m}_0 + \ldots + \mathbf{m}_s = \mathbf{m}$. By example 3.2.1.5 of chapter 3, the variety (5.12) is isomorphic to

$$S^{\mathbf{m}_0}\left(\left(\mathbf{A}_U^{n(-M_\iota,N_\iota)}\right)_{\iota\in\mathcal{I}}\right) \times \prod_{j=1}^{s} \mathbf{A}_{\{x_j\}}^{n(\alpha_{x_j} - M_{\mathbf{m}_j}, \beta_{x_j} + N_{\mathbf{m}_j})}.$$

Remark 5.3.1.6. Though these definitions depend on the choice of U, the ring

$$\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)}$$

which we will consider later won't depend on it.

5.3.2 The fibres of the domains of definition

Let $D \in S^{\mathbf{m}}C$, with residue field $\kappa(D)$. We want to describe the fibre $\varpi_{\mathbf{m}}^{-1}(D)$ above the point D. We know that $S^{\mathbf{m}}C$ is the disjoint union of locally closed subsets isomorphic to

$$S^{\mathbf{m}_0}U \times S^{\mathbf{m}-\mathbf{m}_0}\Sigma$$

for all $\mathbf{m}_0 \in \mathcal{I}_0$ such that $\mathbf{m}_0 \leq \mathbf{m}$. Let $\mathbf{m}_0 \leq \mathbf{m}$ be such that D belongs to the subset corresponding to $\mathbf{m}_0$. Since the field k is algebraically closed, $S^{\mathbf{m}-\mathbf{m}_0}\Sigma$ is a disjoint union of a finite number of closed points, and therefore the variety $S^{\mathbf{m}_0}U \times S^{\mathbf{m}-\mathbf{m}_0}\Sigma$ has a finite number of connected components each corresponding to a point of $S^{\mathbf{m}-\mathbf{m}_0}\Sigma$. Thus, the schematic point D of $S^{\mathbf{m}}C$ is of the form (D_U, D_Σ), where $D_U \in S^{\mathbf{m}_0}U$ and $D_\Sigma \in S^{\mathbf{m}-\mathbf{m}_0}\Sigma$. Let $\pi = (m_\iota)_{\iota\in\mathcal{I}}$ be the partition of $\mathbf{m}_0$ such that $D_U \in S^\pi U$. On the other hand, D_Σ is an effective zero-cycle with coefficients in $\mathcal{I}_0$ and with support contained in Σ, so it may be written in the form

$$D_\Sigma = \mathbf{m}_1 x_1 + \ldots + \mathbf{m}_s x_s \in S^{\mathbf{m}-\mathbf{m}_0}\Sigma$$

for $\mathbf{m}_1, \ldots, \mathbf{m}_s \in \mathcal{I}_0$ such that $\mathbf{m}_0 + \ldots + \mathbf{m}_s = \mathbf{m}$. Using remark 5.3.1.5 as well as proposition 3.4.0.1 from section 3.4 of chapter 3, the fibre above D is of the form

$$\prod_{\iota\in\mathcal{I}} \mathbf{A}_{\kappa(D)}^{m_\iota n(N_\iota+M_\iota)} \times_{\kappa(D)} \prod_{j=1}^{s} \mathbf{A}_{\kappa(D)}^{n(\alpha_{x_j}-M_{\mathbf{m}_j},\beta_{x_j}+N_{\mathbf{m}_j})}. \tag{5.13}$$

More precisely, we have the diagram

$$\begin{array}{ccc} \left(\prod_{\iota\in\mathcal{I}} U^{m_\iota}\right)_* & \longleftarrow & \left(\prod_{\iota\in\mathcal{I}} \mathbf{A}_U^{m_\iota n(-M_\iota,N_\iota)}\right)_{*,U} \\ \downarrow & & \downarrow \\ S^\pi U & \longleftarrow & S^\pi((\mathbf{A}_U^{m_\iota n(-M_\iota,N_\iota)})_{\iota\in\mathcal{I}}) \end{array}$$

where the vertical maps are the quotient morphisms, the upper horizontal line is a trivial vector bundle, and the lower line is a vector bundle. Let D'_U be a point of $\left(\prod_{\iota\in\mathcal{I}} U^{m_\iota}\right)_*$ lifting $D_U \in S^\pi U$. Taking fibres above D_U and D'_U and denoting by K the residue field of D'_U (so that K is a finite extension of $\kappa(D)$), the diagram becomes

$$\begin{array}{ccc} \operatorname{Spec} K & \longleftarrow & \prod_{\iota\in\mathcal{I}} \mathbf{A}_K^{m_\iota n(-M_\iota,N_\iota)} \\ \downarrow & & \downarrow \\ \operatorname{Spec}\kappa(D) & \longleftarrow & \prod_{\iota\in\mathcal{I}} \mathbf{A}_{\kappa(D)}^{m_\iota n(N_\iota+M_\iota)} \end{array}$$

so that we have a linear K-isomorphism

$$\prod_{\iota\in\mathcal{I}} \mathbf{A}_K^{m_\iota n(-M_\iota,N_\iota)} \simeq \prod_{\iota\in\mathcal{I}} \mathbf{A}_{\kappa(D)}^{m_\iota n(N_\iota+M_\iota)} \otimes_{\kappa(D)} K.$$

In other words, the fibre above D_U is a *twisted form* of the variety

$$\prod_{\iota\in\mathcal{I}} \mathbf{A}_{\kappa(D)}^{m_\iota n(-M_\iota,N_\iota)}$$

which splits above the finite extension K of $\kappa(D)$.

We may conclude that the fibre $\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)_D$ above D may be seen as the domain of definition of a Schwartz-Bruhat function, up to extension of scalars to some finite extension of $\kappa(D)$.

Remark 5.3.2.1. We write $\mathbf{A}_{\kappa(D)}^{m_\iota n(N_\iota+M_\iota)}$ instead of $\mathbf{A}_{\kappa(D)}^{m_\iota n(M_\iota,N_\iota)}$ to signify that through the quotient morphism, the chosen identification of the form (5.6) is twisted. Therefore, when looking at functions on such a fibre $\omega_{\mathbf{m}}^{-1}(D)$ later, for example in section 5.5.1, we will pull them back via the quotient morphism before performing operations on them which via (5.6) can be understood as analogues of operations from classical Fourier theory .

Let us make the particular case where $D \in S^{\mathbf{m}}C(k)$ more explicit. Recall k is algebraically closed. Thus, D may be seen as an effective zero-cycle $\sum \iota_v v$ for points $v \in C(k)$ and $\iota_v \in \mathcal{I}_0$, and (5.13) may be written in the form

$$\prod_{v\in C} \mathbf{A}_k^{n(\alpha_v - M_{\iota_v}, \beta_v + N_{\iota_v})}$$

because the field k is algebraically closed.

5.3.3 Uniform choice of uniformisers

In section 5.3.1, we have defined families of domains of definition of Schwartz-Bruhat functions: the product $\prod_v \mathbf{A}_k^{(M_v,N_v)}$ has to be understood as representing $\prod_v t_v^{M_v}\mathcal{O}_v / t_v^{N_v}\mathcal{O}_v$. However, this identification depends on the choice of the uniformisers t_v at each place v, and therefore so will some of the operations we are going to perform in what follows. This choice has to be made as uniformly as possible, so that these operations remain algebraic. We explain in this section how this can be done.

Lemma 5.3.3.1. *Fix a non-constant element $t \in k(C)$. Then there is a dense open set $U \subset C$ such that for all $v \in U$, the function $t_v = t - t(v)$ is a local parameter at v.*

Proof. Denote by U_0 an open dense set of C on which t is regular. We therefore get a holomorphic differential $\mathrm{d}t$ on U_0. It is non-vanishing when restricted to some open dense subset U of U_0. At any $v \in U$, the function t_v is an element of the maximal ideal $\mathfrak{m}_v$, and its differential $\mathrm{d}(t_v) = \mathrm{d}t$ is non-zero, so it is a local parameter. □

From now on, we fix such an element $t \in k(C)$. The uniformiser at any place v in the open set U furnished by the lemma will be given by $t_v = t - t(v)$. For $v \in C \backslash U$, we fix some arbitrary uniformiser t_v.

Lemma 5.3.3.2. *Let $f \in k(C)$. For any $v \in C$, write the t_v−adic expansion of f as*

$$\sum_{p\in\mathbf{Z}} a_p(f,v) t_v^p \in F_v.$$

There is an open dense subset U' of U such that for any integer p, the map

$$\begin{array}{rcl} a_p(f,\cdot): U' & \longrightarrow & k \\ v & \mapsto & a_p(f,v) \end{array}$$

is a regular function.

Proof. Denote by U' an open dense subset of U over which f is regular, so that the a_i with $i < 0$ are identically zero. For any $v \in U'$, we have by definition $f(v) = a_0(f, v)$, and therefore $a_0(f, \cdot) = f_{|U'}$ is a regular map. Now, the differential $\mathrm{d}f$ is a holomorphic differential on U', so there is a regular function $f_1 : U' \longrightarrow k$ such that $\mathrm{d}f = f_1 \mathrm{d}t$. On the other hand, differentiating the t_v-adic expansion of f, we get, since $\mathrm{d}t = \mathrm{d}t_v$,

$$\mathrm{d}f = (a_1(f, v) + 2a_2(f, v)t_v + 3a_2(f, v)t_v^2 + \ldots)\mathrm{d}t$$

Thus, since the differential $\mathrm{d}t$ doesn't vanish on U', we have $a_1(f, \cdot) = f_1$, which is regular. To prove regularity of $a_2(f, \cdot)$, replace f by f_1 and proceed in the same way. By induction, we get regularity of all a_p's. □

5.3.4 Families of Schwartz-Bruhat functions

Definition 5.3.4.1. Let $\mathbf{m} \in \mathscr{I}_0$. Let $\alpha, be : C \to \mathbf{Z}$ be almost zero functions such that $\alpha \leq 0 \leq \beta$, and let $M = (M_\iota)_{\iota \in \mathscr{I}_0}$, $N = (N_\iota)_{\iota \in \mathscr{I}_0}$ be two families of non-negative integers such that $M_0 = N_0 = 0$. The elements of $\mathscr{E}xp\mathscr{M}_{\mathscr{A}_\mathbf{m}(\alpha,\beta,M,N)}$ are called constructible families of Schwartz-Bruhat functions of level $\mathbf{m}$.

Let $\Phi \in \mathscr{E}xp\mathscr{M}_{\mathscr{A}_\mathbf{m}(\alpha,\beta,M,N)}$ be such a family of functions, and let D be a schematic point of $S^\mathbf{m}C$. The fibre of $\mathscr{A}_\mathbf{m}(\alpha, \beta, M, N)$ above D has been computed in (5.13). Restricting Φ to it, we obtain, up to extension of scalars to a finite extension of the residue field of D, a Schwartz-Bruhat function Φ_D in the sense of Hrushovski and Kazhdan. Thus, Φ gives rise to a family of "twisted" Schwartz-Bruhat functions $(\Phi_D)_{D \in S^\mathbf{m}C}$ indexed by $D \in S^\mathbf{m}C$.

In the particular case when $D \in S^\mathbf{m}C(k)$, denoting by $|D|$ the support of the effective zero-cycle D, in the notation of section 5.2.5, we have

$$\Phi_D \in \mathscr{S}\left(\prod_{v \in |D| \cup \Sigma} k(C)_v^n, (\alpha_v - M_{\iota_v}, \beta_v + N_{\iota_v})_{v \in |D| \cup \Sigma}\right),$$

and $k(C)_v$ is the completion of the function field $k(C)$ at the place v.

5.3.5 Uniformly smooth or uniformly compactly supported families

In a similar manner to (5.7) and (5.8), we may define constructible morphisms

$$p : \mathbf{A}_C^{(\alpha - M_\iota, \beta + N_\iota)} \to \mathbf{A}_C^{(\alpha - M_\iota, \beta)}$$

and

$$i : \mathbf{A}_C^{(\alpha, \beta + N_\iota)} \to \mathbf{A}_C^{(\alpha - M_\iota, \beta + N_\iota)}$$

for every $\iota \in \mathscr{I}_0$. Above $v \in C$, the first one is a projection on the first $\beta_v - \alpha_v + M_\iota$ coordinates, and the second one is

$$(x_{\alpha_v}, \ldots, x_{\beta_v + N_\iota - 1}) \mapsto (\underbrace{0, \ldots, 0}_{M_\iota \text{ coordinates}}, x_{\alpha_v}, \ldots, x_{\beta_v + N_\iota - 1}).$$

Taking symmetric products, we get, for every $\mathbf{m} \in \mathcal{I}_0$, morphisms

$$p_{\mathbf{m}} : S^{\mathbf{m}}((\mathbf{A}_C^{(\alpha-M_\iota,\beta+N_\iota)})_{\iota\in\mathcal{I}_0}) \to S^{\mathbf{m}}((\mathbf{A}_C^{(\alpha-M_\iota,\beta)})_{\iota\in\mathcal{I}_0}),$$

$$i_{\mathbf{m}} : S^{\mathbf{m}}((\mathbf{A}_C^{(\alpha,\beta+N_\iota)})_{\iota\in\mathcal{I}_0}) \to S^{\mathbf{m}}((\mathbf{A}_C^{(\alpha-M_\iota,\beta+N_\iota)})_{\iota\in\mathcal{I}_0}),$$

that is,

$$p_{\mathbf{m}} : \mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N) \to \mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,0)$$

and

$$i_{\mathbf{m}} : \mathscr{A}_{\mathbf{m}}(\alpha,\beta,0,N) \to \mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N).$$

Those induce injective ring morphisms

$$p_{\mathbf{m}}^* : \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,0)} \to \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)}$$

and

$$i_{\mathbf{m},!} : \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,0,N)} \to \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)}.$$

Definition 5.3.5.1. Let $\Phi \in \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)}$ be a constructible family of Schwartz-Bruhat functions of level $\mathbf{m}$. It is said to be

— *uniformly smooth* if it belongs to the image of the morphism $p_{\mathbf{m}}^*$.

— *uniformly compactly supported* if it belongs to the image of the morphism $i_{\mathbf{m},*}$.

A word of explanation on this terminology, which is inherited from the classical p-adic setting: let Φ be a constructible family of functions. If Φ is uniformly smooth, then every Φ_D for $D = \sum_v \iota_v v \in S^{\mathbf{m}}C(k)$ may be seen as an element of $\mathscr{E}xp\mathscr{M}_{\prod_v \mathbf{A}_k^{n(\alpha_v - M_{\iota_v},\beta_v)}}$, that is as a function on $\prod_v t^{\alpha_v - M_{\iota_v}}\mathcal{O}_v$, invariant modulo $\prod_v t^{\beta_v}\mathcal{O}_v$, this invariance domain being independent of D. In the same manner, if Φ is uniformly compactly supported, all Φ_D are supported inside $\prod_v t^{\alpha_v}\mathcal{O}_v$ independently of D.

5.4 Fourier transformation in families

5.4.1 Local construction of the Fourier kernel in families

Let $\omega \in \Omega_{F/k}$ be a non-zero meromorphic differential form. For every closed point $v \in C$ we write $\nu_v = -\mathrm{ord}_v\omega$, so that $\mathrm{div}\,\omega = -\sum_{v\in C}\nu_v v$. Then for every v we get a k-linear map $r_v : F_v \longrightarrow k$ given by

$$r_v(x) = \mathrm{res}_v(x\omega).$$

It is non-zero, and its conductor, that is, the least integer a such that $r_{v|t^a\mathcal{O}_v}$ is zero, is equal to ν_v.

Let $M \leq N$ be constructible functions as in definition 5.3.1.3, with the additional assumption that for every v, we have $\nu_v \leq N_v$. Using lemma 5.3.3.2, we see that the map r

gives rise to a piecewise morphism $r^{(M,N)} : \mathbf{A}_C^{(M,N)} \longrightarrow \mathbf{A}_k^1$, sending an element (v,x) to $r_v(x)$.

Fix two additional constructible functions $M', N' : C \longrightarrow \mathbf{Z}$ such that $M' \leq N'$. For every v, the product map $F_v \times F_v \longrightarrow F_v$ defines a morphism

$$\mathbf{A}_{\kappa(v)}^{(M_v,N_v)} \times_{\kappa(v)} \mathbf{A}_{\kappa(v)}^{(M'_v,N'_v)} \longrightarrow \mathbf{A}_{\kappa(v)}^{(M_v+M'_v,N''_v)} \tag{5.14}$$

where $N'' = \min\{M+N', M'+N\}$ (see (5.9)). More precisely, there is a morphism of constructible sets over C

$$\mathbf{A}_C^{(M,N)} \times_C \mathbf{A}_C^{(M',N')} \longrightarrow \mathbf{A}_C^{(M+M',N'')}$$

such that for every $v \in C$ the induced morphism on the fibre above v is (5.14). When $N'' \geq \nu$, for example when $M' = \nu - N$ and $N' = \nu - M$, this can be composed with $r^{(M+M',N'')}$ to get a constructible morphism

$$\mathbf{A}_C^{(M,N)} \times_C \mathbf{A}_C^{(M',N')} \longrightarrow \mathbf{A}^1. \tag{5.15}$$

The restriction to the fibre above v is given by the Fourier kernel $(x,y) \mapsto r_v(xy)$ from (5.10). Thus, the map (5.15) may be interpreted as a parametrisation of all local Fourier kernel maps induced by the differential form ω.

5.4.2 Global construction of the Fourier kernel

Fix two almost zero functions $\alpha \leq 0 \leq \beta$, and two non-negative families of integers $M = (M_\iota)_{\iota\in\mathcal{I}_0}$ and $N = (N_\iota)_{\iota\in\mathcal{I}_0}$ such that $M_0 = N_0 = 0$. According to the discussion in the previous paragraph, for any ι there exists a constructible Fourier kernel morphism

$$\mathbf{A}_C^{n(\alpha-M_\iota,\beta+N_\iota)} \times_C \mathbf{A}_C^{n(\nu-\beta-N_\iota,\nu-\alpha+M_\iota)} \to \mathbf{A}^1$$

Taking symmetric products, we get morphisms

$$r_{\mathbf{m}} : \mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N) \times_{S^{\mathbf{m}}C} \mathscr{A}_{\mathbf{m}}(\nu-\beta,\nu-\alpha,N,M) \to \mathbf{A}^1. \tag{5.16}$$

for any $\mathbf{m} \in \mathcal{I}_0$, using the following straightforward lemma:

Lemma 5.4.2.1. *Let $\mathscr{X} = (X_\iota)$ and $\mathscr{Y} = (Y_\iota)_\iota$ be families of constructible sets over X, and assume that for every ι we are given a constructible morphism $f_\iota : X_\iota \times_X Y_\iota \to \mathbf{A}^1$. Denote by $\mathscr{X} \times_X \mathscr{Y}$ the family $(X_\iota \times_X Y_\iota)_\iota$. Then for every $\pi \in \mathbf{N}^{(\mathcal{I})}$ there is a natural piecewise isomorphism*

$$S^\pi(\mathscr{X} \times_X \mathscr{Y}) \simeq S^\pi(\mathscr{X}) \times_{S^\pi(X)} S^\pi(\mathscr{Y})$$

given by

$$\sum_\iota \iota((x_{\iota,1},y_{\iota,1}),\ldots,(x_{\iota,n_\iota},y_{\iota,n_\iota})) \mapsto \left(\sum_\iota \iota(x_{\iota,1}+\ldots+x_{\iota,n_\iota}), \sum_\iota \iota(y_{\iota,1}+\ldots+y_{\iota,n_\iota})\right),$$

through which the morphism $f^{(\pi)} : S^\pi(\mathscr{X} \times_X \mathscr{Y}) \to \mathbf{A}^1$ becomes

$$\left(\sum_\iota \iota(x_{\iota,1}+\ldots+x_{\iota,n_\iota}), \sum_\iota \iota(y_{\iota,1}+\ldots+y_{\iota,n_\iota})\right) \mapsto \sum_\iota (f(x_{\iota,1},y_{\iota,1})+\ldots f(x_{\iota,n_\iota},y_{\iota,n_\iota})).$$

Using the description of the fibre above D from 5.3.2, we see that for every $D \in S^{\mathbf{m}}C$, the morphism

$$r_D : \mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)_D \times_{\kappa(D)} \mathscr{A}_{\mathbf{m}}(\nu-\beta,\nu-\alpha,N,M)_D \to \mathbf{A}^1$$

induced by $r_{\mathbf{m}}$ on the fibre above D is a twisted form of the Fourier kernel of Hrushovski and Kazhdan. The twisting being linear, r_D is a $\kappa(D)$-bilinear map.

5.4.3 Fourier transform

To define the Fourier transform of a family of Schwartz-Bruhat functions

$$\Phi \in \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)},$$

we start by defining the factor we need to normalise it by, so that it does not depend on the choice of β and N.

For this, we start with the family of C-varieties $(\mathbf{A}_C^{n(\beta+N_\iota)})_{\iota\in\mathcal{I}_0}$. Taking symmetric products, we get a constructible morphism

$$S^{\mathbf{m}}((\mathbf{A}_C^{n(\beta+N_\iota)})_{\iota\in\mathcal{I}_0}) \to S^{\mathbf{m}}C. \tag{5.17}$$

Using the notation in remark 5.3.1.5 and section 5.3.2, U an open dense subset of C above which β is zero, $\Sigma = C\setminus U$, $\mathbf{m}_0\in\mathcal{I}_0$ such that $\mathbf{m}_0\leq\mathbf{m}$, $D_\Sigma = \sum_{v\in\Sigma}\iota_v[v]\in S^{\mathbf{m}-\mathbf{m}_0}\Sigma$ and $\pi = (m_\iota)_{\iota\in\mathcal{I}}$ a partition of $\mathbf{m}_0$ we get, by proposition 3.4.0.1 from section 3.4 of chapter 3, that the restriction of (5.17) above the locally closed subset $S^\pi U\times\{D_\Sigma\}$ of $S^{\mathbf{m}_0}C$ is a vector bundle of rank $\sum_\iota nm_\iota N_\iota + \sum_{v\in\Sigma} n(\beta_v+N_{\iota_v})$. Thus, the class of (5.17) in $\mathscr{E}xp\mathscr{M}_{S^{\mathbf{m}}C}$ is

$$\sum_{\mathbf{m}_0\leq\mathbf{m}}\ \sum_{\substack{D_\Sigma\in S^{\mathbf{m}-\mathbf{m}_0}\Sigma\\ D_\Sigma=\sum_{v\in\Sigma}\iota_v[v]}}\ \sum_{\substack{\pi=(m_\iota)_{\iota\in\mathcal{I}}\\ \sum_{\iota\in\mathcal{I}}m_\iota\iota=\mathbf{m}_0}} \mathbf{L}^{\sum_\iota nm_\iota N_\iota+\sum_{v\in\Sigma}n(\beta_v+N_{\iota_v})}[S^\pi U\times\{D_\Sigma\}\to S^{\mathbf{m}}C],$$

and it makes sense to consider

$$[S^{\mathbf{m}}((\mathbf{A}_C^{n(\beta+N_\iota)})_{\iota\in\mathcal{I}_0})]^{-1}\in\mathscr{E}xp\mathscr{M}_{S^{\mathbf{m}}C}$$

defined by the formula

$$\sum_{\mathbf{m}_0\leq\mathbf{m}}\ \sum_{\substack{D_\Sigma\in S^{\mathbf{m}-\mathbf{m}_0}\Sigma\\ D_\Sigma=\sum_{v\in\Sigma}\iota_v[v]}}\ \sum_{\substack{\pi=(m_\iota)_{\iota\in\mathcal{I}}\\ \sum_{\iota\in\mathcal{I}}m_\iota\iota=\mathbf{m}_0}} \mathbf{L}^{-\sum_\iota nm_\iota N_\iota-\sum_{v\in\Sigma}n(\beta_v+N_{\iota_v})}[S^\pi U\times\{D_\Sigma\}\to S^{\mathbf{m}}C],$$

that is, the same one as above, but with the powers of $\mathbf{L}$ inverted.

Remark 5.4.3.1. This element is indeed the inverse of $[S^{\mathbf{m}}((\mathbf{A}_C^{n(\beta+N_\iota)})_{\iota\in\mathcal{I}_0})]$ in $\mathscr{E}xp\mathscr{M}_{S^{\mathbf{m}}C}$, so our notation is consistent.

We denote by $R_{\mathbf{m}}$ the element of the Grothendieck ring

$$\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)\times_{S^{\mathbf{m}}C}\mathscr{A}_{\mathbf{m}}(\nu-\beta,\nu-\alpha,N,M)}$$

given by

$$R_{\mathbf{m}} := [\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)\times_{S^{\mathbf{m}}C}\mathscr{A}_{\mathbf{m}}(\nu-\beta,\nu-\alpha,N,M), r_{\mathbf{m}}].$$

Moreover, we denote by $\mathrm{pr}_1, \mathrm{pr}_2$ the projections

$$\mathrm{pr}_1 : \mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)\times_{S^{\mathbf{m}}C}\mathscr{A}_{\mathbf{m}}(\nu-\beta,\nu-\alpha,N,M)\to\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)$$

and

$$\mathrm{pr}_2 : \mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)\times_{S^{\mathbf{m}}C}\mathscr{A}_{\mathbf{m}}(\nu-\beta,\nu-\alpha,N,M)\to\mathscr{A}_{\mathbf{m}}(\nu-\beta,\nu-\alpha,N,M).$$

Let $\Phi\in\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)}$ be a constructible family of Schwartz-Bruhat functions. The family $\mathscr{F}\Phi\in\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\nu-\beta,\nu-\alpha,N,M)}$ is defined by the formula

$$\mathscr{F}\Phi := [(S^{\mathbf{m}}((\mathbf{A}_C^{n(\beta+N_\iota)})_{\iota\in\mathcal{I}_0}))]^{-1}(\mathrm{pr}_2)_!((\mathrm{pr}_1)^*\Phi\cdot R_{\mathbf{m}})\in\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\nu-\beta,\nu-\alpha,N,M)}$$

where $\cdot$ is the product in the Grothendieck ring $\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)\times_{S^{\mathbf{m}}C}\mathscr{A}_{\mathbf{m}}(\nu-\beta,\nu-\alpha,N,M)}$.

Explicitly, if $\Phi=[V,f]\in\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)}$, then $\mathscr{F}\Phi$ is given by

$$[V\times_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)}\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)\times_{S^{\mathbf{m}}C}\mathscr{A}_{\mathbf{m}}(\nu-\beta,\nu-\alpha,N,M), f\circ\mathrm{pr}_1+r_{\mathbf{m}}(\mathrm{pr}_2\cdot\mathrm{pr}_3)]$$

multiplied by the normalisation factor $[(S^{\mathbf{m}}((\mathbf{A}_C^{n(\beta+N_\iota)})_{\iota\in\mathcal{I}_0}))]^{-1}$.

Remark 5.4.3.2. Taking $N=0$ (resp. $M=0$) one can see that the Fourier transform of a family of uniformly smooth (resp. uniformly compactly supported) functions is a family of uniformly compactly supported (resp. uniformly smooth) functions.

5.4.4 Compatibility between symmetric products and Fourier transformation

This section deals with the special case where $\Phi\in\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)}$ is given by a symmetric product. More precisely, suppose we are given, for any $\iota\in\mathcal{I}_o$, an element $\varphi_\iota\in\mathscr{E}xp\mathscr{M}_{\mathbf{A}_C^{n(\alpha-M_\iota,\beta+N_\iota)}}$, and that Φ is the symmetric product $S^{\mathbf{m}}((\varphi_\iota)_{\iota\in\mathcal{I}_0})$. There is a natural notion of Fourier transformation for elements of the ring $\mathscr{E}xp\mathscr{M}_{\mathbf{A}_C^{n(\alpha-M_\iota,\beta+N_\iota)}}$, defined in the following manner: denote by r_ι the Fourier kernel

$$\mathbf{A}_C^{n(\alpha-M_\iota,\beta+N_\iota)}\times_C\mathbf{A}_C^{n(\nu-\beta+N_\iota,\nu-\alpha+M_\iota)}\to\mathbf{A}^1$$

defined in (5.15) and by R_ι the element of the Grothendieck ring

$$\mathscr{E}xp\mathscr{M}_{\mathbf{A}_C^{n(\alpha-M_\iota,\beta+N_\iota)}\times_C\mathbf{A}_C^{n(\nu-\beta+N_\iota,\nu-\alpha+M_\iota)}}$$

given by

$$R_\iota = [\mathbf{A}_C^{n(\alpha-M_\iota,\beta+N_\iota)} \times_C \mathbf{A}_C^{n(\nu-\beta+N_\iota,\nu-\alpha+M_\iota)}, r_\iota].$$

Moreover, denoting again by U the open subset of C on which α and β are zero and by Σ its complement, we define the element $\left(\mathbf{A}_C^{n(\beta+N_i)}\right)^{-1}$ of $\mathscr{E}xp\mathscr{M}_C$ by

$$\mathbf{L}^{-nN_\iota}[U \to C] + \sum_{v\in\Sigma} \mathbf{L}^{-n(\beta_v - N_i)}[\{v\} \to C].$$

We denote by pr_1, pr_2 the projections

$$\mathrm{pr}_1 : \mathbf{A}_C^{n(\alpha-M_\iota,\beta+N_\iota)} \times_C \mathbf{A}_C^{n(\nu-\beta+N_\iota,\nu-\alpha+M_\iota)} \to \mathbf{A}_C^{n(\alpha-M_\iota,\beta+N_\iota)}$$

and

$$\mathrm{pr}_2 : \mathbf{A}_C^{n(\alpha-M_\iota,\beta+N_\iota)} \times_C \mathbf{A}_C^{n(\nu-\beta+N_\iota,\nu-\alpha+M_\iota)} \to \mathbf{A}_C^{n(\nu-\beta+N_\iota,\nu-\alpha+M_\iota)}.$$

We then define

$$\mathscr{F}\varphi_i = \left(\mathbf{A}_C^{n(\beta+N_i)}\right)^{-1} (\mathrm{pr}_2)_!((\mathrm{pr}_1)^*\varphi_i \cdot R_\iota) \in \mathscr{E}xp\mathscr{M}_{\mathbf{A}_C^{n(\nu-\beta+N_\iota,\nu-\alpha+M_\iota)}}$$

where $\cdot$ is the product in the ring $\mathscr{E}xp\mathscr{M}_{\mathbf{A}_C^{n(\alpha-M_\iota,\beta+N_\iota)}\times_C \mathbf{A}_C^{n(\nu-\beta+N_\iota,\nu-\alpha+M_\iota)}}$.

Remark 5.4.4.1. For every $v \in C$, denote by $\varphi_{\iota,v} \in \mathscr{E}xp\mathscr{M}_{\mathbf{A}_{\kappa(v)}^{n(\alpha_v-M_\iota,\beta_v+N_\iota}}$ the local Schwartz-Bruhat function obtained from φ_ι by restriction to the fibre $\mathbf{A}_{\kappa(v)}^{n(\alpha_v-M_\iota,\beta_v+N_\iota)}$ of $\mathbf{A}_C^{n(\alpha-M_\iota,\beta+N_\iota)}$ above v. By definition of the Fourier transform of a local Schwartz-Bruhat function (see section 5.2.4) as well as of the Fourier kernel r_ι (see section 5.4.1), we see that

$$\mathscr{F}(\varphi_{\iota,v}) = (\mathscr{F}\varphi_i)_v$$

in $\mathscr{E}xp\mathscr{M}_{\mathbf{A}_{\kappa(v)}^{n(\nu_v-\beta_v+N_\iota,\nu_v-\alpha_v+M_\iota)}}$, where the right-hand side is the restriction of $\mathscr{F}\varphi_i$ to the fibre $\mathbf{A}_{\kappa(v)}^{n(\nu_v-\beta_v+N_\iota,\nu_v-\alpha_v+M_\iota)}$. Thus, the operation we just defined performs Fourier transformation on families of local Schwartz-Bruhat functions parametrised by C, with level constant except at a finite number of closed points.

Proposition 5.4.4.2. *We have*

$$\mathscr{F}S^{\mathbf{m}}((\varphi_\iota)_{\iota\in\mathcal{I}_0}) = S^{\mathbf{m}}((\mathscr{F}\varphi_\iota)_{\iota\in\mathcal{I}_0}).$$

Proof. By definition of $r_{\mathbf{m}}$ and of Φ, with the notations from the previous sections we have

$$(\mathrm{pr}_2)_!((\mathrm{pr}_1)^*S^{\mathbf{m}}((\varphi_\iota)_{\iota\in\mathcal{I}_0}) \cdot R_{\mathbf{m}}) = S^{\mathbf{m}}\left(((\mathrm{pr}_2)_!((\mathrm{pr}_1)^*\varphi_i \cdot R_\iota))_{\iota\in\mathcal{I}_0}\right),$$

since the projections in the previous section are obtained from those in the previous section by taking symmetric products. Therefore, by lemma 5.4.2.1 we get the result, comparing the normalisation factors. ∎

5.4.5 Inversion formula

This section will not be used in what follows, but we include it for the sake of completeness.

For every $\iota \in \mathcal{I}_0$, define a constructible morphism $\mathbf{A}_C^{(\alpha-M_\iota,\beta+N_\iota)} \to \mathbf{A}_C^{(\alpha-M_\iota,\beta+N_\iota)}$ of C-varieties by

$$(v, x_{\alpha_v-M_\iota}, \dots, x_{\beta_v+N_\iota-1}) \to (v, -x_{\alpha_v-M_\iota}, \dots, -x_{\beta_v+N_\iota-1}).$$

Taking symmetric products, it induces a constructible morphism of $S^{\mathbf{m}}C$-varieties

$$\mathrm{inv}_m : \mathscr{A}_{\mathbf{m}}(\alpha, \beta, M, N) \to \mathscr{A}_{\mathbf{m}}(\alpha, \beta, M, N).$$

This Fourier transform satisfies the following Fourier inversion formula:

Proposition 5.4.5.1. *For every Φ in $\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)}$, we have*

$$\mathscr{F}\mathscr{F}\Phi = \mathbf{L}^{n(2g-2)}\Phi \circ \mathrm{inv}_{\mathbf{m}}$$

in $\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)}$.

Proof. Note that looking at the fibres of the constructions in the previous paragraphs above every rational point $D \in S^{\mathbf{m}}C(k)$, we recover the theory from [ChL], described in section 5.2. For a general schematic point $D \in S^{\mathbf{m}}C$, we recover a twisted version of this theory. Theorem 1.2.9 in [ChL] implies that we have

$$\mathscr{F}\mathscr{F}\Phi_D(x) = \mathbf{L}^{n(2g-2)}\Phi_D(-x)$$

for all rational points $D \in S^{\mathbf{m}}C(k)$. The heart of the proof of theorem 1.2.9 is the fact that the domains of definition of our functions are affine spaces, that the Fourier kernel is bilinear and the use of the relation $[\mathbf{A}^1, \mathrm{id}] = 0$. Therefore, this proof generalises easily to the twisted setting, and the Fourier inversion formula is valid for Φ_D for any schematic point $D \in S^{\mathbf{m}}C$. We may conclude using lemma 2.1.3.1. □

5.5 Summation over $k(C)^n$

5.5.1 Summation for twisted Schwartz-Bruhat functions

We defined constructible sets $\mathscr{A}_{\mathbf{m}}(\alpha, \beta, M, N)$ lying above the symmetric power $S^{\mathbf{m}}C$ in section 5.3.1, and gave an explicit description of the fibre above a point $D \in S^{\mathbf{m}}C$ in section 5.3.2, which shows that it may be seen as a twisted version of the domain of definition of a Schwartz-Bruhat function in the sense of Hrushovski and Kazhdan's theory. A function Φ on $\mathscr{A}_{\mathbf{m}}(\alpha, \beta, M, N)$ may therefore be seen as a family $(\Phi_D)_{D\in S^{\mathbf{m}}C}$, each Φ_D being a function on $\mathscr{A}_{\mathbf{m}}(\alpha, \beta, M, N)_D$. In this section, we explain how Hrushovski and Kazhdan's operation of summation over rational points from section 5.2.6 may be extended to the functions Φ_D.

In the notation of section 5.3.2, we have $D = (D_U, D_\Sigma) \in S^{\mathbf{m}_0}U \times S^{\mathbf{m}-\mathbf{m}_0}\Sigma$ where U is a dense open subset over which α and β are zero, and $\Sigma = C \setminus U$, a finite union of closed points, is its complement. Moreover, for some partition $\pi = (m_\iota)_{\iota\in\mathcal{I}}$ of $\mathbf{m}_0$, we have $D_U \in S^\pi U$. Via the quotient morphism

$$\left(\prod_{\iota\in\mathcal{I}} U^{m_\iota}\right)_* \to S^\pi U,$$

the schematic point $D_U \in S^\pi U$ pulls back to some K-point D'_U of $(\prod_{\iota\in\mathcal{I}} U^{m_\iota})_*$, where K is a finite extension of the residue field $\kappa(D)$ of D. We choose D'_U so that K is of minimal degree. We denote by $v_{\iota,j}$ the projection of D'_U on the j-th copy of U in the factor U^{m_ι}. This gives us a collection $\{v_{\iota,j}\}_{\substack{\iota\in\mathcal{I}\\ j\in\{1,\ldots,m_\iota\}}}$ of K-points of the open subset U of the curve C, all distinct because they come as projections from a point in the complement of the diagonal.

Remark 5.5.1.1. A different choice of D'_U amounts to a permutation of the points $v_{\iota,j}$ via the $G = \prod_{\iota\in\mathcal{I}} \mathfrak{S}_{m_\iota}$-action on $(\prod_{\iota\in\mathcal{I}} U^{m_\iota})_*$. More precisely, the quotient morphism $(\prod_{\iota\in\mathcal{I}} U^{m_\iota})_* \to S^\pi U$ being étale, the fibre product $(\prod_{i\in\mathcal{I}} U^{m_\iota})_* \times_{S^\pi U} \operatorname{Spec}\kappa(D)$ is the spectrum of an étale algebra $\mathcal{E}$ over $\kappa(D)$, endowed with a G-action such that $\mathcal{E}^G = \kappa(D)$. The étale algebra $\mathcal{E}$ is isomorphic to some power of K, and the point D'_U is one of the irreducible components of $\operatorname{Spec}\mathcal{E}$. If we denote by H the subgroup of G stabilising D'_U, then the invariant field K^H is $\kappa(D)$ (see e.g. proposition 3.1.2.3 in chapter 3). Consequently, in the commutative diagram

$$\begin{array}{ccc}
\operatorname{Spec} K & \longleftarrow & \prod_{\iota\in\mathcal{I}} \mathbf{A}_K^{m_\iota n(-M_\iota, N_\iota)} \\
\downarrow & & \downarrow{\scriptstyle q_{D_U}} \\
\operatorname{Spec}\kappa(D) & \longleftarrow & \prod_{\iota\in\mathcal{I}} \mathbf{A}_{\kappa(D)}^{m_\iota n(M_\iota+N_\iota)}
\end{array}$$

the vertical morphisms, induced by the quotient morphisms on the fibres above D_U and D'_U, are exactly the quotients by the action of the finite group H. The diagram gives an equivariant K-linear isomorphism

$$\prod_{\iota\in\mathcal{I}} \mathbf{A}_K^{m_\iota n(-M_\iota,N_\iota)} \simeq \prod_{\iota\in\mathcal{I}} \mathbf{A}_{\kappa(D)}^{m_\iota n(M_\iota+N_\iota)} \times_{\kappa(D)} K.$$

This induces a $\kappa(D)$-linear isomorphism between the $\kappa(D)$-points of the left-hand side (that is, the points invariant by the H-action) and the affine space $\prod_{\iota\in\mathcal{I}} \mathbf{A}_{\kappa(D)}^{m_\iota n(M_\iota+N_\iota)}$. In other words, the morphism q_{D_U} induces a $\kappa(D)$-linear isomorphism on $\kappa(D)$-points.

We define an effective zero-cycle E_D on the curve $C_{\kappa(D)}$, that is, the curve C seen as a curve over the field $\kappa(D)$, by

$$E_D := \sum_{\iota\in\mathcal{I}} M_\iota(v_{\iota,1} + \ldots + v_{\iota,m_\iota}) - \sum_{v\in\Sigma}(\alpha_v - M_{\iota_v})v$$

where the ι_v are given by $D_\Sigma = \sum_{v\in\Sigma} \iota_v[v]$. The zero-cycle E_D has the same field of definition as D, that is, $\kappa(D)$, and an associated Riemann-Roch space

$$L_{\kappa(D)}(E_D) = \{0\} \cup \{f \in \kappa(D)(C), \mathrm{div} f \geq -E_D\},$$

which is a finite-dimensional vector space over $\kappa(D)$.

Then we may define a morphism

$$\theta_D : L_{\kappa(D)}(E_D)^n \to \mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)_D = \prod_{\iota\in\mathcal{I}} \mathbf{A}_{\kappa(D)}^{nm_\iota(N_\iota+M_\iota)} \times_{\kappa(D)} \prod_{v\in\Sigma} \mathbf{A}_{\kappa(D)}^{n(\alpha_v - M_{\iota_v},\beta_v+N_{\iota_v})}$$

in the following manner. We start by defining an intermediate morphism

$$\theta'_D : L_{\kappa(D)}(E_D)^n \to \prod_{\iota\in\mathcal{I}} \mathbf{A}_K^{nm_\iota(-M_\iota,N_\iota)} \times_K \prod_{v\in\Sigma} \mathbf{A}_K^{n(\alpha_v-M_{\iota_v},\beta_v+N_{\iota_v})}.$$

For simplicity, assume $n=1$. Then, for any $f \in L_{\kappa(D)}(E_D)$:

1. **Image of f in the component $\prod_{v\in\Sigma}\mathbf{A}_K^{(\alpha_v-M_{\iota_v},\beta_v+N_{\iota_v})}$:** it is given, for each $v\in\Sigma$, by the coefficients of the v-adic expansion of f in the range $\alpha_v - M_{\iota_v},\ldots,\beta_v+N_{\iota_v}-1$.
2. **Image of f in the component $\prod_{\iota\in\mathcal{I}}\mathbf{A}_K^{m_\iota(-M_\iota,N_\iota)}$:** We may consider f as an element of the function field $K(C)$ of the curve C seen as a curve over K. We therefore send f to the point of
$$\prod_{\iota\in\mathcal{I}}\prod_{j=1}^{m_\iota}\mathbf{A}_K^{(-M_\iota,N_\iota)}$$
with (ι,j)-component given by the coefficients of orders $-M_\iota,\ldots,N_\iota-1$ of the $v_{i,j}$-adic expansion of f. Note that f will define a $\kappa(D)$-point of this above affine space.

Then we compose θ'_D with the quotient morphism

$$q_D : \prod_{\iota\in\mathcal{I}} \mathbf{A}_K^{m_\iota n(-M_\iota,N_\iota)} \times_K \prod_{v\in\Sigma} \mathbf{A}_K^{n(\alpha_v-M_{\iota_v},\beta_v+N_{\iota_v})} \to \mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)_D$$

to get $\theta_D = q_D\circ\theta'_D$.

Remark 5.5.1.2. Because of the composition with the quotient morphism, a different choice of D'_U gives the same θ_D.

Definition 5.5.1.3. For $\Phi_D \in \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)_D}$, we define its *summation over rational points*, denoted $\sum_{x\in\kappa(D)(C)}\Phi_D(x)$, to be the class of $\theta_D^*\Phi_D$ in $\mathscr{E}xp\mathscr{M}_{\kappa(D)}$.

Lemma 5.5.1.4 (Poisson formula)**.** *For $\Phi_D \in \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)_D}$, we have the equality*

$$\sum_{x\in\kappa(D)(C)^n}\Phi_D(x) = \mathbf{L}^{(1-g)n}\sum_{x\in\kappa(D)(C)^n}\mathscr{F}\Phi_D(x).$$

Proof. By the same kind of reduction as in the proof of theorem 1.3.10 in [ChL], it suffices to prove the formula in the case where Φ_D is given by a class $[\{a\} \to \mathscr{A}(\alpha, \beta, M, N)_D]$ where a is a rational point of $\mathscr{A}(\alpha, \beta, M, N)_D$. We may also assume $n = 1$.

We are going to use the notations from section 5.5.1 throughout the proof, denoting with a tilde the objects pertaining to the Fourier side: $\tilde{q}_D$ for the quotient morphism

$$\prod_{\iota \in \mathcal{I}} \mathbf{A}_K^{(-N_\iota, M_\iota)} \times_K \prod_{v \in \Sigma} \mathbf{A}_K^{(\nu_v - \beta_v - N_{\iota_v}, \nu_v - \alpha_v + M_{\iota_v})} \to \mathscr{A}_{\mathbf{m}}(\nu - \beta, \nu - \alpha, N, M)_D,$$

$\tilde{E}_D$ for the divisor

$$\tilde{E}_D = \sum_{\iota \in \mathcal{I}} N_\iota (v_{\iota,1} + \ldots + v_{\iota, m_\iota}) - \sum_{v \in \Sigma} (\nu_v - \beta_v - N_{\iota_v}) v,$$

$\tilde{\theta}_D, \tilde{\theta}'_D$ for the summation morphisms, etc.

Define the zero-cycle

$$\Lambda_D = \sum_{\iota, j} N_\iota v_{\iota,j} + \sum_{v \in \Sigma} (\beta_v + N_{\iota_v}) v = \tilde{E}_D - \operatorname{div} \omega$$

on $C_{\kappa(D)}$. It has the same field of definition as D, namely $\kappa(D)$.

The proof is essentially the same as the proof of theorem 1.3.10 in [ChL]: it will boil down to the theorem of Riemann-Roch and Serre duality for the divisor Λ_D on the curve $C_{\kappa(D)}$ over the field $\kappa(D)$. We refer to [ChL], 1.3.7 for reminders on these results.

Denote by F_D the function field $\kappa(D)(C)$ of the curve $C_{\kappa(D)}$. For any divisor E on $C_{\kappa(D)}$ define

$$\Omega(E) = \{\omega \in \Omega_{F_D/\kappa(D)}, \operatorname{div} \omega \geq E\},$$
$$\mathbb{A}_{F_D}(E) = \{a \in \mathbb{A}_{F_D}, \operatorname{div} a \geq -E\}.$$

Recall that Serre's duality theorem says that for any divisor E on $C_{\kappa(D)}$, the morphism

$$\Omega_{F_D/\kappa(D)} \to \operatorname{Hom}(\mathbb{A}_{F_D}, \kappa(D))$$

given by

$$\omega \mapsto \left((x_s)_s \mapsto \sum_s \operatorname{res}_s(x_s \omega) \right)$$

induces an isomorphism

$$\Omega(E) \to \operatorname{Hom}(\mathbb{A}_{F_D} / (\mathbb{A}_{F_D}(E) + F_D), \kappa(D))$$

identifying $\Omega(E)$ with the orthogonal subspace of $\mathbb{A}_{F_D}(E) + F_D$ in $\operatorname{Hom}(\mathbb{A}_{F_D}, \kappa(D))$, which itself is isomorphic to the dual of the cohomology group $H^1(\mathscr{L}(E))$.

By remark 5.5.1.1, a rational point $a \in \mathscr{A}_{\mathbf{m}}(\alpha, \beta, M, N)_D$ comes from a $\kappa(D)$-point b of the fibre

$$\prod_{\iota \in \mathcal{I}} \mathbf{A}_K^{m_\iota(-M_\iota, N_\iota)} \times_k \prod_{v \in \Sigma} \mathbf{A}_k^{(\alpha_v - M_{\iota_v}, \beta_v + N_{\iota_v})}$$

via the quotient morphism

$$q_D : \prod_{\iota \in \mathcal{I}} \mathbf{A}_K^{m_\iota(-M_\iota, N_\iota)} \times_k \prod_{v \in \Sigma} \mathbf{A}_k^{(\alpha_v - M_{\iota v}, \beta_v + N_{\iota v})} \to \mathscr{A}_{\mathbf{m}}(\alpha, \beta, M, N)_D.$$

Thus, it corresponds to the characteristic function of a polydisc inside the adeles $\mathbb{A}_{K(C)}$ with $\kappa(D)$-rational centre b and radius described by the divisor Λ_D. The Fourier transform $\mathscr{F}\Phi_D$ is defined on the $\kappa(D)$-variety $\mathscr{A}_{\mathbf{m}}(\nu - \beta, \nu - \alpha, N, M)_D$, which is the image of the quotient map

$$\tilde{q}_D : \prod_{\iota \in \mathcal{I}} \mathbf{A}_K^{m_\iota(-N_\iota, M_\iota)} \times_k \prod_{v \in \Sigma} \mathbf{A}_k^{(\nu - \beta_v - N_{\iota v}, \nu - \alpha_v - M_{\iota v})} \to \mathscr{A}_{\mathbf{m}}(\nu - \beta, \nu - \alpha, N, M)_D.$$

Let us compute the right-hand side of the Poisson formula. For this, recall that the Fourier kernel

$$r_D : \mathscr{A}_{\mathbf{m}}(\alpha, \beta, M, N) \times_{\kappa(D)} \mathscr{A}_{\mathbf{m}}(\nu - \beta, \nu - \alpha, N, M) \to \mathbf{A}^1$$

is a $\kappa(D)$-bilinear morphism satisfying

$$r_D(q_D(u), \tilde{q}_D(v)) = r(u, v)$$

for any $\kappa(D)$-rational points u, v of the fibres described above, where r is the Fourier kernel associated to the differential form ω on the adeles $\mathbb{A}_{K(C)}$.

By definition, for any $y \in \mathscr{A}_{\mathbf{m}}(\nu - \beta, \nu - \alpha, N, M)_D(\kappa(D))$, we have

$$\begin{aligned}
\mathscr{F}\Phi_D(y) &= \mathbf{L}^{-\deg \Lambda_D}[\{a\} \times_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)_D} \mathscr{A}_{\mathbf{m}}(\alpha, \beta, M, N)_D \times_{\kappa(D)} \{y\}, r_D(\mathrm{pr}_2, \mathrm{pr}_3)] \\
&= \mathbf{L}^{-\deg \Lambda_D}[\operatorname{Spec} k, r_D(a, y)] \\
&= \mathbf{L}^{-\deg \Lambda_D}\psi(r_D(a, y))
\end{aligned}$$

For any $f \in L_{\kappa(D)}(\operatorname{div} \omega + \Lambda_D)$, we have

$$r_D(a, \tilde{\theta}_D(f)) = r(b, \tilde{\theta}'_D(f)) = \sum_s \operatorname{res}_s(b_s f \omega)$$

where the sum goes over the points of the curve C_K. Note that the map $f \mapsto f\omega$ identifies $L_{\kappa(D)}(\operatorname{div}(\omega) + \Lambda_D)$ with $\Omega(-\Lambda_D)$. By invariance of the residue, $f \mapsto r_D(a, \theta_D(f))$ is identically zero on $L_{\kappa(D)}(\operatorname{div} \omega + \Lambda_D)$ if and only if $b \in \Omega(-\Lambda_D)^\perp$. By lemma 1.1.11 in [ChL], we have

$$\begin{aligned}
\sum_{x \in F_D} \mathscr{F}\Phi_D(x) &= \mathbf{L}^{-\deg \Lambda_D} \sum_{f \in L(\operatorname{div}(w) + \Lambda_D)} \psi(r_D(a, \tilde{\theta}_D(f))) \\
&= \begin{cases} \mathbf{L}^{-\deg \Lambda_D + \dim L_{\kappa(D)}(\operatorname{div} \omega + \Lambda_D)} & \text{if } b \in \Omega(-\Lambda_D)^\perp \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus, by the Riemann-Roch theorem applied to Λ_D on the curve $C_{\kappa(D)}$ we have:

$$\mathbf{L}^{1-g} \sum_{x \in F_D} \mathscr{F}\Phi_D(x) = \begin{cases} \mathbf{L}^{\dim L(-\Lambda_D)} & \text{if } b \in \Omega(-\Lambda_D)^\perp \\ 0 & \text{otherwise} \end{cases}$$

We now compute the left-hand side of the Poisson formula. If $b = (b_s)_s \in \Omega(-\Lambda_D)^\perp = \mathbb{A}_{F_D}(-\Lambda_D) + F_D$, there exists $c \in F_D$ such that $\mathrm{div}(c - b) \geq \Lambda_D$. In other words, there exists an element c of F_D in the polydisc of centre b with radii controlled by the divisor Λ_D. Then the intersection of F_D with this polydisc is exactly $c + L(-\Lambda_D)$, so that

$$\sum_{x\in F_D} \Phi_D(x) = \sum_{x\in F_D} \Phi_D(x - c) = \sum_{x\in L(-D)} 1 = \mathbf{L}^{\dim L(-D)}.$$

If on the other hand $b \notin \Omega(-\Lambda_D)^\perp$, then the intersection of F_D with the polydisc is empty, and the sum on the left-hand side of the Poisson formula is zero. This concludes the proof. □

5.5.2 Uniform summation for uniformly compactly supported functions

In what follows, we are going to show that summation over $\kappa(D)(C)$ may be done *uniformly* in D if Φ is a family of uniformly compactly supported functions, that is, if all integers in the family M are zero. Let us recall the key point of the construction in this particular case: the zero-cycle E_D from section 5.5.1 is equal to $D_\alpha = -\sum_v \alpha_v v$, and we are interested in the space $L_{\kappa(D)}(D_\alpha)$. Since D_α is defined over k, by flat base change (see e.g. [Milne], theorem 4.2 (a)) we have a $\kappa(D)$-linear canonical isomorphism

$$L_{\kappa(D)}(D_\alpha) \simeq L(D_\alpha) \otimes_k \kappa(D)$$

where

$$L(D_\alpha) = \{0\} \cup \{f \in k(C), \mathrm{div} f \geq -D_\alpha\}.$$

There is a morphism of algebraic varieties over $\kappa(D)$:

$$\theta_D : L(D_\alpha)^n \otimes_k \kappa(D) \longrightarrow \mathscr{A}_{\mathbf{m}}(\alpha, \beta, 0, N)_D,$$

constructed in the previous section.

The aim of the following proposition is to show that, as D varies in $S^{\mathbf{m}}C$, the maps θ_D can be combined into a morphism performing summation over rational points uniformly in D. Of course, our uniform choice of uniformisers will be crucial here.

Proposition 5.5.2.1. *There exists a constructible morphism*

$$\theta_{\mathbf{m}} : L(D_\alpha)^n \times S^{\mathbf{m}}C \longrightarrow \mathscr{A}_{\mathbf{m}}(\alpha, \beta, 0, N)$$

over $S^{\mathbf{m}}C$ such that for any $D \in S^{\mathbf{m}}C$, the induced morphism

$$L(D_\alpha)^n \otimes_k \kappa(D) \longrightarrow \mathscr{A}_{\mathbf{m}}(\alpha, \beta, 0, N)_D$$

on fibres above D is exactly the morphism θ_D constructed above.

Proof. Lemma 5.3.3.2 shows that for any $f \in L(D_\alpha)$ and any $\iota \in \mathcal{I}_0$, there is a constructible morphism

$$\varphi_{f,\iota} : C \longrightarrow \mathbf{A}_C^{(\alpha,\beta+N_\iota)}.$$

over C, sending $v \in C$ to $(v, a_{\alpha_v}(f,v), \dots, a_{\beta_v+N_\iota-1}(f,v))$ (where we use the notation from lemma 5.3.3.2). Taking products over C, for any $f = (f_1, \dots, f_n) \in L(D_\alpha)^n$, and any $\iota \in \mathcal{I}_0$, there is a constructible morphism

$$\varphi_{f,\iota} : C \longrightarrow \mathbf{A}_C^{n(\alpha,\beta+N_\iota)}.$$

Taking symmetric products, for any $\mathbf{m} \in \mathcal{I}_0$ we get morphisms

$$S^{\mathbf{m}}\varphi_f : S^{\mathbf{m}}C \longrightarrow \mathscr{A}_{\mathbf{m}}(\alpha, \beta, 0, N).$$

Finally, $L(D_\alpha)^n$ being a finite-dimensional k-vector space, combining these morphisms for a basis of the latter, we get constructible morphisms

$$\theta_{\mathbf{m}} : L(D_\alpha)^n \times S^{\mathbf{m}}C \longrightarrow \mathscr{A}_{\mathbf{m}}(\alpha, \beta, 0, N).$$

By construction, the restriction to the fibre over a schematic point $D \in S^{\mathbf{m}}C$ corresponds indeed to θ_D. □

We see $L(D_\alpha)^n \times S^{\mathbf{m}}C$ as a variety over $S^{\mathbf{m}}C$ via the second projection, which yields a group morphism

$$\mathscr{E}xp\mathscr{M}_{L(D_\alpha)^n \times S^{\mathbf{m}}C} \longrightarrow \mathscr{E}xp\mathscr{M}_{S^{\mathbf{m}}C},$$

interpreted as "summation over rational points in each fibre". Using this morphism, we can define uniform summation over rational points:

Definition 5.5.2.2. Let $\Phi \in \mathscr{E}xp\mathscr{M}_{\mathscr{A}^{\mathbf{m}}(\alpha,\beta,0,N)}$ be a family of uniformly compactly supported functions. The uniform summation of Φ over rational points, denoted by

$$\left(\sum_{x \in \kappa(D)(C)^n} \Phi_D(x)\right)_{D \in S^{\mathbf{m}}C},$$

is the image in $\mathscr{E}xp\mathscr{M}_{S^{\mathbf{m}}C}$ of the pullback $\theta_{\mathbf{m}}^*\Phi$.

Remark 5.5.2.3 (Order of summation). Starting from a uniformly compactly supported family $\Phi \in \mathscr{A}^{\mathbf{m}}(\alpha, \beta, 0, N)$ and pulling it back to $L(D_\alpha)^n \times S^{\mathbf{m}}C$ via $\theta_{\mathbf{m}}$, we obtain an object of $\mathscr{E}xp\mathscr{M}_{L(D_\alpha)^n \times S^{\mathbf{m}}C}$, from which we can get an object of $\mathscr{E}xp\mathscr{M}_k$, by either projecting first to $S^{\mathbf{m}}C$ and then to k, or first to $L(D_\alpha)$ and then to k. The fact that these two operations commute can be interpreted in terms of motivic sums as the possibility of interchanging the order of summation:

$$\sum_{D \in S^{\mathbf{m}}C} \sum_{x \in \kappa(D)(C)^n} \Phi_D(x) = \sum_{x \in k(C)^n} \sum_{D \in S^{\mathbf{m}}C} \Phi_D(x).$$

Here $\sum_{x \in k(C)^n}$ denotes summation over $L(D_\alpha)$.

5.6 Poisson formula in families

In this section we describe the way in which the Poisson formula is going to be used in section 6.2.6 of chapter 6.

5.6.1 Uniformly summable families

Definition 5.6.1.1. We say that a constructible family of functions

$$\Phi \in \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)}$$

is uniformly summable if there is an element $\Sigma \in \mathscr{E}xp\mathscr{M}_{S^{\mathbf{m}}C}$ such that for any schematic point $D \in S^{\mathbf{m}}(C)$, the pullback of Σ along D is $\sum_{x\in\kappa(D)(C)^n} \Phi_D(x) \in \mathscr{E}xp\mathscr{M}_{\kappa(D)}$.

Remark 5.6.1.2. By lemma 2.1.3.1, such an element is unique.

Notation 5.6.1.3. The element Σ from the definition will be denoted

$$\left(\sum_{x\in\kappa(D)(C)^n} \Phi_D(x)\right)_{D\in S^{\mathbf{m}}C}.$$

Example 5.6.1.4. We showed in the previous section that a uniformly compactly supported family of functions is uniformly summable. The notation in definition 5.5.2.2 is consistent with the one in 5.6.1.3.

5.6.2 Motivic Poisson formula in families

Let $\Phi \in \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\alpha,\beta,M,N)}$ be a constructible family of Schwartz-Bruhat functions, which we assume to be uniformly summable. By the motivic Poisson formula as it is stated in lemma 5.5.1.4, for any schematic point $D \in S^{\mathbf{m}}C$, we have

$$\sum_{x\in\kappa(D)(C)^n} \Phi_D(x) = \mathbf{L}^{(1-g)n} \sum_{y\in\kappa(D)(C)^n} \mathscr{F}\Phi_D(y).$$

We may conclude from this and from lemma 2.1.3.1 that the family

$$\mathscr{F}\Phi \in \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{m}}(\nu-\beta,\nu-\alpha,N,M)}$$

is uniformly summable as well, and that

$$\left(\sum_{x\in\kappa(D)(C)^n} \Phi_D(x)\right)_{D\in S^{\mathbf{m}}C} = \mathbf{L}^{(1-g)n} \left(\sum_{y\in\kappa(D)(C)^n} \mathscr{F}\Phi_D(y)\right)_{D\in S^{\mathbf{m}}C} \tag{5.18}$$

as elements of $\mathscr{E}xp\mathscr{M}_{S^{\mathbf{m}}C}$.

In other words, the Poisson formula from lemma 5.5.1.4 shows that Φ is uniformly summable if and only if $\mathscr{F}\Phi$ is uniformly summable, and that in this case the families of their sums are related by a Poisson formula 5.18.

5.6.3 The case of a uniformly smooth family

In chapter 6, section 6.2.6, we are going to use this formula in the case where Φ is a uniformly smooth family, so that $\mathscr{F}\Phi$ is uniformly compactly supported. By example 5.6.1.4, $\mathscr{F}\Phi$ is then uniformly summable, and section 5.6.2 states that so is Φ, and that we have equality (5.18). Taking classes in $\mathscr{E}xp\mathscr{M}_k$ (written as motivic sums), we then have:

$$\sum_{D\in S^{\mathbf{m}}C}\sum_{x\in\kappa(D)(C)^n}\Phi_D(x) = \mathbf{L}^{(1-g)n}\sum_{D\in S^{\mathbf{m}}C}\sum_{y\in\kappa(D)(C)^n}\mathscr{F}\Phi_D(y). \tag{5.19}$$

5.6.4 Reversing the order of summation

By remark 5.5.2.3, it makes sense to reverse the order of summation in the right-hand-side of (5.19), to get:

$$\sum_{D\in S^{\mathbf{m}}C}\sum_{x\in\kappa(D)(C)^n}\Phi_D(x) = \mathbf{L}^{(1-g)n}\sum_{y\in k(C)^n}\sum_{D\in S^{\mathbf{m}}C}\mathscr{F}\Phi_D(y). \tag{5.20}$$

Remark 5.6.4.1. For simplicity of notation, in chapter 6 we will drop the mention of $\kappa(D)$ in the summations, and write simply $\sum_{x\in k(C)}$.

Chapter 6

Motivic height zeta functions

6.1 Geometric setting

6.1.1 Equivariant compactifications

Let k be an algebraically closed field of characteristic zero. Let C_0 be a smooth quasi-projective curve over k, C be its smooth projective compactification, and $S = C \setminus C_0$. We denote by $F = k(C)$ the function field of C, g its genus, and G the additive group scheme $\mathbf{G}_a^n$. A smooth projective *equivariant compactification* of G_F is a smooth projective F-scheme X containing G_F as a dense open subset, and such that the group law $G_F \times G_F \longrightarrow G_F$ extends to a group action $G_F \times X \longrightarrow X$.

The geometry of such varieties has been investigated in [HT]. We summarise here the main facts that will be used in this chapter.

Proposition 6.1.1.1. *Let X be a smooth projective equivariant compactification of G_F.*

1. *The boundary $X \setminus G_F$ is a divisor.*
2. *The Picard group of X is freely generated by the irreducible components $(D_\alpha)_{\alpha\in\mathscr{A}}$ of $X \setminus G_F$.*
3. *The closed cone of effective divisors $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{Pic}(X)_{\mathbf{R}}$ is given by*
$$\Lambda_{\mathrm{eff}}(X) = \bigoplus_{\alpha\in\mathscr{A}} \mathbf{R}_{\geq 0} D_\alpha.$$
4. *Up to multiplication by a scalar, there is a unique G_F-invariant meromophic differential form ω_X on X. Its restriction to G_F is proportional to the form $\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_n$.*
5. *There exist integers $\rho_\alpha \geq 2$ such that the divisor $-\mathrm{div}(\omega_X)$ is given by*
$$-\mathrm{div}(\omega_X) = \sum_{\alpha\in\mathscr{A}} \rho_\alpha D_\alpha.$$

We will moreover assume that the divisor $X\setminus G_F$ has strict normal crossings. This means that the components D_α are geometrically irreducible, smooth and meet transversally.

6.1.2 Partial compactifications

A *partial compactification* of G_F is a smooth quasi-projective scheme U, containing G_F as an open subset, and endowed with an action of G_F which extends the group law $G_F \times G_F \longrightarrow G_F$. If U is an open subscheme of a projective smooth equivariant compactification X of G_F, globally invariant under the action of G_F, with complement a reduced divisor $D = X \setminus U$, we denote by $\mathscr{A}_D$ the subset of $\mathscr{A}$ such that

$$D = \sum_{\alpha \in \mathscr{A}_D} D_\alpha,$$

where $(D_\alpha)_{\alpha \in \mathscr{A}}$ are the irreducible components of $X \setminus G_F$. The log-anticanonical class of the partial compactification U is the class of the divisor

$$-K_X(D) = -(K_X + D) = \sum_{\alpha \in \mathscr{A}_D} (\rho_\alpha - 1) D_\alpha + \sum_{\alpha \notin \mathscr{A}_D} \rho_\alpha D_\alpha = \sum_{\alpha \in \mathscr{A}} \rho'_\alpha D_\alpha,$$

where $\rho'_\alpha = \rho_\alpha - 1$ for $\alpha \in \mathscr{A}_D$, and ρ_α otherwise. Since $\rho_\alpha \geq 2$, it belongs to the interior of the effective cone of X, and therefore it is big.

6.1.3 Choice of a good model

Let k, C_0, C, F be as in section 6.1.1. We now assume we are in the situation described in the introduction, namely that we are given a projective irreducible k-scheme $\mathscr{X}$ together with a non-constant morphism $\pi : \mathscr{X} \to C$, $\mathscr{U}$ a Zariski open subset of $\mathscr{X}$, and $\mathscr{L}$ a line bundle on $\mathscr{X}$. We make the following assumptions on the generic fibres $X = \mathscr{X}_F$ and $U = \mathscr{U}_F$:

— X is a smooth equivariant compactification of G_F, and U is a partial compactification of G_F.

— the boundary $D = X \setminus U$ is a strict normal crossings divisor.

— the restriction L of the line bundle $\mathscr{L}$ to X is the log-anticanonical line bundle $-K_X(D)$.

We also assume that for all $v \in C_0$ we have $\mathbf{G}(F_v) \cap \mathscr{U}(\mathcal{O}_v) \neq \varnothing$, where F_v is the completion of F at v, and $\mathcal{O}_v$ its ring of integers. According to lemma 3.4.1 in [ChL], up to modifying the models without changing this hypothesis nor the counting problem we are going to deal with, we may assume additionally that in fact $\mathscr{X}$ is a good model, that is, it is smooth over k and the sum of the non-smooth fibres of $\mathscr{X}$ and of the closures $\mathscr{D}_\alpha$ of the irreducible components D_α of $X \setminus U$ is a divisor with strict normal crossings on $\mathscr{X}$. Moreover, we may assume that $\mathscr{U}$ is the complement in $\mathscr{X}$ to a divisor with strict normal crossings. We will make use of the notations concerning equivariant compactifications introduced in section 6.1.1.

6.1.4 Vertical components

For every $v \in C(k)$, we write $\mathscr{B}_v$ for the set of irreducible components of $\pi^{-1}(v)$, and $\mathscr{B}$ for the disjoint union of all $\mathscr{B}_v$, $v \in C(k)$. For $\beta \in \mathscr{B}_v$, we denote by E_β the corresponding component, and by μ_β its multiplicity in the special fibre of $\mathscr{X}$ at v.

The line bundle $\mathscr{L}$ is of the form $\sum_{\alpha\in\mathscr{A}} \rho'_\alpha \mathscr{L}_\alpha$ where for every α, the line bundle $\mathscr{L}_\alpha$ on $\mathscr{X}$ extends D_α. We may write:

$$\mathscr{L}_\alpha = \mathscr{D}_\alpha + \sum_{\beta\in\mathscr{B}} e_{\alpha,\beta} E_\beta$$

where the integers $e_{\alpha,\beta}$ are zero for almost all β. We also define integers ρ_β such that

$$-\mathrm{div}(\omega_X) = \sum_\alpha \rho_\alpha \mathscr{D}_\alpha + \sum_{\beta\in\mathscr{B}} \rho_\beta E_\beta$$

where ω_X is seen as a meromorphic section of $K_{\mathscr{X}/C}$.

6.1.5 Weak Néron models

Let $\mathscr{B}_1$ be the subset of $\mathscr{B}$ consisting of those β for which the multiplicity μ_β is equal to 1. Let $\mathscr{B}_{1,v} = \mathscr{B}_1 \cap \mathscr{B}_v$. Let $\mathscr{X}_1$ be the complement in $\mathscr{X}$ of the union of the components E_β for $\beta \in \mathscr{B} \setminus \mathscr{B}_1$ as well as of the intersections of distinct vertical components. It is a smooth scheme over C. By lemma 3.2.1 in [ChL], the C-scheme $\mathscr{X}_1$ is a weak Néron model of X. This means that for every smooth C-scheme $\mathscr{Z}$, the canonical map

$$\mathrm{Hom}_C(\mathscr{Z}, \mathscr{X}_1) \to \mathrm{Hom}_F(\mathscr{Z}_F, X)$$

is a bijection.

Applying this to $\mathscr{Z} = C$, we see that in particular, any rational point $g : \mathrm{Spec}\, F \to X$ extends to a section $\sigma_g : C \to \mathscr{X}_1$ with image inside $\mathscr{X}_1$.

6.2 Height zeta functions

6.2.1 Definition

Let $\lambda = (\lambda_\alpha)_{\alpha\in\mathscr{A}}$ be a family of positive integers, and let $\mathscr{L}_\lambda = \sum_{\alpha\in\mathscr{A}} \lambda_\alpha \mathscr{L}_\alpha$ be the corresponding line bundle on the good model $\mathscr{X}$. For every integer $n \in \mathbf{Z}$ and any $\mathbf{n} = (n_\alpha)_{\alpha\in\mathscr{A}} \in \mathbf{Z}^{\mathscr{A}}$ let $M_{U,n}$ be the moduli space of sections $\sigma : C \to \mathscr{X}$ such that

— the section σ maps the generic point η_C of C to a point of G_F.

— it represents an S-integral point of U, that is, $\sigma(C_0) \subset \mathscr{U}$.

— $\deg_C(\sigma^*\mathscr{L}_\lambda) = n$

and $M_{U,\mathbf{n}}$ the moduli space of sections $\sigma : C \to \mathscr{X}$ such that

— the section σ maps the generic point η_C of C to a point of G_F.

— it represents an S-integral point of U, that is, $\sigma(C_0) \subset \mathscr{U}$.

— for all $\alpha \in \mathscr{A}$, $\deg_C(\sigma^*\mathscr{L}_\alpha) = n_\alpha$.

According to Proposition 2.2.2 in [ChL], these moduli spaces exist as constructible sets over k, and there exists an integer m such that $M_{U,n}$ (resp. $M_{U,\mathbf{n}}$) is empty when $n < m$

(resp. when $n_\alpha < m$ for all $\alpha \in \mathscr{A}$). Moreover, $M_{U,n}$ can be seen as the disjoint union of all $M_{U,\mathbf{n}}$ such that $\sum_{\alpha\in\mathscr{A}} \lambda_\alpha n_\alpha = n$.

The multivariate motivic height zeta function is given by

$$Z(\mathbf{T}) = \sum_{\mathbf{n}\in\mathbf{Z}^{\mathscr{A}}} [M_{U,\mathbf{n}}]\mathbf{T}^{\mathbf{n}} \in \mathscr{M}_k[[(T_\alpha)_{\alpha\in\mathscr{A}}]]\left[\prod_{\alpha\in\mathscr{A}} T_\alpha^{-1}\right],$$

and the motivic height zeta function associated to the line bundle $\mathscr{L}$, defined as

$$Z_\lambda(T) = \sum_{n\in\mathbf{Z}} [M_{U,n}]T^n \in \mathscr{M}_k[[T]][T^{-1}],$$

can be written in the form

$$Z_\lambda(T) = Z((T^{\lambda_\alpha})) = \sum_{m\in\mathbf{Z}} \left(\sum_{\substack{\mathbf{n}\in\mathbf{Z}^{\mathscr{A}} \\ \sum_\alpha \lambda_\alpha n_\alpha = m}} [M_{U,\mathbf{n}}] \right) T^m.$$

As stated in the introduction, we will investigate the case where $\mathscr{L} = \mathscr{L}_{\rho'} = \sum_{\alpha\in\mathscr{A}} \rho'_\alpha \mathscr{L}_\alpha$ is generically the log-anticanonical line bundle. Denote by $Z(T)$ the corresponding height zeta function.

6.2.2 Local intersection degrees

Let $v \in C(k)$. To every point $g \in G(F_v)$, one can attach local intersection degrees $(g, \mathscr{D}_\alpha)_v \in \mathbf{N}$ for all $\alpha \in \mathscr{A}$ and $(g, E_\beta)_v$ for all $\beta \in \mathscr{B}_v$ such that

1. For every $g \in G(F)$ and every $\alpha \in \mathscr{A}$, one has

$$\deg_C(\sigma_g^*(\mathscr{D}_\alpha)) = \sum_{v\in C(k)} (g, \mathscr{D}_\alpha)_v$$

   where $\sigma_g : C \to \mathscr{X}$ is the canonical section extending g.

2. There is exactly one $\beta \in \mathscr{B}_v$ such that $(g, E_\beta)_v = 1$, and this β has multiplicity one. For any $\beta' \in \mathscr{B}_v$ different from this β, one has $(g, E_{\beta'})_v = 0$ (see section 6.1.5).

We refer to [ChL], 3.3 for details.

6.2.3 Decomposition of $G(F_v)$

We can decompose $G(F_v)$ into definable (in the Denef-Pas language, see section 2.1 of [ClL08]) bounded domains on which all the above intersection degrees are constant: for all $\mathbf{m} \in \mathbf{N}^{\mathscr{A}}$ and all $\beta \in \mathscr{B}_v$, we define

$$G(\mathbf{m}, \beta)_v = \{g \in G(F_v),\ (g, E_\beta)_v = 1 \text{ and } (g, \mathscr{D}_\alpha)_v = m_\alpha \text{ for all } \alpha \in \mathscr{A}\}$$

and $G(\mathbf{m})_v = \cup_{\beta\in\mathscr{B}_v} G(\mathbf{m},\beta)_v$. Lemma 3.3.2 in [ChL] says that for any $\mathbf{m} \in \mathbf{N}^{\mathscr{A}}$ and any $\beta \in \mathscr{B}_v$, the set $G(\mathbf{m},\beta)_v$ is a bounded definable subset of $G(F_v)$, and that $G(F_v)$ is the disjoint union of all the $G(\mathbf{m},\beta)_v$ for $\mathbf{m} \in \mathbf{N}^{\mathscr{A}}$ and $\beta \in \mathscr{B}_v$.

Moreover, Lemmas 3.3.3 and 3.3.4 from [ChL] can be summarised in the following proposition:

Proposition 6.2.3.1. *There exists a dense open subset C_1 of C_0 such that for every $v \in C_1(k)$:*

(i) The set $\mathscr{B}_v$ contains only one element β_v.

(ii) The set $G(\mathbf{0},\beta)_v$ is equal to the subgroup $G(\mathcal{O}_v)$ of $G(F_v)$.

Moreover, there is an almost zero function $s : C \to \mathbf{Z}$ such that for every $v \in C(k)$, $\mathbf{m} \in \mathbf{N}^{\mathscr{A}}$ and $\beta \in \mathscr{B}_v$, the set $G(\mathbf{m},\beta)_v$ is invariant under the subgroup $G(\mathfrak{m}_v^{s_v})$ of $G(\mathcal{O}_v)$, where $\mathfrak{m}_v$ is the maximal ideal of $\mathcal{O}_v$, and one can take $s_v = 0$ for all $v \in C_1(k)$.

As a consequence of this, as in Corollary 3.3.5 from [ChL], the characteristic function of each set $G(\mathbf{m},\beta)_v$ may be seen as motivic Schwartz-Bruhat functions on $G(F_v)$ in the sense of 5.2.1, with $N = s_v$ and with M such that $G(\mathbf{m},\beta)_v \subset (t^M\mathcal{O}_v)^n$ (which exists by boundedness).

Notation 6.2.3.2. In what follows, it will be convenient for us to consider an even smaller set C_1. Namely, from now on C_1 denotes the open subset of places $v \in C_0$ satisfying

— the conditions in proposition 6.2.3.1;

— $e_{\alpha,\beta_v} = 0$ for all $\alpha \in \mathscr{A}$;

— $\rho_{\beta_v} = 0$

— For all $\alpha \in \mathscr{A} \setminus \mathscr{A}_D$, $\mathscr{D}_\alpha \times_C C_1 \to C_1$ is smooth.

Proposition 6.2.3.3. *There exist almost zero functions s and s', an unbounded family $N = (N_{\mathbf{m}})_{\mathbf{m}\in\mathbf{N}^{\mathscr{A}}}$ with $N_{\mathbf{0}} = 0$, and for every $\mathbf{m} \in \mathbf{N}^{\mathscr{A}}$ and every $\beta \in \prod_v \mathscr{B}_v$, a constructible subset $G_{\mathbf{m},\beta}$ of $\mathbf{A}_C^{n(s'-N_{\mathbf{m}},s)}$, the fibre of which at every $v \in C(k)$ is exactly $G(\mathbf{m},\beta_v)_v$.*

Proof. The definable boundedness of $G(\mathbf{m},\beta)_v$ means that there exists an almost zero function $s' : C \to \mathbf{Z}$, and, for every $\mathbf{m} \in \mathbf{N}^{\mathscr{A}}$, a non-negative integer $N_{\mathbf{m}}$ such that

$$G(\mathbf{m},\beta)_v \subset (t_v^{s'_v - N_{\mathbf{m}}}\mathcal{O}_v)^n,$$

for all $v \in C$. Statement (ii) in proposition 6.2.3.1 implies that we may take $N_{\mathbf{0}} = 0$. Since $G(F_v)$ is the union of all the $G(\mathbf{m},\beta)_v$, the family $(N_{\mathbf{m}})_{\mathbf{m}}$ is necessarily unbounded. Taking the almost zero function s from proposition 6.2.3.1, we see that $G(\mathbf{m},\beta)_v$ defines naturally a constructible subset of $\mathbf{A}_k^{n(s'_v-N_{\mathbf{m}},s_v)}$: the conditions on the intersection degrees will indeed translate into Zariski open and Zariski closed polynomial conditions on the coordinates of $\mathbf{A}_k^{n(s'_v-N_{\mathbf{m}},s_v)}$. Let $\mathscr{Y}$ be an affine subset of $\mathscr{X}$ such that $\pi_{|\mathscr{Y}} : \mathscr{Y} \to C$ is non-constant. Let C' be an open dense subset of C contained in the intersection of the image of $\pi_{|\mathscr{Y}}$ and of the open subset C_1 from proposition 6.2.3.1. For every $\alpha \in \mathscr{D}_\alpha$, let f_α be a local equation for $\mathscr{D}_\alpha$ in $\mathscr{Y}$. Then the condition that $(g,\mathscr{D}_\alpha)_v = m_\alpha$ may be written $\operatorname{ord}(f_\alpha(g)) = m_\alpha$ for all v in the image of $\pi_{|\mathscr{Y}}$. We thus see that we may take the same integer $N_{\mathbf{m}}$ for all v, and that for all $v\in C'$, the equations defining $G(\mathbf{m})_v$ inside $\mathbf{A}_k^{n(s'_v-N_{\mathbf{m}},s_v)}$ are uniform in v (in the sense that they are defined by the same formula in the Denef-Pas Language for every such v). This, together with the fact that the $G(\mathbf{m},\beta)_v$, for $v\in C\setminus C'$ are constructible, guarantees the existence of a constructible subset of $\mathbf{A}_C^{n(s'-N_{\mathbf{m}},s)}$ as in the statement of the proposition. $\square$

6.2.4 Integral points

The complement of the model $\mathscr{U}$ inside $\mathscr{X}$ is the union of the divisors $\mathscr{D}_\alpha$ for $\alpha\in\mathscr{A}_D$, and of the vertical components E_β for a finite subset $\mathscr{B}^0$ of $\mathscr{B}$. We then set $\mathscr{B}^0_v=\mathscr{B}^0\cap\mathscr{B}_v$ for every $v\in C(k)$, and define

$$\mathscr{B}_0=\mathscr{B}_1\setminus\left(\cup_{v\in C_0}\mathscr{B}^0_v\right),$$

and $\mathscr{B}_{0,v}=\mathscr{B}_0\cap\mathscr{B}_v$. In other words, $\mathscr{B}_0$ corresponds to vertical components of multiplicity one which either lie above S or are contained in $\mathscr{U}$. Thus in particular $\mathscr{B}_{0,v}=\mathscr{B}_{1,v}$ for $v\in S$.

Let $\mathbf{m}_v\in\mathbf{N}^{\mathscr{A}}$ and $\beta_v\in\mathscr{B}_v$. We say that the pair $(\mathbf{m}_v,\beta_v)$ is *v-integral* if

— either $v\in S$

— or $v\in C_0$, $\beta_v\in\mathscr{B}_0$ and $m_{\alpha,v}=0$ for every $\alpha\in\mathscr{A}_D$.

In other words, the union of the sets $G(\mathbf{m}_v,\beta_v)_v$ for all v-integral pairs $(\mathbf{m}_v,\beta_v)$ is equal to $\mathscr{U}(\mathcal{O}_v)$ if $v\in C_0$, and $G(F_v)$ otherwise.

For any $(\mathbf{m}_v,\beta_v)\in\mathbf{N}^{\mathscr{A}}\times\mathscr{B}_v$, define

$$H(\mathbf{m}_v,\beta_v)_v=\begin{cases} G(\mathbf{m}_v,\beta_v)_v & \text{if } (\mathbf{m}_v,\beta_v) \text{ is } v\text{-integral}\\ \varnothing & \text{else.}\end{cases}$$

Then the union of the sets $H(\mathbf{m}_v,\beta_v)_v$ for all $(\mathbf{m}_v,\beta_v)$ is equal to $\mathscr{U}(\mathcal{O}_v)$ if $v\in C_0$, and $G(F_v)$ otherwise. We also define $H(\mathbf{m}_v)_v=\cup_{\beta\in\mathscr{B}_v}H(\mathbf{m}_v,\beta)_v$. Let $\mathbf{m}=(\mathbf{m}_v)_v$ and $\beta=(\beta_v)_v$ be families indexed by $v\in C(k)$, where $\mathbf{m}_v=(m_{\alpha,v})\in\mathbf{N}^{\mathscr{A}}$ and $\beta_v\in\mathscr{B}_v$ for all v, such that $\mathbf{m}_v=0$ for almost all v. The element $\mathbf{m}$ must be seen as an effective zero-cycle on C with coefficients in $\mathbf{N}^{\mathscr{A}}$. We say $(\mathbf{m},\beta)$ is *integral* if $(\mathbf{m}_v,\beta_v)$ is v-integral for every v.

Remark 6.2.4.1. For fixed $\mathbf{n}\in\mathbf{N}^{\mathscr{A}}$ and $\beta\in\prod_v\mathscr{B}_{0,v}$, families $\mathbf{m}=(\mathbf{m}_v)_v$ such that the pair $(\mathbf{m},\beta)$ is integral and such that $\sum_{v\in C}\mathbf{m}_v=\mathbf{n}$ are parametrised by the symmetric product $S^{\mathbf{n}'}(C\setminus C_0)\times S^{\mathbf{n}''}(C)$ (where $\mathbf{n}'=(n_\alpha)_{\alpha\in\mathscr{A}_D}$ and $\mathbf{n}''=(n_\alpha)_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}$) which may naturally be seen as a constructible subset of $S^{\mathbf{n}}(C)$.

For any pair $(\mathbf{m},\beta)$, the characteristic functions of the subsets

$$G(\mathbf{m},\beta)=\prod_{v\in C(k)}G(\mathbf{m}_v,\beta_v)_v\subset G(\mathbb{A}_F)$$

and

$$H(\mathbf{m},\beta)=\prod_{v\in C(k)} H(\mathbf{m}_v,\beta_v)_v\subset G(\mathbb{A}_F)$$

may be seen as global motivic Schwartz-Bruhat functions by proposition 6.2.3.1. More precisely, using the notation of proposition 6.2.3.3, they may be seen as elements of

$$\mathscr{E}xp\mathscr{M}_{\prod_v \mathbf{A}_k^{n(s'_v-N_{\mathbf{m}_v},s_v)}}.$$

We have $H(\mathbf{m},\beta)=G(\mathbf{m},\beta)$ if $(\mathbf{m},\beta)$ is integral, and $H(\mathbf{m},\beta)=\varnothing$ else.

In the same manner as the $G(\mathbf{m},\beta)_v$, the $H(\mathbf{m},\beta)_v$ may be combined into a constructible set $H_{\mathbf{m},\beta}$:

Proposition 6.2.4.2. *For any $\mathbf{m}\in\mathbf{N}^{\mathscr{A}}$ and any $\beta=(\beta_v)_v\in\prod_v\mathscr{B}_v$, there is a constructible subset $H_{\mathbf{m},\beta}\subset G_{\mathbf{m},\beta}$ the fibre of which at any $v\in C(k)$ is exactly $H(\mathbf{m},\beta_v)_v$.*

Proof. There are two cases to consider. Assume first that $\mathbf{m}$ is such that $\mathbf{m}_\alpha=0$ for all $\alpha\in\mathscr{A}_D$. Then $H_{\mathbf{m},\beta}$ is obtained from $G_{\mathbf{m},\beta}$ by removing the fibres above the finite number of points $v\in C(k)$ such that $(\mathbf{m},\beta_v)$ is not v-integral, that is, such that $v\in C_0$ and $\beta_v\notin\mathscr{B}_0$. If on the contrary there exists $\alpha\in\mathscr{A}_D$ such that $\mathbf{m}_\alpha\neq 0$, then $H_{\mathbf{m},\beta}$ is the restriction of $G_{\mathbf{m},\beta}$ to the finite set of points S. □

6.2.5 Two constructible families of Schwartz-Bruhat functions

In propositions 6.2.3.3 and 6.2.4.2 we have combined, for any $\mathbf{m}\in\mathbf{N}^{\mathscr{A}}$ and any $\beta=(\beta_v)_v\in\prod_v\mathscr{B}_v$, the sets $G(\mathbf{m},\beta_v)_v$ (resp. $H(\mathbf{m},\beta_v)_v$) into a familiy $G_{\mathbf{m},\beta}\subset\mathbf{A}_C^{n(s'-N_{\mathbf{m}},s)}$ (resp. $H_{\mathbf{m},\beta}\subset G_{\mathbf{m},\beta}$) above C. The symmetric product construction then allows us to consider, for any $\mathbf{n}\in\mathbf{N}^{\mathscr{A}}$ and any $\beta\in\prod_v\mathscr{B}_v$, constructible subsets

$$S^{\mathbf{n}}((H_{\mathbf{m},\beta})_{\mathbf{m}\in\mathbf{N}^{\mathscr{A}}})\subset S^{\mathbf{n}}((G_{\mathbf{m},\beta})_{\mathbf{m}\in\mathbf{N}^{\mathscr{A}}})\subset S^{\mathbf{n}}\left(\left(\mathbf{A}_C^{n(s'-N_{\mathbf{m}},s)}\right)_{\mathbf{m}\in\mathbf{N}^{\mathscr{A}}}\right)=\mathscr{A}_{\mathbf{n}}(s',s,N,0).$$

with the notation of section 5.3.1. Therefore, using the terminology of section 5.3.5, this defines two uniformly smooth constructible families of Schwartz-Bruhat functions of level $\mathbf{n}$. They parametrise the characteristic functions of the adelic sets $H(\mathbf{m},\beta)$ and $G(\mathbf{m},\beta)$ for fixed $\beta\in\prod_v\mathscr{B}_v$ and with $\mathbf{m}$ varying inside $S^{\mathbf{n}}C$: we will therefore denote these families $(\mathbf{1}_{H(\mathbf{m},\beta)})_{\mathbf{m}\in S^{\mathbf{n}}C}$ and $(\mathbf{1}_{G(\mathbf{m},\beta)})_{\mathbf{m}\in S^{\mathbf{n}}C}$. Their Fourier transforms will then be uniformly compactly supported constructible families of functions, defined on $\mathscr{A}_{\mathbf{n}}(\nu-s,\nu-s',0,N)$.

6.2.6 Applying the Poisson summation formula

Let $\mathbf{n}\in\mathbf{Z}^{\mathscr{A}}$. For any $\beta=(\beta_v)_v\in\prod_v\mathscr{B}_{0,v}$, and $\alpha\in\mathscr{A}$, put $n_\alpha^\beta:=n_\alpha-\sum_v e_{\alpha,\beta_v}$. We define $M_{\mathbf{n},\beta}$ to be the constructible subset of $M_{\mathbf{n}}$ of sections σ_g such that for all v, $(g,E_{\beta_v})_v=1$ (so that $(g,E_{\beta'_v})=0$ for all $\beta'_v\neq\beta_v$). By definition, these sections satisfy $\deg(\sigma_g^*\mathscr{D}_\alpha)=n_\alpha^\beta$ for all $\alpha\in\mathscr{A}$. Thus, since the $\mathscr{D}_\alpha$ are effective, for any $\mathbf{n}$ such that $M_{\mathbf{n},\beta}\neq\varnothing$, the n_α^β, $\alpha\in\mathscr{A}$, are non-negative integers.

Lemma 6.2.6.1. *Let* $\mathbf{n} \in \mathbf{Z}^{\mathscr{A}}$ *and* $\beta \in \prod_v \mathscr{B}_{0,v}$ *be such that* $M_{\mathbf{n},\beta}$ *is non-empty. There is a morphism of constructible sets defined by*

$$\begin{array}{ccc} M_{\mathbf{n},\beta} & \to & S^{\mathbf{n}^\beta} C \\ \sigma_g & \mapsto & \sum_{v \in C(k)} ((g, \mathscr{D}_\alpha)_v)_{\alpha \in \mathscr{A}} [v] \end{array}$$

Proof. We will start by making some reductions. To simplify notations, write $M_{\mathbf{n},\beta} = M$, and $\mathbf{n}^\beta = \mathbf{n}$. Since $S^{\mathbf{n}} C = \prod_{\alpha \in \mathscr{A}} S^{n_\alpha} C$, it suffices to prove constructibility for the map

$$\begin{array}{ccc} M & \to & S^{n_\alpha} C \\ \sigma_g & \mapsto & \sum_v (g, \mathscr{D}_\alpha)_v [v] \end{array}$$

associated to one $\mathscr{D}_\alpha$. To simplify notations further, write $n_\alpha = n$ and $\mathscr{D}_\alpha = \mathscr{D}$. By definition of the moduli space of sections, there is a morphism

$$\begin{array}{cccc} \epsilon : & C \times M & \to & \mathscr{X} \\ & (v, \sigma) & \mapsto & \sigma(v) \end{array}$$

Denote by $s_{\mathscr{D}}$ the canonical section of the line bundle $\mathcal{O}_{\mathscr{X}}(\mathscr{D})$ and put $\Delta = \operatorname{div}(\epsilon^* s_{\mathscr{D}})$. This is a closed subscheme of $C \times M$, finite over M. By generic flatness, we may stratify M into locally closed subsets U_i such that for every i, $\Delta \times_M U_i \subset C \times U_i$ is flat over U_i. By definition of Hilbert schemes, it therefore defines a morphism $U_i \to \operatorname{Hilb}(C)$ to the Hilbert scheme of points of C. Moreover, for every $\sigma \in M$, the fibre $\Delta_\sigma = \operatorname{div}(\sigma^* s_{\mathscr{D}})$ is a zero-dimensional subscheme of C of length n. Thus, the image of the above morphism is in fact contained in the Hilbert scheme of n points of C, which we may identify with the symmetric product $S^n C$. The constructible morphism we want is then obtained by combining these morphisms $U_i \to S^n C$ for every i. □

Recall that a point in $M_{\mathbf{n},\beta}$ represents a section that intersects components in $\mathscr{A}_D$ only above places in $S = C \setminus C_0$. Thus, in fact, $M_{\mathbf{n},\beta}$ lies above the constructible subset of $S^{\mathbf{n}^\beta} C$ consisting of zero-cycles $\mathbf{m} = (\mathbf{m}_v)_v$ with components with respect to $\alpha \in \mathscr{A}_D$ supported inside S: in other words, these are the zero-cycles $\mathbf{m}$ such that $(\mathbf{m}, \beta)$ is integral.

The exact correspondence between sections $\sigma : C \to \mathscr{X}$ and elements of $X(F)$ via $\sigma \mapsto \sigma(\eta_C)$ restricts to an exact correspondence between sections $\sigma \in M_{\mathbf{n},\beta}$ lying above $\mathbf{m} \in S^{\mathbf{n}^\beta} C$ and elements in $G(F) \cap H(\mathbf{m}, \beta)$, where $H(\mathbf{m}, \beta)$ is the adelic set defined in section 6.2.4. We denote $H_F(\mathbf{m}, \beta) := G(F) \cap H(\mathbf{m}, \beta)$, which is a constructible set over k. Note that by definition of summation over rational points (see section 5.5.1 in chapter 5), we have, for every $\mathbf{m} \in S^{\mathbf{n}^\beta} C$

$$\sum_{x \in \kappa(\mathbf{m})(C)^n} \mathbf{1}_{H(\mathbf{m},\beta)}(x) = [H_F(\mathbf{m}, \beta)] = (M_{\mathbf{n},\beta})_{\mathbf{m}} \in \mathscr{M}_{\kappa(\mathbf{m})}.$$

With the notation of section 6.2.5, the uniformly smooth family $(\mathbf{1}_{H(\mathbf{m},\beta)})_{\mathbf{m} \in S^{\mathbf{n}^\beta} C}$ is uniformly summable (see section 5.6.1 of chapter 5), and its sum is exactly the class of $M_{\mathbf{n},\beta}$ in $\mathscr{M}_{S^{\mathbf{n}^\beta} C}$. Taking classes in $\mathscr{M}_k$, we may therefore write the motivic summation

$$[M_{\mathbf{n},\beta}] = \sum_{\mathbf{m} \in S^{\mathbf{n}^\beta} C} [H_F(\mathbf{m}, \beta)].$$

Remark 6.2.6.2. Here the existence of the moduli spaces $M_{\mathbf{n},\beta}$ shows that the family of functions $(\mathbf{1}_{H(\mathbf{m},\beta)})_{\mathbf{m}\in S^{\mathbf{n}^\beta}C}$ is uniformly summable, independently of section 5.6.3. By uniqueness of the sum (remark 5.6.1.2), this sum (the class of $M_{\mathbf{n},\beta}$ in $\mathscr{E}xp\mathscr{M}_{S^{\mathbf{n}^\beta}C}$) is equal to the one given by the Poisson formula as stated in 5.6.3.

Note that

$$\mathbf{T}^{\mathbf{n}} = \prod_{\alpha\in\mathscr{A}} T_\alpha^{n_\alpha} = \prod_{\alpha\in\mathscr{A}} T_\alpha^{\sum_v e_{\alpha,\beta_v}} \prod_{\alpha\in\mathscr{A}} \mathbf{T}_\alpha^{n_\alpha^\beta}.$$

Write $\mathbf{T}^{||\beta||}$ for $\prod_{\alpha\in\mathscr{A}} T_\alpha^{\sum_v e_{\alpha,\beta_v}}$. Then

$$\begin{aligned} Z(\mathbf{T}) = \sum_{\mathbf{n}\in\mathbf{Z}^{\mathscr{A}}}[M_{\mathbf{n}}]\mathbf{T}^{\mathbf{n}} &= \sum_{\mathbf{n}\in\mathbf{Z}^{\mathscr{A}}}\sum_{\beta\in\prod_v\mathscr{B}_{0,v}}[M_{\mathbf{n},\beta}]\mathbf{T}^{\mathbf{n}} \\ &= \sum_{\mathbf{n}\in\mathbf{Z}^{\mathscr{A}}}\left(\sum_{\beta\in\prod_v\mathscr{B}_{0,v}}\sum_{\mathbf{m}\in S^{\mathbf{n}^\beta}C}[H_F(\mathbf{m},\beta)]\right)\mathbf{T}^{\mathbf{n}} \\ &\underset{\mathbf{n}\leftarrow\mathbf{n}^\beta}{=} \sum_{\beta\in\prod_v\mathscr{B}_{0,v}}\mathbf{T}^{||\beta||}\sum_{\mathbf{n}\in\mathbf{N}^{\mathscr{A}}}\sum_{\mathbf{m}\in S^{\mathbf{n}}C}[H_F(\mathbf{m},\beta)]\mathbf{T}^{\mathbf{n}} \end{aligned}$$

For clarity, let us remark that in this summation, we have three different types of sums: the sum over $\prod_v \mathscr{B}_{0,v}$, which is a finite sum, the one over $\mathbf{Z}^{\mathscr{A}}$ or over $\mathbf{N}^{\mathscr{A}}$ which is the sum of the formal series, and the motivic sum over $\mathbf{m}\in S^{\mathbf{n}}C$.

The Poisson summation formula (see section 5.6, and especially 5.6.3 and 5.6.4; see also remark 5.6.4.1), applied to the uniformly smooth constructible family of Schwartz-Bruhat functions $(\mathbf{1}_{H(\mathbf{m},\beta)})_{\mathbf{m}\in S^{\mathbf{n}}C}$ (see the end of section 6.2.5) gives, for any $\mathbf{n}\in\mathbf{N}^{\mathscr{A}}$:

$$\begin{aligned} \sum_{\mathbf{m}\in S^{\mathbf{n}}C}[H_F(\mathbf{m},\beta)] &= \sum_{\mathbf{m}\in S^{\mathbf{n}}C}\sum_{x\in k(C)^n}\mathbf{1}_{H(\mathbf{m},\beta)}(x) \\ &= \sum_{\mathbf{m}\in S^{\mathbf{n}}C}\left(\mathbf{L}^{(1-g)n}\sum_{\xi\in k(C)^n}\mathscr{F}(\mathbf{1}_{H(\mathbf{m},\beta)})(\xi)\right) \\ &= \mathbf{L}^{(1-g)n}\sum_{\xi\in k(C)^n}\sum_{\mathbf{m}\in S^{\mathbf{n}}C}\mathscr{F}(\mathbf{1}_{H(\mathbf{m},\beta)})(\xi) \end{aligned}$$

Then $Z(\mathbf{T})$ may me rewritten in the form

$$Z(\mathbf{T}) = \mathbf{L}^{(1-g)n}\sum_{\xi\in k(C)^n}\sum_{\beta\in\prod_v\mathscr{B}_{0,v}}\mathbf{T}^{||\beta||}\sum_{\mathbf{n}\in\mathbf{N}^{\mathscr{A}}}\sum_{\mathbf{m}\in S^{\mathbf{n}}C}\mathscr{F}(\mathbf{1}_{H(\mathbf{m},\beta)})(\xi)\mathbf{T}^{\mathbf{n}}.$$

By the definition of Euler products, we have

$$\sum_{\mathbf{n}\in\mathbf{N}^{\mathscr{A}}}\sum_{\mathbf{m}\in S^{\mathbf{n}}C}\mathscr{F}(\mathbf{1}_{H(\mathbf{m},\beta)})(\xi)\mathbf{T}^{\mathbf{n}} = \prod_{v\in C}\left(\sum_{\mathbf{m}_v\in\mathbf{N}^{\mathscr{A}}}\mathscr{F}(\mathbf{1}_{H(\mathbf{m}_v,\beta_v)})(\xi_v)\mathbf{T}^{\mathbf{m}_v}\right). \tag{6.1}$$

Indeed, we are dealing here with the uniformly compactly supported family

$$(\mathscr{F}(\mathbf{1}_{H(\mathbf{m},\beta)}))_{\mathbf{m}\in S^{\mathbf{n}}C}\in\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{n}}(\nu-s,\nu-s',0,N_\iota)}$$

(see end of section 6.2.5). The summation over $k(C)^n$ is therefore in fact a summation on the n-th power of the Riemann-Roch space associated to the divisor $-\sum_v(\nu_v - s_v)[v]$. For every point ξ of this space, as explained in the beginning of the proof of lemma 5.5.2.1, ξ induces constructible morphisms

$$\varphi_{\xi,\iota} : C \to \mathbf{A}_C^{(\nu-s,\nu-s'+N_\iota)}$$

for every $\iota \in \mathbf{N}^{\mathscr{A}}$, which after taking symmetric products, give constructible morphisms

$$S^{\mathbf{n}}\varphi_\xi : S^{\mathbf{n}}C \to \mathscr{A}_{\mathbf{n}}(s', s, 0, N)$$

such that

$$(\mathscr{F}(\mathbf{1}_{H(\mathbf{m},\beta)})(\xi))_{\mathbf{m}\in S^{\mathbf{n}}C} = (S^{\mathbf{n}}\varphi_\xi)^*(\mathscr{F}(\mathbf{1}_{H(\mathbf{m},\beta)}))_{\mathbf{m}\in S^{\mathbf{n}}C}.$$

By section 5.4.4 (especially proposition 5.4.4.2), we have the equality

$$(\mathscr{F}(\mathbf{1}_{H(\mathbf{m},\beta)}))_{\mathbf{m}\in S^{\mathbf{n}}C} = S^{\mathbf{n}}((\mathscr{F}\mathbf{1}_{H_{\iota,\beta}})_{\iota\in\mathbf{N}^{\mathscr{A}}})$$

in $\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{n}}(\nu-s,\nu-s',0,N)}$. Pulling back via $S^{\mathbf{n}}\varphi_\xi = S^{\mathbf{n}}((\varphi_{\xi,\iota})_{\iota\in\mathbf{N}^{\mathscr{A}}})$ gives the equality

$$(\mathscr{F}(\mathbf{1}_{H(\mathbf{m},\beta)})(\xi))_{\mathbf{m}\in S^{\mathbf{n}}C} = S^{\mathbf{n}}((\varphi_{\xi,\iota}^*\mathscr{F}(\mathbf{1}_{H_{\iota,\beta}})_{\iota\in\mathbf{N}^{\mathscr{A}}})$$

in $\mathscr{E}xp\mathscr{M}_{S^{\mathbf{n}}C}$. The pullback to any $v \in C$ of the ι-th element $\varphi_{\xi,\iota}^*\mathscr{F}\mathbf{1}_{H_{\iota,\beta}} \in \mathscr{E}xp\mathscr{M}_C$ of the family on the right-hand side is exactly $\mathscr{F}(\mathbf{1}_{H(\iota,\beta_v)})(\xi_v)$, by the definition of $\varphi_{\xi,\iota}$ and by remark 5.4.4.1, so this should be the ι-th coefficient of the local factor of the Euler product corresponding to v.

Notation 6.2.6.3. We use the notation from [ChL], 3.6: for every $v \in C$, $\alpha \in \mathscr{A}$, $\mathbf{m}_v \in \mathbf{N}^{\mathscr{A}}$ and $\beta \in \mathscr{B}_v$, we define

$$||\mathbf{m}_v, \beta_v||_\alpha := m_{\alpha,v} + e_{\alpha,\beta_v}$$

and $\mathbf{T}^{||\mathbf{m}_v,\beta_v||} := \prod_{\alpha\in\mathscr{A}} T_\alpha^{||\mathbf{m}_v,\beta_v||}$. Note that for $g \in G(F_v)\cap G(\mathbf{m}_v, \beta_v)_v$, the local intersection degree $(g, \mathscr{L}_\alpha)_v$ is exactly

$$(g, \mathscr{L}_\alpha)_v = (g, \mathscr{D}_\alpha)_v + \sum_{\beta\in\mathscr{B}_v} e_{\alpha,\beta}(g, E_\beta)_v = m_{\alpha,v} + e_{\alpha,\beta_v} = ||\mathbf{m}_v, \beta_v||_\alpha.$$

Finally, we have

$$\begin{aligned} Z(\mathbf{T}) &= \mathbf{L}^{(1-g)n} \sum_{\xi\in k(C)^n} \sum_{\beta\in\prod_v \mathscr{B}_{0,v}} \prod_{v\in C} \prod_{\alpha\in\mathscr{A}} T_\alpha^{e_{\alpha,\beta_v}} \prod_{v\in C} \left(\sum_{\mathbf{m}_v\in\mathbf{N}^{\mathscr{A}}} \mathscr{F}(\mathbf{1}_{H(\mathbf{m}_v,\beta_v)})(\xi_v)\mathbf{T}^{\mathbf{m}_v} \right) \\ &= \mathbf{L}^{(1-g)n} \sum_{\xi\in k(C)^n} \prod_{v\in C} \left(\sum_{\beta_v\in\mathscr{B}_{0,v}} \prod_{\alpha\in\mathscr{A}} T_\alpha^{e_{\alpha,\beta_v}} \sum_{\mathbf{m}_v\in\mathbf{N}^{\mathscr{A}}} \mathscr{F}(\mathbf{1}_{H(\mathbf{m}_v,\beta_v)})(\xi_v)\mathbf{T}^{\mathbf{m}_v} \right) \\ &= \mathbf{L}^{(1-g)n} \sum_{\xi\in k(C)^n} \prod_{v\in C} \left(\sum_{\substack{\mathbf{m}_v\in\mathbf{N}^{\mathscr{A}} \\ \beta_v\in\mathscr{B}_{0,v}}} \mathscr{F}(\mathbf{1}_{H(\mathbf{m}_v,\beta_v)})(\xi_v)\mathbf{T}^{||\mathbf{m}_v,\beta_v||} \right) \end{aligned}$$

Thus, we have written $Z(\mathbf{T})$ in the form

$$Z(\mathbf{T}) = \mathbf{L}^{(1-g)n} \sum_{\xi \in k(C)^n} Z(\mathbf{T}, \xi) \tag{6.2}$$

where $Z(\mathbf{T}, \xi)$ has an Euler product decomposition with local factors

$$Z_v(\mathbf{T}, \xi) := \sum_{\substack{\mathbf{m}_v \in \mathbf{N}^{\mathscr{A}} \\ \beta_v \in \mathscr{B}_{0,v}}} \mathscr{F}(\mathbf{1}_{H(\mathbf{m}_v, \beta_v)})(\xi_v) \mathbf{T}^{||\mathbf{m}_v, \beta_v||} = \sum_{(\mathbf{m}_v, \beta_v)\ v-\text{integral}} \mathscr{F}(\mathbf{1}_{G(\mathbf{m}_v, \beta_v)})(\xi_v) \mathbf{T}^{||\mathbf{m}_v, \beta_v||}.$$

More precisely, it is the product of the finite number of factors corresponding to $v \in C \setminus C_1$, and of the Euler product of the series

$$Z_{C_1}(\mathbf{T}, \xi) := \sum_{\mathbf{m} \in \mathbf{N}^{\mathscr{A}}} \mathscr{F}(\mathbf{1}_{H_{\mathbf{m},\beta} \times_C C_1})(\xi) \mathbf{T}^{\mathbf{m}} \in \mathscr{E}xp\mathscr{M}_{C_1}[[T]]$$

where $\mathbf{T}^{\mathbf{m}} = \prod_{\alpha \in \mathscr{A}} T_\alpha^{m_\alpha}$. In what follows, we will study these local factors to be able to apply lemma 4.7.2.1 to the series $Z_{C_1}(\mathbf{T}, \xi)$. Combined with estimates for the remaining local factors, this will give us the convergence properties of the series $Z(\mathbf{T})$.

Remark 6.2.6.4 (Restriction of the summation domain). As noted in the end of section 6.2.5, the Fourier transforms of the families $(\mathbf{1}_{H(\mathbf{m},\beta)})_{\mathbf{m} \in S^{\mathbf{m}}C}$ and $(\mathbf{1}_{G(\mathbf{m},\beta)})_{\mathbf{m} \in S^{\mathbf{m}}C}$ are uniformly compactly supported. We may conclude, as in [ChL], 4.2, that there is a finite-dimensional k-vector space V (given by an appropriate Riemann-Roch space), and a linear F-morphism $\mathfrak{a} : V_F \to G_F$ such that the summation (6.2) restricts in fact to

$$Z(\mathbf{T}) = \mathbf{L}^{(1-g)n} \sum_{\xi \in \mathfrak{a}(V)} Z(\mathbf{T}, \xi).$$

6.3 Analysis of local factors and convergence

The aim of this section is to study the local factors

$$\begin{aligned} Z_v(\mathbf{T}, \xi) &= \sum_{\substack{\mathbf{m}_v \in \mathbf{N}^{\mathscr{A}} \\ \beta \in \mathscr{B}_{0,v}}} \mathscr{F}(\mathbf{1}_{H(\mathbf{m}_v, \beta_v)})(\xi_v) \mathbf{T}^{||\mathbf{m}_v, \beta_v||} \\ &= \sum_{\substack{\mathbf{m}_v \in \mathbf{N}^{\mathscr{A}} \\ \beta \in \mathscr{B}_{0,v}}} \int_{H(\mathbf{m}_v, \beta_v)} \prod_{\alpha \in \mathscr{A}} T_\alpha^{(g, \mathscr{L}_\alpha)_v} \mathrm{e}(\langle g, \xi \rangle) \mathrm{d}g. \end{aligned}$$

(We use here the notation e from [ChL], which is a substitute for the notation $\psi \circ r$ from section 5.2.4.) We follow [ChL] and rewrite this integral as an integral with respect to the motivic measure on the arc space $\mathscr{L}(\mathscr{X})$. We then give estimates for it, first in the case when ξ is the trivial character, then in the case when it is non-trivial. This allows us to study convergence of the Euler product $Z(T, \xi)$.

In this section, we will often omit the index v once the place v is fixed.

6.3.1 Motivic functions and integrals

In this section, we consider a field k of characteristic zero, $R = k[[t]]$ and $K = k((t))$. For this and the next section only, let $\mathscr{X}$ be a smooth and flat R-scheme of finite type and of pure relative dimension n. We are going to use the notion of *motivic residual function* on the arc space $\mathscr{L}(\mathscr{X})$, introduced in section 2.4 of [ChL].

Recall that the spaces of m-jets $\mathscr{L}_m(\mathscr{X})$ for $m \geq 0$ come with natural affine morphisms $p_m^{m+1} : \mathscr{L}_{m+1}(\mathscr{X}) \to \mathscr{L}_m(\mathscr{X})$ which turn the collection of relative Grothendieck rings $\mathscr{E}xp\mathscr{M}_{\mathscr{L}_m(\mathscr{X})}$ into an inductive system via the induced ring morphisms

$$\left(p_m^{m+1}\right)^* : \mathscr{E}xp\mathscr{M}_{\mathscr{L}_m(\mathscr{X})} \to \mathscr{E}xp\mathscr{M}_{\mathscr{L}_{m+1}(\mathscr{X})}$$

sending the class of a variety $H \to \mathscr{L}_m(\mathscr{X})$ to the class of $H \times_{\mathscr{L}_m(\mathscr{X})} \mathscr{L}_{m+1}(\mathscr{X}) \to \mathscr{L}_{m+1}(\mathscr{X})$ with the appropriate operation on exponentials.

Definition 6.3.1.1. The ring of motivic residual functions on $\mathscr{L}(\mathscr{X})$ is the inductive limit of all Grothendieck rings $\mathscr{E}xp\mathscr{M}_{\mathscr{L}_m(\mathscr{X})}$, $m \geq 0$.

For example, take a constructible subset W of $\mathscr{L}(\mathscr{X})$, that is, a subset of the form $p_m^{-1}(W_m)$ where W_m is a constructible subset of $\mathscr{L}_m(\mathscr{X})$ and $p_m : \mathscr{L}(\mathscr{X}) \to \mathscr{L}_m(\mathscr{X})$ is the projection morphism. Then the characteristic function of W may be seen as a motivic residual function.

Let h be a motivic residual function. Assume it to be of the form $[H, f]$ where H is a variety over $\mathscr{L}_m(\mathscr{X})$ for some m, and $f : H \to \mathbf{A}^1$ a morphism. Then the integral of h over the arc space $\mathscr{L}(\mathscr{X})$ is defined to be

$$\int_{\mathscr{L}(\mathscr{X})} h(x)\mathrm{d}x = \int_{\mathscr{L}(\mathscr{X})} H_x \psi(r(f(x)))\mathrm{d}x := \mathbf{L}^{-(m+1)n}[H, f]_k \in \mathscr{E}xp\mathscr{M}_k.$$

This does not depend on m because for any $m' \geq m$, the projection morphism $\mathscr{L}_{m'}(\mathscr{X}) \to \mathscr{L}_m(\mathscr{X})$ is a locally trivial fibration with fibre $\mathbf{A}_k^{(m'-m)n}$. More generally, one can consider integrals $\int_W$ over constructible subsets of $\mathscr{L}(\mathscr{X})$ by multiplying the integrand by the characteristic function $\mathbf{1}_W$.

Example 6.3.1.2. If $W = p_m^{-1}(W_m)$ is a constructible subset of $\mathscr{L}(\mathscr{X})$ for some $m \geq 0$, then one may define the volume of W to be

$$\mathrm{vol}(W) := \int_{\mathscr{L}(\mathscr{X})} \mathbf{1}_W(x)\mathrm{d}x = \mathbf{L}^{-(m+1)n}[W_m, 0] \in \mathscr{E}xp\mathscr{M}_k.$$

In particular, if $W = \mathscr{L}(\mathscr{X}) = p_0^{-1}(\mathscr{X}_k)$ where $\mathscr{X}_k$ is the special fibre of $\mathscr{X}$, then

$$\mathrm{vol}(\mathscr{L}(\mathscr{X})) = \mathbf{L}^{-n}[\mathscr{X}_k, 0]. \tag{6.3}$$

Another useful special case is the volume of the subspace $\mathscr{L}(\mathbf{A}^1, 0)$ of $\mathscr{L}(\mathbf{A}^1)$ of arcs with origin in $0 \in \mathbf{A}^1$. We have:

$$\mathrm{vol}(\mathscr{L}(\mathbf{A}^1, 0)) = \mathbf{L}^{-1}. \tag{6.4}$$

Combining (6.3) and (6.4), we also get that the volume of the subspace of arcs of order 0 in $\mathscr{L}(\mathbf{A}^1)$ is

$$\mathrm{vol}(\{x \in \mathscr{L}(\mathbf{A}^1),\ \mathrm{ord}(x) = 0\}) = \mathrm{vol}(\mathscr{L}(\mathbf{A}^1)) - \mathrm{vol}(\mathscr{L}(\mathbf{A}^1, 0)) = 1 - \mathbf{L}^{-1} \tag{6.5}$$

Remark 6.3.1.3. Let $W = p^{-1}(W_m)$ be a constructible subset of $\mathscr{L}(\mathscr{X})$ for some $W_m \subset \mathscr{L}_m(\mathscr{X})$ and some $m \geq 0$ and let $h = [\mathscr{L}_m(\mathscr{X}), f]$. Then by the triangular inequality for weights (chapter 4, lemma 4.6.3.3), as well as lemma 4.6.3.1:

$$\begin{aligned} w\left(\int_W h(x)\mathrm{d}x\right) &= w\left(\int_{\mathscr{L}(\mathscr{X})} \mathbf{1}_W(x)\mathrm{e}(f(x))\mathrm{d}x\right) \\ &= w(\mathbf{L}^{-(m+1)n}[W, f_{|W}]) \\ &\leq -2(m+1)n + w([W, 0]) \\ &= w(\mathrm{vol}(W)) \end{aligned}$$

We are going to use this property repeatedly.

6.3.2 Some computations of motivic integrals

We keep the notations from the previous section. Let $r : K \to k$ be the linear map defined by $r(a) = \mathrm{res}_0(a\mathrm{d}t)$, so that $r(t^{-1}) = 1$ and $r(t^n) = 0$ for any $n \neq -1$.

Lemma 6.3.2.1. *Let d be a non-zero integer and let $\xi \in K$ be such that $\mathrm{ord}(\xi) = 0$. Let $Q = a_0 + a_1 x + \ldots + a_d x^d \in k((t))[x]$ be a non-zero polynomial of degree $\leq d$ such that for all $i \in \{1, \ldots, d\}$, $\mathrm{ord}(a_i) > \mathrm{ord}(a_0)$. Then for any $n \in \mathbf{N}$ such that $-2n \leq \mathrm{ord}(a_0) < -n$, one has*

$$\int_{\xi + t^n k[[t]]} \psi(r(Q(x)x^{-d}))\mathrm{d}x = 0.$$

Proof. The proof goes along the same lines as the proof of lemma 5.1.1 in [ChL]. The condition on the orders of the coefficients of Q implies that:

— $\mathrm{ord}(Q(\xi)) = \mathrm{ord}(a_0)$.

— For all $i \in \{1, \ldots, d\}$,

$$\mathrm{ord}(Q^{(i)}(\xi)) \geq \min_{i \leq j \leq d} \mathrm{ord}(a_j) > \mathrm{ord}(Q(\xi)) \geq -2n. \tag{6.6}$$

Write $x = \xi(1 + t^n u)$ for some u with $\mathrm{ord}(u) \geq 0$, so that we have

$$\int_{\xi + t^n k[[t]]} \psi(r(Q(x)x^{-d}))\mathrm{d}x = \mathbf{L}^{-n} \int_{k[[t]]} \psi(r(Q(\xi(1 + t^n u))(\xi(1 + t^n u))^{-d})\mathrm{d}u.$$

We may expand

$$Q(\xi(1 + t^n u)) = Q(\xi) + Q'(x)\xi t^n u + Q''(x)(\xi t^n u)^2 + \ldots$$

and

$$(\xi(1+t^n u))^{-d} = \xi^{-d}\left(1 - dt^n u + \binom{-d}{2} t^{2n}u^2 + \dots\right)$$

Taking the product and using (6.6), we have

$$r\left(Q(\xi(1+t^n u))\xi^{-d}(1+t^n u)\right) = r(Q(\xi)\xi^{-d}) + r\left(Q'(\xi)\xi^{1-d}t^n u - Q(\xi)\xi^{-d}dt^n u\right),$$

since all the other terms belong to the maximal ideal $tk[[t]]$. We therefore have

$$\begin{aligned}&\int_{k[[t]]} \psi(r(Q(\xi(1+t^n u))(\xi(1+t^n u))^{-d})\mathrm{d}u \\ &= [\operatorname{Spec} k, r(Q(\xi)\xi^{-d})] \int_{k[[t]]} \psi(r\left(Q'(\xi)\xi^{1-d}t^n u - Q(\xi)\xi^{-d}dt^n u\right))\mathrm{d}u.\end{aligned}$$

Now, $\operatorname{ord}(Q'(\xi)\xi^{1-d}t^n - Q(\xi)d\xi^{-d}t^n) = \operatorname{ord}(Q(\xi)t^n) < 0$, and therefore the integral in the right-hand side is zero. □

For $m \in \mathbf{Z}$, let C_m be the annulus defined by $\operatorname{ord}(x) = m$.

Lemma 6.3.2.2. *Let m and d be positive integers and $P \in k[[t]][x]$ a non-zero polynomial such that* $\operatorname{ord}(P(0)) = 0$*. Then*

$$\int_{C_m} \psi(r(P(x)x^{-d}))\mathrm{d}x = \begin{cases} -\mathbf{L}^{-2} & \text{if } m = d = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $I(m,d,P)$ be the above integral. A change of variable allows us to write

$$I(m,d,P) = \mathbf{L}^{-m}\int_{C_0} \psi(r(P(t^m u)t^{-md}u^{-d}))\mathrm{d}u.$$

Thus, we have $I(m,d,P) = I(0,d,Q)$ where $Q(u) = a_0 + a_1 u + \ldots \in k((t))[u]$ is the polynomial given by $Q(u) = P(t^m u)t^{-md}$. Since P has integral coefficients, and $\operatorname{ord}(P(0)) = 0$, we have

— $\operatorname{ord}(a_0) = \operatorname{ord}(P(0)t^{-md}) = -md$
— for all $i \geq 1$, $\operatorname{ord}(a_i) \geq mi - md > -md = \operatorname{ord}(a_0)$.

If $md > 1$, then there exists a positive integer n such that $-2n \leq -md < -n$, and choosing such an n, the previous lemma tells us that

$$I(m,d,P) = \mathbf{L}^{-m}\int_{C_0}\int_{k[[t]]} \psi(r(Q(u)(u+t^n y)^{-d})\mathrm{d}y\,\mathrm{d}u = 0.$$

Assume now $m = d = 1$. Then, writing $P(0) = a$, we have

$$\begin{aligned} I(1,1,P) &= \mathbf{L}^{-1}\int_{C_0} \psi(r(P(tu)t^{-1}u^{-1}))\mathrm{d}u = \mathbf{L}^{-1}\int_{C_0}\psi(r(at^{-1}u^{-1}))\mathrm{d}u \\ &= \mathbf{L}^{-1}\int_{C_0}\psi(r(at^{-1}u))\mathrm{d}u \\ &= \mathbf{L}^{-1}\left(\int_{k[[t]]}\psi(r(at^{-1}u))\mathrm{d}u - \int_{tk[[t]]}\psi(r(at^{-1}u))\mathrm{d}u\right)\end{aligned}$$

which gives the result, since the first term in the parenthesis is zero (using again that $\operatorname{ord}(a) = 0$), and the second one is equal to $\mathbf{L}^{-1}$. □

6.3.3 An integral over the arc space

We now go back to the setting and notations of section 6.1. In this section we recall briefly, following [ChL], how integrals of motivic Schwartz-Bruhat functions can be rewritten as integrals on arc spaces.

Lemma 6.3.3.1. *([ChL], lemma 6.1.1) Let $\Phi \in \mathscr{S}(F_v^n)$ be a local motivic Schwartz-Bruhat function. Then the integral $\int_{G(F_v)} \Phi(g)\mathrm{d}g$ can be rewritten as*

$$\int_{\mathscr{L}(\mathscr{X})} \Phi(x)\mathbf{L}^{-\mathrm{ord}_\omega(x)}\mathrm{d}x$$

where $\mathrm{d}x$ denotes the motivic measure on the arc space $\mathscr{L}(\mathscr{X})$.

For every subset $A \subset \mathscr{A}$ and every $\beta \in \mathscr{B}_{1,v}$ for some v, we denote by $\Delta(A,\beta)$ the locally closed subset of the special fibre $\mathscr{X}_v := \mathscr{X} \times_R \mathrm{Spec}\,(k)$ of points belonging to the divisors $\mathscr{D}_\alpha,\ \alpha \in A$ and no other, as well as to the vertical divisor E_β, and no other.

The special fibre $\mathscr{X}_v$ identifies with the jet scheme $\mathscr{L}_0(\mathscr{X})$ of order 0, and therefore there is a specialisation morphism $\mathscr{L}(\mathscr{X}) \longrightarrow \mathscr{X}_v$. We denote by $\Omega(A,\beta)$ the preimage in $\mathscr{L}(\mathscr{X})$ of $\Delta(A,\beta)$. Lemma 5.2.6 of [ChL] then states the following:

Lemma 6.3.3.2. *Let A be a subset of $\mathscr{A}$ and let B be a set of cardinality $n - \mathrm{Card}(A)$. There exists a mesure-preserving definable isomorphism θ from $\Delta(A,\beta) \times \mathscr{L}(\mathbf{A}^1,0)^A \times \mathscr{L}(\mathbf{A}^1,0)^B$ with coordinates $x_\alpha,\ \alpha \in A$ and $y_\beta,\ \beta \in B$, to $\Omega(A,\beta)$, such that $\mathrm{ord}_{\mathscr{D}_\alpha}(\theta(x)) = \mathrm{ord}(x_\alpha)$ for $\alpha \in A$, and $\mathrm{ord}_{\mathscr{D}_\alpha}(\theta(x)) = 0$ for $\alpha \notin A$.*

Remark 6.3.3.3. We changed the normalisation slightly with respect to [ChL]. Note that this is consistent with example 6.3.1.2: the volume of $\Omega(A,\beta)$ is given by $[\Delta(A,\beta)]\mathbf{L}^{-n}$.

In what follows, we therefore identify a point of $\Omega(A,\beta)$ with a triple

$$(w,x,y) \in \Delta(A,\beta) \times \mathscr{L}(\mathbf{A}^1,0)^A \times \mathscr{L}(\mathbf{A}^1,0)^B.$$

We also recall Lemma 6.2.6 from [ChL], which uses this isomorphism to rewrite the motivic Fourier transforms $Z_v(\mathbf{T},\xi)$ as sums of motivic integrals over arc spaces:

Lemma 6.3.3.4. *For every motivic residual function h on $\mathscr{L}(\mathscr{X})$ and every $\xi \in G(F_v)$, one has*

$$\int_{G(F_v)} \prod_{\alpha\in\mathscr{A}} T_\alpha^{(g,\mathscr{L}_\alpha)_v} h(g)\mathrm{e}(\langle g,\xi\rangle)\mathrm{d}g$$

$$= \sum_{\substack{A\subset\mathscr{A}\\ \beta\in\mathscr{B}_1}} \prod_{\alpha\in\mathscr{A}} T_\alpha^{e_{\alpha,\beta}}\mathbf{L}^{\rho_\beta} \int_{\Omega(A,\beta)} \prod_{\alpha\in A} (\mathbf{L}^{\rho_\alpha}T_\alpha)^{\mathrm{ord}(x_\alpha)} h(x)\mathrm{e}(\langle x,\xi\rangle)\mathrm{d}x.$$

6.3.4 A few words on convergence of Euler products in our setting

If in our general convergence result, proposition 4.7.2.1, we take X to be a curve (so that $w(X) = 2$), $\epsilon = \frac{1}{2}$ and $\beta = 0$ (and replacing α by the letter c to avoid confusion with the indices of the components D_α), the obtain the following particular case which will be the one we are going to use:

Lemma 6.3.4.1. *Let X be a curve over $\mathbf{C}$. Assume $F(T) = 1 + \sum_{i\geq 1} X_i t^i \in \mathscr{E}xp\mathscr{M}_X[[T]]$ is such that there exist an integer $M \geq 1$ and a real number $c < 1$ such that*

— *for all $i \in \{1, \ldots, M\}$, $w_X(X_i) \leq 2i - 2$*

— *for all $i \geq M$, $w_X(X_i) \leq 2ci - 1$.*

Then there exists $\delta > 0$ such that the Euler product $\prod_{v\in X} F_v(T) \in \mathscr{E}xp\mathscr{M}_k[[T]]$ converges for $|T| < \mathbf{L}^{-1+\delta}$.

In practice, X will be some dense open subset of our original curve C, and the convergence of the remaining factors will be checked separately.

Remark 6.3.4.2. Note that by lemma 4.6.3.2, to bound $w_X(X_i)$, it suffices to bound $2\dim_X(X_i) + \dim X$. We are going to use this remark for almost all terms except the very few first ones where a bound on weights rather than dimensions is crucial.

Remark 6.3.4.3. Whenever the series $F(T)$ is in fact a polynomial, we may take M to be its degree and then it suffices to check the bound $w_X(X_i) \leq 2i - 2$ for all i, taking c to be zero in the statement of the lemma. In the case when we want to check it on dimensions, this boils down to the inequality $\dim_X(X_i) \leq i - 2$.

This remark motivates the following terminology which we will use to discard terms that won't obstruct convergence:

Definition 6.3.4.4. Let X be a k-variety and $i \geq 0$ be an integer. A polynomial $F = \sum a_i T^i \in \mathscr{E}xp\mathscr{M}_X[T]$ is said to be admissible if $\dim_X(a_i) \leq i - 2$ for all $i \geq 0$. A polynomial $F = \sum_{\mathbf{m}\in\mathbf{N}^{\mathscr{A}}} a_{\mathbf{m}} \mathbf{T}^{\mathbf{m}} \in \mathscr{E}xp\mathscr{M}_X[\mathbf{T}]$ is said to be ρ'-admissible if $F((T^{\rho'_\alpha})_{\alpha\in\mathscr{A}})$ is admissible.

Note that $F \in \mathscr{E}xp\mathscr{M}_X[T]$ is admissible if and only if for all $v \in X$, $F_v = \sum a_{i,v} T^i \in \mathscr{E}xp\mathscr{M}_k[T]$ is admissible, so admissibility may be checked locally.

In what follows, we are going to use the weight function from chapter 4. Therefore, unless explicitly stated, in all what follows, the base field k will be the field of complex numbers $\mathbf{C}$.

6.3.5 Places in C_0

Let v be a place in C_0. In this case, for any character ξ, $Z_v(\mathbf{T}, \xi)$ is given by lemma 6.3.3.4, taking h to be the characteristic function of the set $\mathscr{U}(\mathcal{O}_v)$ inside $G(F_v)$. In other words, one has $h = 0$ on $\Omega(A, \beta)$ whenever $A \cap \mathscr{A}_D \neq \varnothing$ or $\beta \notin \mathscr{B}_0$, and $h = 1$ otherwise.

Therefore,

$$Z_v(\mathbf{T},\xi)=\sum_{\substack{A\subset\mathscr{A}\setminus\mathscr{A}_D\\ \beta\in\mathscr{B}_{0,v}}}\prod_{\alpha\in\mathscr{A}}T_\alpha^{e_{\alpha,\beta}}\mathbf{L}^{\rho_\beta}\int_{\Omega(A,\beta)}\prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha}T_\alpha)^{\mathrm{ord}(x_\alpha)}\mathrm{e}(\langle x,\xi\rangle)\mathrm{d}x. \tag{6.7}$$

We are going to study the local factors in this form. For all but a finite number of places in C_0, a precise analysis is required to prove meromorphic continuation of the Euler product (specialised for $\mathbf{T}=(T^{\rho'_\alpha})_{\alpha\in\mathscr{A}}$) for $|T|<\mathbf{L}^{-1+\delta}$, with a pole at $\mathbf{L}^{-1}$. For the finite number of remaining places, a coarser estimate suffices, given by the following lemma:

Lemma 6.3.5.1. *Let $v\in C_0$. The local factor $Z_v((T^{\rho'_\alpha})_\alpha,\xi)$ converges for $|T|<\mathbf{L}^{-1+\delta}$ for some $\delta>0$.*

Proof. Write

$$Z_v(\mathbf{T},\xi)=\sum_{\beta\in\mathscr{B}_{0,v}}\prod_{\alpha\in\mathscr{A}}T_\alpha^{e_{\alpha,\beta}}\mathbf{L}^{\rho_\beta}\sum_{A\subset\mathscr{A}\setminus\mathscr{A}_D}Z_{A,\beta}(\mathbf{T},\xi),$$

with

$$Z_{A,\beta}(\mathbf{T},\xi)=\int_{\Omega(A,\beta)}\prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha}T_\alpha)^{\mathrm{ord}(x_\alpha)}\mathrm{e}(\langle x,\xi\rangle)\mathrm{d}x.$$

It suffices to check convergence of each $Z_{A,\beta}(\xi)$. Using the notation f_ξ for the linear form $\langle\cdot,\xi\rangle$ on G_F and the rational function it induces on X, we have

$$\begin{aligned}Z_{A,\beta}(\mathbf{T},\xi)&=\sum_{\mathbf{m}\in\mathbf{N}^A_{>0}}\prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha}T_\alpha)^{m_\alpha}\int_{\substack{\Delta(A,\beta)\times\mathscr{L}(\mathbf{A}^1,0)^{n-|A|}\times\mathscr{L}(\mathbf{A}^1,0)^{|A|}\\ \mathrm{ord}(x_\alpha)=m_\alpha}}\mathrm{e}(f_\xi(x,y))\mathrm{d}x\mathrm{d}y\\ &=\sum_{\mathbf{m}\in\mathbf{N}^A_{>0}}\prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha-1}T_\alpha)^{m_\alpha}\int_{\substack{\Delta(A,\beta)\times\mathscr{L}(\mathbf{A}^1)^{n-|A|}\times\mathscr{L}(\mathbf{A}^1,0)^{|A|}\\ \mathrm{ord}(x_\alpha)=0}}\mathrm{e}(f_\xi(t^{\mathbf{m}}x,y))\mathrm{d}x\mathrm{d}y.\end{aligned}$$

By remark 6.3.1.3, each integral is of weight at most the weight of the volume of the domain of integration, which, by (6.4) and (6.5) is

$$\begin{aligned}w([\Delta(A,\beta)](1-\mathbf{L}^{-1})^{|A|}\mathbf{L}^{-n+|A|})&\leq 2\dim[\Delta(A,\beta)]+2(-n+|A|)\\ &\leq 0\end{aligned}$$

because the divisors D_α intersect transversely. Thus, the series converges if the series

$$\sum_{\mathbf{m}\in\mathbf{N}^A_{>0}}\prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha-1}T_\alpha)^{m_\alpha}.$$

converges. This series is equal to

$$\prod_{\alpha\in A}\frac{\mathbf{L}^{\rho_\alpha-1}T_\alpha}{1-\mathbf{L}^{\rho_\alpha-1}T_\alpha}.$$

Specialising to $\mathbf{T}=(T^{\rho_\alpha})_{\alpha\in A}$ we get the result. □

For $\xi=0$, we need to be more precise and to check the exact order of the pole, therefore we are going to use this lemma only for $\xi\neq 0$.

6.3.5.1 Trivial character

Assume $\xi = 0$. Then the factor $\mathrm{e}(\langle x, \xi\rangle)$ equals 1, and we may compute directly (see [ChL], 6.4):

$$Z_v(\mathbf{T},0) = \sum_{\beta\in\mathscr{B}_{0,v}} \prod_{\alpha\in\mathscr{A}} T_\alpha^{e_{\alpha,\beta}}\mathbf{L}^{\rho_\beta} \sum_{A\subset\mathscr{A}\setminus\mathscr{A}_D} [\Delta(A,\beta)]\mathbf{L}^{-n+|A|}(1-\mathbf{L}^{-1})^{|A|}\prod_{\alpha\in A}\frac{\mathbf{L}^{\rho_\alpha-1}T_\alpha}{1-\mathbf{L}^{\rho_\alpha-1}T_\alpha}.$$

Therefore, $Z_v(\mathbf{T},0)$ is a rational function, and $Z_v(\mathbf{T},0)\prod_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}(1-\mathbf{L}^{\rho_\alpha-1}T_\alpha)$ is a Laurent polynomial (and a polynomial for almost all v).

Proposition 6.3.5.2. *There is a real number $\delta > 0$ such that the product*

$$\prod_{v\in C_0}\left(Z_v((T^{\rho'_\alpha})_\alpha,0)\prod_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}(1-\mathbf{L}^{\rho_\alpha-1}T^{\rho_\alpha})\right)$$

converges for $|T| < \mathbf{L}^{-1+\delta}$ and takes a non-zero effective value in $\widehat{\mathscr{M}_k}$ at $T = \mathbf{L}^{-1}$.

Proof. It suffices to check convergence of the product over v inside the dense open subset C_1 of C_0 (see notation 6.2.3.2). Thus, we may assume that $\mathscr{B}_v = \{\beta\}$ (and denote $\Delta(A,\beta)$ simply by $\Delta(A)$) and that the integers $e_{\alpha,\beta}$ and ρ_β are zero. Then the above formula simplifies to

$$Z_v(\mathbf{T},0) = \sum_{A\subset\mathscr{A}\setminus\mathscr{A}_D}[\Delta(A)]\mathbf{L}^{-n+|A|}(1-\mathbf{L}^{-1})^{|A|}\prod_{\alpha\in A}\frac{\mathbf{L}^{\rho_\alpha-1}T_\alpha}{1-\mathbf{L}^{\rho_\alpha-1}T_\alpha}.$$

Put $F_v(\mathbf{T},0) := Z_v(\mathbf{T},0)\prod_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}(1-\mathbf{L}^{\rho_\alpha-1}T_\alpha)$. It is a polynomial. For A such that $|A|\geq 2$ we have $\dim\Delta(A)\leq n-|A|$ so that

$$\begin{aligned} F_v(\mathbf{T},0) &= 1-\sum_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}\mathbf{L}^{\rho_\alpha-1}T_\alpha + \sum_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}[\Delta(\{\alpha\})]\mathbf{L}^{1-n}\mathbf{L}^{\rho_\alpha-1}T_\alpha + P_v(\mathbf{T}) \\ &= 1-\sum_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}([\mathscr{D}_{\alpha,v}]-\mathbf{L}^{n-1})\mathbf{L}^{-n}\mathbf{L}^{\rho_\alpha}T_\alpha + P_v(\mathbf{T})\end{aligned}$$

where $P_v(\mathbf{T})\in\mathscr{M}_k[\mathbf{T}]$ is a ρ'-admissible polynomial (see definition 6.3.4.4). This computation is uniform in $v\in C_1$, meaning that there is a polynomial $F(\mathbf{T},0)\in\mathscr{M}_{C_1}[\mathbf{T}]$ and a ρ'-admissible polynomial $P(\mathbf{T})\in\mathscr{M}_{C_1}[\mathbf{T}]$ such that, denoting by $v : \operatorname{Spec} k\to C_1$ the morphism defining the point v, we have $v^*F = F_v$, $v^*P = P_v$, and

$$F(\mathbf{T},0) = 1-\sum_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}([\mathscr{D}_\alpha]-\mathbf{L}^{n-1})\mathbf{L}^{-n}\mathbf{L}^{\rho_\alpha}T_\alpha + P(\mathbf{T})$$

in $\mathscr{M}_{C_1}[\mathbf{T}]$. By lemma 4.6.3.4, we have

$$w_{C_1}(([\mathscr{D}_\alpha]-\mathbf{L}^{n-1})\mathbf{L}^{-n}\mathbf{L}^{\rho_\alpha}) \leq 2(n-1)-2n+2\rho_\alpha \leq 2(\rho_\alpha-1).$$

Specialising T_α to T^{ρ_α}, we see that we can apply lemma 6.3.4.1 with $X = C_1$, as explained in remark 6.3.4.3.

Thus, there is a real number $\delta > 0$ such that the infinite product $\prod_{v\in C_1} F_v((T^{\rho'_\alpha})_\alpha, 0)$ converges for $|T| < \mathbf{L}^{-1+\delta}$ and has a non-zero effective value at $\mathbf{L}^{-1}$ in $\widehat{\mathscr{M}_k}$, which is of the form $1 + a$ with $w(a) < 0$. Multiplying by the finite number of factors $v \in (C_0 \setminus C_1)(k)$ does not change convergence. Moreover, for $v \in C_0 \setminus C_1(k)$ the value of $F_v((T^{\rho'_\alpha})_\alpha, 0)$ at $T = \mathbf{L}^{-1}$ is exactly:

$$(1-\mathbf{L}^{-1})^{|\mathscr{A}\setminus\mathscr{A}_D|} \sum_{\substack{A\subset\mathscr{A}\setminus\mathscr{A}_D\\ \beta\in\mathscr{B}_0}} \mathbf{L}^{\rho_\beta - \sum_{\alpha\in\mathscr{A}}\rho'_\alpha e_{\alpha,\beta}}[\Delta(A,\beta)]\mathbf{L}^{-n}, \tag{6.8}$$

which is clearly effective. It is non-zero unless $\Delta(A,\beta) = \varnothing$ for all $A \subset \mathscr{A} \setminus \mathscr{A}_D$ and all $\beta \in \mathscr{B}_{0,v}$, which would mean that $\mathscr{U}(\mathcal{O}_v) = \varnothing$. The latter was ruled out by our assumption of existence of local sections, see 6.1.3. We may conclude that the product

$$\prod_{v\in C_0} F_v((T^{\rho'_\alpha})_\alpha, 0)$$

converges for $|T| < \mathbf{L}^{-1+\delta}$ and has a non-zero effective value at $\mathbf{L}^{-1}$ in $\widehat{\mathscr{M}_{\mathbf{C}}}$. Moreover, we may give a formula for this value. For all $v \in C_1$, the expression in (6.8) simplifies to

$$(1-\mathbf{L}^{-1})^{|\mathscr{A}\setminus\mathscr{A}_D|} \sum_{A\subset\mathscr{A}\setminus\mathscr{A}_D} [\Delta(A)]\mathbf{L}^{-n}$$

$$= (1-\mathbf{L}^{-1})^{|\mathscr{A}\setminus\mathscr{A}_D|}\left(1 + \mathbf{L}^{-n}\sum_{\varnothing\neq A\subset\mathscr{A}\setminus\mathscr{A}_D}[\Delta(A)]\right)$$

because $[\Delta(\varnothing)] = [G(k)] = \mathbf{L}^n$, so that the total value at $\mathbf{L}^{-1}$ is:

$$\prod_{v\in C_1}(1-\mathbf{L}^{-1})^{|\mathscr{A}\setminus\mathscr{A}_D|}\left(1 + \mathbf{L}^{-n}\sum_{\varnothing\neq A\subset\mathscr{A}\setminus\mathscr{A}_D}[\Delta(A)]\right)$$

$$\times \prod_{v\in C_0\setminus C_1}(1-\mathbf{L}^{-1})^{|\mathscr{A}\setminus\mathscr{A}_D|}\left(\sum_{\substack{A\subset\mathscr{A}\setminus\mathscr{A}_D\\ \beta\in\mathscr{B}_0}} \mathbf{L}^{\rho_\beta - \sum_{\alpha\in\mathscr{A}}\rho'_\alpha e_{\alpha,\beta}}[\Delta(A,\beta)]\mathbf{L}^{-n}\right).$$

□

Using hypothesis 3, this result implies that the infinite product $\prod_{v\in C_0} Z_v(0, (T^{\rho'_\alpha})_\alpha)$ has a meromorphic continuation for $|T| < \mathbf{L}^{-1+\delta}$, its only pole being a pole of order $\mathrm{Card}(\mathscr{A}\setminus\mathscr{A}_D) = \mathrm{rk}\ \mathrm{Pic}(U)$ at $T = \mathbf{L}^{-1}$.

6.3.5.2 Non-trivial characters

For every $\xi \in G(F_v)$, the linear form $x \mapsto \langle x, \xi \rangle$ on G_{F_v} defines a meromorphic function f_ξ on X. The support of its divisor of poles is contained in $\bigcup_\alpha D_\alpha$, so that we can write

$$\operatorname{div} f_\xi = E(\xi) - \sum_{\alpha \in \mathscr{A}} d_\alpha(\xi) D_\alpha,$$

where $E(\xi)$ is an effective divisor, and the integers $d_\alpha(\xi)$ are non-negative. Once ξ is fixed, we will simply write E and d_α, omitting the mention of ξ. In the following analysis, the place v is fixed, so that $Z_v(\mathbf{T}, \xi)$ will be simply denoted $Z(\mathbf{T}, \xi)$. Moreover, using lemma 6.3.5.1, we may assume that v belongs to the open subset C_1 of C (see notation 6.2.3.2). We write $\Omega(A)$ for $\Omega(A, \beta)$. Recall that because of the restriction of the domain of summation (section 6.2.6.4), ξ is an element of $(t^{\nu_v} k[[t]])^n = (k[[t]])^n$.

We have

$$Z(\mathbf{T}, \xi) = \sum_{A \subset \mathscr{A} \setminus \mathscr{A}_D} Z_A(\mathbf{T}, \xi)$$

where

$$Z_A(\mathbf{T}, \xi) = \int_{\Omega(A)} \prod_{\alpha \in A} (\mathbf{L}^{\rho_\alpha} T_\alpha)^{\operatorname{ord} x_\alpha} \operatorname{e}(f_\xi(x)) \mathrm{d}x.$$

The case $A = \varnothing$ The set $\Omega(\varnothing)$ corresponds to arcs with origin contained in none of the D_α, that is, contained in G. Then we get

$$Z_\varnothing(\mathbf{T}, \xi) = \int_{\mathscr{L}(\mathbf{G}_{\mathrm{a}}^n)} \operatorname{e}(\langle x, \xi \rangle) \mathrm{d}x$$

Since ξ is an element of $(t^{\nu_v} k[[t]])^n$, for all $x \in \mathscr{L}(\mathbf{G}_{\mathrm{a}}^n)(k) = k[[t]]^n$, we have

$$\operatorname{ord}(\langle x, \xi \rangle) = \operatorname{ord}(x_1 \xi_1 + \ldots + x_n \xi_n) \geq \nu_v.$$

Thus, in fact $r(\langle x, \xi \rangle) = 0$, and the integral is equal to 1.

The case $A = \{\alpha\}$ We are going to cut the integral into two pieces: arcs with origin outside or inside the divisor E:

$$\begin{aligned} Z_{v,\{\alpha\}}(\mathbf{T}, \xi) &= \int_{\mathscr{L}(\mathscr{X}, D_\alpha^\circ \setminus E)} (\mathbf{L}^{\rho_\alpha} T_\alpha)^{\operatorname{ord} x_\alpha} \operatorname{e}(f_\xi(x_\alpha, \mathbf{y})) \mathrm{d}x_\alpha \mathrm{d}\mathbf{y} \\ &\quad + \int_{\mathscr{L}(\mathscr{X}, D_\alpha^\circ \cap E)} (\mathbf{L}^{\rho_\alpha} T_\alpha)^{\operatorname{ord} x_\alpha} \operatorname{e}(f_\xi(x_\alpha, \mathbf{y})) \mathrm{d}x_\alpha \mathrm{d}\mathbf{y}. \end{aligned}$$

By the equality $X(k((t))) = \mathscr{X}(k[[t]])$, the rational function f_ξ on X induces a rational function f_ξ on $\mathscr{L}(\mathscr{X})$. On the subspace $\mathscr{L}(\mathscr{X}, D_\alpha^\circ \setminus E)$, f_ξ is of the form $f_\xi(x, \mathbf{y}) = g_\xi(x, \mathbf{y}) x^{-d}$, where g_ξ is a regular function on $\mathscr{L}(\mathscr{X}, D_\alpha^\circ \setminus E)$. We may expand the function

g_ξ as a converging and non-vanishing power series in $x \in \mathscr{L}^1(\mathbf{A}^1, 0)$ and $\mathbf{y} \in (\mathscr{L}^1(\mathbf{A}^1, 0))^B$ where $B = A\backslash\{\alpha\}$:

$$g_\xi(x, \mathbf{y}) = \sum_{\substack{p \geq 0 \\ \mathbf{q} \in \mathbf{N}^B}} g_{p,q} x^p \mathbf{y}^{\mathbf{q}}, \tag{6.9}$$

with $g_{p,q} \in \mathcal{O}(D_\alpha^\circ)[[t]]$. Since we consider only arcs with origin outside E, we have $\mathrm{ord}(g_0) = 0$, and more generally, $\mathrm{ord}(g_\xi(x, \mathbf{y})) = 0$ for all $x, \mathbf{y}$.

There are several cases to consider, depending on the order $d = d_\alpha$ of the pole of f_ξ at D_α. Define

$$\mathscr{A}_0(\xi)^D = \{\alpha \in \mathscr{A} \setminus \mathscr{A}_D,\ d_\alpha = 0\}$$

and

$$\mathscr{A}_1(\xi)^D = \{\alpha \in \mathscr{A} \setminus \mathscr{A}_D,\ d_\alpha = 1\}.$$

The order of the pole at D_α is zero Here we assume that $\alpha \in \mathscr{A}_0^D(\xi)$, so that $d = 0$. Since $\mathrm{ord}(g) \geq 0$, we have $r \circ g = 0$. Therefore

$$\begin{aligned}
&\int_{\mathscr{L}(\mathscr{X}, D_\alpha^\circ \setminus E)} (\mathbf{L}^{\rho_\alpha} T_\alpha)^{\mathrm{ord} x_\alpha} \mathrm{e}(f_\xi(x, \mathbf{y})) \mathrm{d}x \mathrm{d}\mathbf{y} \\
= &\int_{(D_\alpha^\circ \setminus E) \times \mathscr{L}(\mathbf{A}^1, 0)^{n-1}} \sum_{m \geq 1} (\mathbf{L}^{\rho_\alpha} T_\alpha)^m \left(\int_{\substack{\mathscr{L}(\mathbf{A}^1, 0) \\ \mathrm{ord} x = m}} \psi(r(g_\xi(x, \mathbf{y}))) \mathrm{d}x \right) \mathrm{d}\mathbf{y} \\
= &\int_{(D_\alpha^\circ \setminus E) \times \mathscr{L}(\mathbf{A}^1, 0)^{n-1}} \sum_{m \geq 1} (\mathbf{L}^{\rho_\alpha} T_\alpha)^m \mathbf{L}^{-m} (1 - \mathbf{L}^{-1}) \\
= &(1 - \mathbf{L}^{-1}) \frac{\mathbf{L}^{\rho_\alpha - 1} T_\alpha}{1 - \mathbf{L}^{\rho_\alpha - 1} T_\alpha} [D_\alpha^\circ \setminus E] \mathbf{L}^{-n+1}
\end{aligned}$$

The order of the pole at D_α is positive Assume now $d \geq 1$. Then:

$$\begin{aligned}
&\int_{\mathscr{L}(\mathscr{X}, D_\alpha^\circ \setminus E)} (\mathbf{L}^{\rho_\alpha} T_\alpha)^{\mathrm{ord} x_\alpha} \mathrm{e}(f_\xi(x, \mathbf{y})) \mathrm{d}x \mathrm{d}\mathbf{y} \\
= &\sum_{m \geq 1} (\mathbf{L}^{\rho_\alpha} T_\alpha)^m \int_{\substack{\mathscr{L}(\mathscr{X}, D_\alpha^\circ \setminus E) \\ \mathrm{ord} x = m}} \mathrm{e}(g_\xi(x, \mathbf{y}) x^{-d}) \mathrm{d}x \mathrm{d}\mathbf{y} \\
= &\int_{(D_\alpha^\circ \setminus E) \times \mathscr{L}(\mathbf{A}^1, 0)^{n-1}} \sum_{m \geq 1} (\mathbf{L}^{\rho_\alpha} T_\alpha)^m \left(\int_{\substack{\mathscr{L}(\mathbf{A}^1, 0) \\ \mathrm{ord} x = m}} \psi(r(g_\xi(x, \mathbf{y}) x^{-d})) \mathrm{d}x \right) \mathrm{d}\mathbf{y}.
\end{aligned}$$

Fixing $\mathbf{y}$ and viewing $g_\xi(x, \mathbf{y})$ as a power series in x, we may apply lemma 6.3.2.2: indeed, first of all we may drop all the terms of degree in x greater than d because of the invariance of r, so that we get a polynomial in x with coefficients in $k[[t]]$. Moreover, its constant term is $g_\xi(0, \mathbf{y})$ which by a remark above is of order 0. This shows that this expression is zero when $d > 1$. When $d = 1$, only the term for $m = 1$ remains, and it is equal to

$$[D_\alpha^\circ \setminus E] \times \mathbf{L}^{1-n} \mathbf{L}^{\rho_\alpha} T_\alpha (-\mathbf{L}^{-2}),$$

which is ρ'_α-admissible, as $\dim[D_\alpha^\circ \setminus E] = n - 1$.

Arcs with origin in E The term corresponding to arcs with origin in E may be rewritten as

$$\begin{aligned}
&\sum_{m\geq 1}(\mathbf{L}^{\rho_\alpha}T_\alpha)^m\int_{\substack{\mathscr{L}(\mathscr{X},D_\alpha^\circ\cap E)\\ \mathrm{ord}x=m}}\mathrm{e}(f_\xi(x,\mathbf{y}))\mathrm{d}x\mathrm{d}\mathbf{y}\\
=\ &\sum_{m\geq 1}(\mathbf{L}^{\rho_\alpha}T_\alpha)^m\int_{\substack{D_\alpha^\circ\cap E\times\mathscr{L}(\mathbf{A}^1)\mathscr{L}(\mathbf{A}^1,0)^{n-1}\\ \mathrm{ord}x=m}}\mathrm{e}(f_\xi(t^m x,\mathbf{y}))\mathrm{d}x\mathrm{d}\mathbf{y}
\end{aligned}$$

Using remark 6.3.1.3 again, the weight of the coefficient of degree m is bounded by the the weight of $\mathbf{L}^{m\rho_\alpha}[D_\alpha^\circ\cap E]\times\mathbf{L}^{-m}(1-\mathbf{L}^{-1})\mathbf{L}^{-n+1}$. The dimension of the latter is smaller than

$$m\rho_\alpha+(n-2)-m-(n-1)=m\rho_\alpha-m-1$$

Since $m\geq 1$, we see that all these terms are ρ'-admissible. It remains to bound all terms of sufficiently large degree as in lemma 6.3.4.1. For this, put

$$c=\max_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}\left(1-\frac{1}{2\rho_\alpha}\right)<1, \tag{6.10}$$

so that for all $\alpha\in\mathscr{A}\setminus\mathscr{A}_D$, $\rho_\alpha-\frac{1}{2}\leq c\rho_\alpha$. Using remark 6.3.4.2, we see that

$$2(m\rho_\alpha-m-1)+\dim C_1\leq 2(mc\rho_\alpha-1)+1=2c(m\rho_\alpha)-1$$

so if T_α is specialised to T^{ρ_α} for all $\alpha\in\mathscr{A}\setminus\mathscr{A}_D$, we are in the situation of lemma 6.3.4.1. Note that here we in fact could have taken a smaller c, namely $c=\max_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}(1-\rho_\alpha^{-1})$. The definition we chose will be important in the next case.

The case $\#A>2$ Then Z_A can be rewritten as:

$$Z_{v,A}(\mathbf{T},\xi)\ =\ \sum_{\mathbf{m}\in\mathbf{N}_{>0}^A}\prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha}T_\alpha)^{m_\alpha}\int_{\mathrm{ord}x_\alpha=m_\alpha}\mathrm{e}(f_\xi(x,y))\mathrm{d}x\mathrm{d}y$$

We proceed as in the previous case. The integral in each term is over the constructible subset of $\Omega(A)$ of points satisfying $\mathrm{ord}(x_\alpha)=m_\alpha$, which has motivic volume

$$[D_A^\circ]\prod_{\alpha\in A}\left(\mathbf{L}^{-m_\alpha}(1-\mathbf{L}^{-1})\right)\mathbf{L}^{-n+\#A}.$$

The constructible set D_A° is given as an open subset of an intersection of the $|A|$ divisors D_α, $\alpha\in A$, and therefore is of dimension at most $n-|A|$. We may conclude that the dimension of the element of $\mathscr{E}xp\mathscr{M}_k$ given by the integral is at most $-\sum_{\alpha\in A}m_\alpha\leq -2$. Thus, all terms are ρ'-admissible, and it remains to give a stronger bound for all terms of sufficiently large degree. Using remark 6.3.4.2 again, with c given by (6.10), we have

$$\begin{aligned}
2\sum_{\alpha\in A}m_\alpha(\rho_\alpha-1)+\dim C_1\ &\leq\ 2\sum_{\alpha\in A}m_\alpha\left(\rho_\alpha-\frac{1}{2}\right)-\sum_{\alpha\in A}m_\alpha+1\\
&\leq\ 2c\sum_{\alpha\in A}m_\alpha\rho_\alpha-1.
\end{aligned}$$

Thus, here again we are in the situation of lemma 6.3.4.1.

Putting everything together We decomposed the zeta function at v in the following manner:

$$\begin{aligned} Z_v(\mathbf{T},\xi) &= 1+\sum_{\alpha\in\mathscr{A}_0} Z_{v,\alpha}(\mathbf{T},\xi)+\sum_{\alpha\in\mathscr{A}_1} Z_{v,\alpha}(\mathbf{T},\xi)+\sum_{\#A>2} Z_{v,A}(\mathbf{T},\xi) \\ &= 1+\mathbf{L}^{-n}\sum_{\alpha\in\mathscr{A}_0^D(\xi)} [D_\alpha^\circ\setminus E]\frac{\mathbf{L}^{\rho_\alpha}T_\alpha}{1-\mathbf{L}^{\rho_\alpha-1}T_\alpha}+\text{terms satisfying the bounds of lemma 6.3.4.1 for } c \text{ given by (6.10)} \end{aligned}$$

Since $\rho_\alpha - 1 \leq c\rho_\alpha$, multiplication by $\mathbf{L}^{\rho_\alpha-1}T^{\rho_\alpha}$ preserves the bounds of lemma 6.3.4.1. Thus, multiplying everything by

$$\prod_{\alpha\in\mathscr{A}_0^D(\xi)}(1-\mathbf{L}^{\rho_\alpha-1}T_\alpha)$$

and keeping only potentially non-admissible terms, we get:

$$\begin{aligned} \prod_{\alpha\in\mathscr{A}_0^D(\xi)}(1-\mathbf{L}^{\rho_\alpha-1}T_\alpha)Z_v(\mathbf{T},\xi) &= 1-\sum_{\alpha\in\mathscr{A}_0^D(\xi)}\mathbf{L}^{\rho_\alpha-1}T_\alpha & (6.11)\\ &+\mathbf{L}^{1-n}\sum_{\alpha\in\mathscr{A}_0^D(\xi)}[D_\alpha^\circ\setminus E](\mathbf{L}^{\rho_\alpha-1}T_\alpha) & (6.12)\\ &+\text{terms satisfying the bounds of lemma 6.3.4.1 for } c \text{ given by (6.10)} & (6.13) \end{aligned}$$

Proposition 6.3.5.3. *There is a real number $\delta > 0$ such that the product*

$$\prod_{v\in C_0}\left(Z_v((T^{\rho'_\alpha})_\alpha,\xi)\prod_{\alpha\in\mathscr{A}_0^D(\xi)}(1-\mathbf{L}^{\rho_\alpha-1}T^{\rho_\alpha})\right)$$

converges for $|T| < \mathbf{L}^{-1+\delta}$.

Proof. The above calculations give explicit formulas for the main terms of all $v \in C_1$. Using lemma 6.3.5.1, it suffices to check convergence for these places, by showing that they satisfy the conditions of lemma 6.3.4.1 for c given by (6.10) and some sufficiently large M. According to the above, it suffices to bound the weights of the terms in (6.11) and (6.12), which is done exactly as in the proof of proposition 6.3.5.2. The conclusion follows. □

6.3.6 Places in S

Let v be a place in $S = C\setminus C_0$. In this case, for any character ξ, $Z(\mathbf{T},\xi)$ is given by lemma 6.3.3.4, taking $h = 1$:

$$Z(\mathbf{T},\xi)=\sum_{\substack{A\subset\mathscr{A}\\ \beta\in\mathscr{B}_0}}\prod_{\alpha\in\mathscr{A}}T_\alpha^{e_{\alpha,\beta}}\mathbf{L}^{\rho_\beta}\int_{\Omega(A,\beta)}\prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha}T_\alpha)^{\mathrm{ord}(x_\alpha)}\mathrm{e}(\langle x,\xi\rangle)\mathrm{d}x.$$

In this section we are going to use Clemens complexes: see [ChL], section 2.3, for the definition.

6.3.6.1 Trivial character

Assume $\xi = 0$, so that, as in section 6.4 of [ChL],

$$Z_v(\mathbf{T},0) = \sum_{\substack{A\subset\mathscr{A}\\ \beta\in\mathscr{B}_{1,v}}} \left(\prod_{\alpha\in\mathscr{A}} T_\alpha^{e_{\alpha,\beta}}\mathbf{L}^{\rho_\beta}\right)[\Delta(A,\beta)](1-\mathbf{L}^{-1})^{|A|}\mathbf{L}^{-n+|A|}\prod_{\alpha\in A}\frac{\mathbf{L}^{\rho_\alpha-1}T_\alpha}{1-\mathbf{L}^{\rho_\alpha-1}T_\alpha}. \quad (6.14)$$

This case has essentially been treated in section 6.4 and proposition 4.3.2 of [ChL]. The only modification we have to make is to adapt to the case where D is not all of $X\setminus G$, so that the $\alpha\notin\mathscr{A}_D$ do not contribute to the pole. For this, we proceed as in [ChL], fixing for every pair (A,β) a maximal subset A_0 of $\mathscr{A}_D$ such that $A\cap\mathscr{A}_D\subset A_0$ and $\Delta(A_0,\beta)\neq\emptyset$. We collect the terms in equation 6.14 corresponding to pairs (A,β) associated with any given A_0:

$$Z_v(\mathbf{T},0) = \sum_{A_0\in\mathrm{Cl}_v^{\mathrm{an,max}}(X,D)}\ \sum_{\substack{A\subset\mathscr{A},\beta\in\mathscr{B}_{1,v}\\ (A,\beta)\mapsto A_0}} \left(\prod_{\alpha\in\mathscr{A}} T_\alpha^{e_{\alpha,\beta}}\mathbf{L}^{\rho_\beta}\right)[\Delta(A,\beta)]$$

$$\times(1-\mathbf{L}^{-1})^{|A|}\mathbf{L}^{-n+|A|}\prod_{\alpha\in A\setminus\mathscr{A}_D}\frac{\mathbf{L}^{\rho_\alpha-1}T_\alpha}{1-\mathbf{L}^{\rho_\alpha-1}T_\alpha}\prod_{\alpha\in A\cap\mathscr{A}_D}\frac{\mathbf{L}^{\rho_\alpha-1}T_\alpha}{1-\mathbf{L}^{\rho_\alpha-1}T_\alpha}.$$

Thus, there exists a family of Laurent polynomials $(P_{v,A})$ with coefficients in $\mathscr{M}_k$ indexed by the set of maximal faces A of $\mathrm{Cl}_v^{\mathrm{an}}(X,D)$ such that

$$Z_v(\mathbf{T},0) = \sum_{A\in\mathrm{Cl}_v^{\mathrm{an,max}}(X,D)}\frac{P_{v,A}(\mathbf{T})}{\prod_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}(1-\mathbf{L}^{\rho_\alpha-1}T_\alpha)}\prod_{\alpha\in A}\frac{1}{1-\mathbf{L}^{\rho_\alpha-1}T_\alpha},$$

with $P_{A,v}(\mathbf{T})$ congruent to some non-zero effective element of $\mathscr{M}_k$ modulo the ideal generated by the polynomials $1-\mathbf{L}^{\rho_\alpha-1}T_\alpha$ for $\alpha\in\mathscr{A}$. Putting $T_\alpha = T^{\rho'_\alpha}$ for every $\alpha\in\mathscr{A}$, we may deduce from this that there is a family of Laurent series $(F_{v,A})$ with coefficients in $\mathscr{M}_k$, indexed by the set of maximal faces A of $\mathrm{Cl}_v^{\mathrm{an}}(X,D)$, converging for $|T| < \mathbf{L}^{-1+\min_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}\frac{1}{\rho_\alpha}}$, taking a non-zero effective value in $\widehat{\mathscr{M}_k}$ at $\mathbf{L}^{-1}$, and such that

$$Z_v(\mathbf{T},0) = \sum_{A\in\mathrm{Cl}_v^{\mathrm{an,max}}(X,D)} F_{v,A}(T)\prod_{\alpha\in A}\frac{1}{1-(\mathbf{L}T)^{\rho_\alpha-1}}.$$

In particular, setting $d_v = 1+\dim\mathrm{Cl}_v^{\mathrm{an}}(X,D)$, we may deduce the following result:

Proposition 6.3.6.1. *There is a real number $\delta>0$ such that for every non-zero common multiple a of the integers $\rho_\alpha-1$, $\alpha\in\mathscr{A}_D$, the Laurent series $(1-\mathbf{L}^aT^a)^{d_v}(Z_v(T,0))$ converges for $|T|<\mathbf{L}^{-1+\delta}$ and takes a non-zero effective value at $\mathbf{L}^{-1}$.*

Remark 6.3.6.2. According to the above calculations, one may take $\delta = \min_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}\frac{1}{\rho_\alpha}$.

6.3.6.2 Non-trivial characters

For this case we refer to proposition 4.3.4 and section 6.5 in [ChL]. Recall from 6.2.6.4 that we restricted the summation domain to a finite-dimensional k-vector space V. For any v, denote by $\mathfrak{a} : V_{F_v} \to G_{F_v}$ the corresponding F_v-linear inclusion. For every v, we then have a Laurent series $Z_v(T, \mathfrak{a}(\cdot)) \in \mathscr{E}xp\mathscr{M}_V[[T]][T^{-1}]$, and we ask for its convergence properties. Section 6.5 of [ChL] gives an argument, based on ideas from section 3.4 of [CLT12], to describe the convergence properties of $Z_v(T, \mathfrak{a}(\cdot))$, uniformly on the strata of a constructible partition of V. First of all, Chambert-Loir and Loeser show lemma 6.5.1, which allows them to resolve indeterminacies of the function f_ξ uniformly on each stratum P of such a partition. Then they apply change of variables to show that one can compute the integral giving $Z_v(\mathbf{T}, \xi)$ on the fibre above ξ of such a resolution. On the other hand, they have a general result, proposition 5.3.1, giving a formula for motivic Igusa zeta functions without indeterminacies. It states that the poles of such an Igusa zeta function are controlled by the set of maximal faces of the subcomplex $\mathrm{Cl}_v^{\mathrm{an}}(X, D)_\xi$ of the complex $\mathrm{Cl}_v^{\mathrm{an}}(X, D)$ where we only keep vertices $\alpha \in \mathscr{A}_D$ such that $d_\alpha(\xi) = 0$. This argument works in exactly the same way in our setting, the only difference being that in Chambert-Loir and Loeser's paper, the set $\mathscr{A}_D$ is equal to $\mathscr{A}$, which is not necessarily the case here. Therefore, proposition 4.3.4 from [ChL] adapts to our setting in the following form:

Proposition 6.3.6.3. *Let $v \in S$ and let $d_v(\xi) = 1 + \dim \mathrm{Cl}_v^{\mathrm{an}}(X, D)_\xi$. There exists a constructible partition $(U_{v,i})$ of $V \setminus \{0\}$ on each stratum of which $\xi \mapsto d_v(\xi)$ is constant equal to some integer $d_{v,i}$, and, for every i, an element $P_{v,i} \in \mathscr{E}xp\mathscr{M}_{U_{v,i}}[\mathbf{T}, \mathbf{T}^{-1}]$ and finite families $(a_{v,i,j})$, $(b_{v,i,j})$ where $a_{v,i,j} \in \mathbf{N}$, $b_{v,i,j} \in \mathbf{N}^{\mathscr{A}}$, such that the restriction of $Z_v(\mathbf{T}, \mathfrak{a}(\cdot))$ to $U_{v,i}$ equals*

$$\prod_j (1 - \mathbf{L}^{a_{v,i,j}} \mathbf{T}^{b_{v,i,j}})^{-1} P_{v,i}(\mathbf{T}; \cdot).$$

Moreover, there exist integers $a_{v,i} \geq 1$ and a real number $\delta > 0$ such that the restriction to $U_{v,i}$ of $(1 - (\mathbf{L}T)^{a_{v,i}})^{d_{v,i}} Z_v(T, \mathfrak{a}(\cdot))$ converges for $|T| < \mathbf{L}^{-1+\delta}$.

6.4 Proof of theorem 1 and corollary 2

According to (6.2), we may write the multivariate zeta function $Z(\mathbf{T})$ in the form

$$Z(\mathbf{T}) = \mathbf{L}^{(1-g)n} Z(\mathbf{T}, 0) + \mathbf{L}^{(1-g)n} \sum_{\xi \in V \setminus \{0\}} Z(\mathbf{T}, \xi)$$

where V is a finite-dimensional k-vector space contained in $k(C)^n$ and g is the genus of the smooth projective curve C. We are interested in the convergence properties of

$$Z(T) = \mathbf{L}^{(1-g)n} Z(T, 0) + \mathbf{L}^{(1-g)n} \sum_{\xi \in V \setminus \{0\}} Z(T, \xi)$$

where for all ξ, we have $Z(T, \xi) = Z((T^{\rho'_\alpha})_\alpha, \xi)$. Recall we still assume $k = \mathbf{C}$.

6.4.1 The function $Z(T,0)$

In section 6.3.5.1 we showed the convergence beyond $\mathbf{L}^{-1}$ of the product of local factors $Z_v(T,0)$ over $v\in C_0$ after multiplication by the polynomial

$$\prod_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}(1-\mathbf{L}^{\rho_\alpha-1}T^{\rho_\alpha})$$

at each place v. To derive a meromorphic continuation for $\prod_{v\in C_0}Z_v(T,0)$, it therefore suffices to describe the convergence of the product

$$\prod_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}\prod_{v\in C_0}(1-\mathbf{L}^{\rho_\alpha-1}T^{\rho_\alpha})^{-1}.$$

In the latter, we recognise for each $\alpha\in\mathscr{A}\setminus\mathscr{A}_D$ the Euler product decomposition of the motivic zeta function of C_0 at $\mathbf{L}^{\rho_\alpha-1}T^{\rho_\alpha}$.

Denote by $Z_C(T)=\sum_{m\geq 0}[S^mC]T^m\in\mathscr{M}_k[[T]]$ the motivic zeta function of the smooth projective curve C. Since $k=\mathbf{C}$ is algebraically closed, we have, by theorem 1.1.9 in [Kapr], that

$$Z_C(T)=\frac{P_C(T)}{(1-T)(1-\mathbf{L}T)}$$

where $P_C(T)\in\mathscr{M}_k[T]$ is a polynomial of degree $2g$ such that $P_C(\mathbf{L}^{-1})=\mathbf{L}^{-g}[J(C)]$, with $J(C)$ being the Jacobian of C. Moreover, the zeta function $Z_{C_0}(T)$ of the open dense subset $C_0\subset C$ is given by

$$Z_{C_0}(T)=\prod_{v\in C_0}(1-T)^{-1}=Z_C(T)\prod_{v\in C\setminus C_0}(1-T).$$

We have

$$\begin{aligned}\prod_{v\in C}\prod_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}(1-\mathbf{L}^{\rho_\alpha-1}T^{\rho_\alpha})^{-1} &= \prod_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}Z_C(\mathbf{L}^{\rho_\alpha-1}T^{\rho_\alpha})\\ &= \prod_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}\frac{P_C(\mathbf{L}^{\rho_\alpha-1}T^{\rho_\alpha})}{(1-\mathbf{L}^{\rho_\alpha-1}T^{\rho_\alpha})(1-(\mathbf{L}T)^{\rho_\alpha})}.\end{aligned}$$

Since $1-\mathbf{L}^{\rho_\alpha-1}T^{\rho_\alpha}$ evaluated at $\mathbf{L}^{-1}$ is $1-\mathbf{L}^{-1}=\mathbf{L}^{-1}[\mathbf{A}^1\setminus\{0\}]$ which is effective, we may conclude that the product over places in C_0

$$\prod_{v\in C_0}\prod_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}(1-\mathbf{L}^{\rho_\alpha-1}T^{\rho_\alpha})^{-1}$$

is of the form

$$\frac{F(T)}{\prod_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}(1-(\mathbf{L}T)^{\rho_\alpha})}$$

where $F\in\mathscr{M}_k[[T]]$ is a rational power series converging for $|T|<\mathbf{L}^{-1+\delta}$ for some $\delta>0$ (more precisely, any $\delta\leq\min_{\alpha\in\mathscr{A}\setminus\mathscr{A}_D}\frac{1}{\rho_\alpha}$ works) and taking a non-zero effective value at $\mathbf{L}^{-1}$.

Thus, by proposition 6.3.5.2, for a common multiple a of the $\rho_\alpha, \alpha \in \mathscr{A} \setminus \mathscr{A}_D$, we may write the product $\prod_{v\in C_0} Z(T,0)$ in the form

$$\frac{F_1(T)}{(1-(\mathbf{L}T)^a)^{|\mathscr{A}\setminus\mathscr{A}_D|}}$$

for $F_1(T) \in \mathscr{M}_k[[T]]$ converging for $|T| < \mathbf{L}^{-1+\delta}$ for some δ and taking a non-zero effective value at $\mathbf{L}^{-1}$. Thus, the product $\prod_{v\in C_0} Z_v(T,0)$ has a pole at $\mathbf{L}^{-1}$ of order $|\mathscr{A}\setminus\mathscr{A}_D| = \text{rk Pic}U$, and a meromorphic continuation beyond $\mathbf{L}^{-1}$.

Combining this with the result of section 6.3.6.1 we see each place $v \in C \setminus C_0$ gives an additional contribution to the pole at $\mathbf{L}^{-1}$, of order exactly $d_v = 1 + \mathrm{Cl}_v^{\mathrm{an}}(X,D)$. Finally, we can conclude that there is a real number $\delta > 0$, a Laurent series $G_0(T) \in \mathscr{M}_k[[T]][T^{-1}]$ converging for $|T| < \mathbf{L}^{-1+\delta}$, and taking a non-zero effective value at $\mathbf{L}^{-1}$, and an integer $a \geq 1$ (we may take a to be any common multiple of the $\rho'_\alpha,\ \alpha \in \mathscr{A}$) such that the product $Z(T,0) = \prod_{v\in C} Z(T,0)$ may be written in the form

$$Z(T,0) = \frac{G_0(T)}{(1-(\mathbf{L}T)^a)^r},$$

where

$$r = \text{rk Pic}(U) + \sum_{v\in C\setminus C_0} (1 + \dim \mathrm{Cl}_v^{\mathrm{an}}(X,D)).$$

Example 6.4.1.1. 1. Assume $U = G$, so that $\mathrm{Pic}(U) = 0$. One then recovers the result from [ChL].

2. Assume $U = X$. Then one may take $C_0 = C$, and the order of the pole is exactly rk $\mathrm{Pic}(X)$.

6.4.2 The function $Z(T,\xi)$

We proceed as in section 6.4.1: according to proposition 6.3.5.3, for every $\xi \in V \setminus \{0\}$ the product

$$\left(\prod_{v\in C_0} Z_v(T,\xi)\right) \prod_{\alpha\in\mathscr{A}_0^D(\xi)} Z_C(\mathbf{L}^{\rho_\alpha-1}T^{\rho_\alpha}) \in \mathscr{E}xp\mathscr{M}_k[[T]][T^{-1}]$$

converges for $|T| < \mathbf{L}^{-1+\delta}$ for some $\delta > 0$. We may apply this to the generic point of $V\setminus\{0\}$ and use spreading-out and induction on the dimension to show that there exists a finite constructible partition of $V\setminus\{0\}$ on which the functions $\xi \mapsto d_\alpha(\xi)$ are constant, and such that this convergence holds uniformly in ξ on each piece of the partition. Proposition 6.3.6.3 on the other hand tells us that the order of the pole of $\prod_{v\in C\setminus C_0} Z(T,\xi)$ at $\mathbf{L}^{-1}$ is at most $\sum_v d'_v$ where $d'_v = 1 + \dim \mathrm{Cl}_v^{\mathrm{an}}(X,D)_0$, again uniformly on the pieces of a constructible partition of $V\setminus\{0\}$. Using a partition of $V\setminus\{0\}$ refining the aforementioned two partitions, we may conclude that for any stratum P of this partition, the order of the pole of

$$\mathbf{L}^{(1-g)n} \sum_{\xi\in\mathfrak{a}(P)} Z(T,\xi)$$

is at most

$$|\mathscr{A}_0^D(\xi))| + \sum_{v\in C\setminus C_0} d_v(\xi)$$

for any $\xi \in \mathfrak{a}(P)$ (recall $\xi \mapsto d_\alpha(\xi)$, and therefore also $\xi \mapsto |\mathscr{A}_0^D(\xi))|$ and $\xi \mapsto d_v(\xi)$ are constant on P). Lemma 3.5.4 in [CLT12] shows that this is strictly less than the order r of the pole of $Z(T,0)$ at $\mathbf{L}^{-1}$.

6.4.3 Conclusion of the proof of theorem 1

Taking a to be the least common multiple of the integers ρ'_α, $\alpha \in \mathscr{A}$ and of the integers $a_{v,i}$ appearing in proposition 6.3.6.3, we have shown that $(1-(\mathbf{L}T)^a)^r Z(T)$ converges for $T = \mathbf{L}^{-1}$, and takes a non-zero effective value at $\mathbf{L}^{-1}$, which concludes the proof of theorem 1.

6.4.4 Proof of corollary 2

Applying lemma 4.7.3.1, we get corollary 2 in the case where $k = \mathbf{C}$. We now explain how we may deduce from this the general case.

All the geometric data in our counting problem involves a finite number of equations over the field k: we may therefore assume that everything is defined over a finitely generated subfield of $\mathbf{C}$. Moreover, the assumption $\mathscr{U}(\mathcal{O}_v) \neq \varnothing$ for all $v \in C_0$ may be reformulated more geometrically by saying that the volume of the arc space $\mathscr{L}(\mathscr{U}_v)$ at the place v should be non-zero. With the notations of section 6.3.3, this volume may be expressed by the formula:

$$\begin{aligned}\mathrm{vol}(\mathscr{L}(\mathscr{U}_v)) &= \sum_{\substack{A\subset \mathscr{A}\setminus\mathscr{A}_D\\ \beta\in\mathscr{B}_{0,v}}} \mathrm{vol}(\Omega(A,\beta))\\ &= \mathbf{L}^{-n}\sum_{\substack{A\subset \mathscr{A}\setminus\mathscr{A}_D\\ \beta\in\mathscr{B}_{0,v}}} [\Delta(A,\beta)].\end{aligned}$$

Thus, at least one of the sets $\Delta(A,\beta)$ for $A\subset \mathscr{A}\setminus\mathscr{A}_D$, $\beta\in\mathscr{B}_{0,v}$ has a k-point. Since it is defined over $\mathbf{C}$, it also has a $\mathbf{C}$-point.

Thus, we have shown that without loss of generality, even if the original problem was stated over some algebraically closed field k of characteristic zero, we may in fact assume everything is defined over $\mathbf{C}$, and apply corollary 2 in this setting. By functoriality of Hilbert schemes, we may then deduce corollary 2 over k.

Bibliography

[BS] P. Balmer, M. Schlichting, Idempotent completion of triangulated categories, Journal of Alg. 236 (2001), 819–834.

[BM] V. Batyrev, Yu. Manin, Sur le nombre des points rationnels de hauteur bornée des variétés algébriques, Math. Ann. 286 (1990), 27-43

[BT95] V. Batyrev, Yu. Tschinkel, Rational points of bounded height on compactifications of anisotropic tori, Internat. Math. Res. Notices 12 (1995), 591-635

[BT96] V. Batyrev, Yu. Tschinkel, Rational points on some Fano cubic bundles, C.R. Acad. Sci. Paris Sér. I Math. 323 (1996), 41-46

[BT98] V. Batyrev, Yu. Tschinkel, Manin's conjecture for toric varieties, Journ. of Alg. Geom., 7, no. 1, 15-53, (1998)

[BBD] A. A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers, in *Analysis and topology on singular spaces, 1 (Luminy 1981)*, 5-171, Astérisque **100**, Soc. Math. France, Paris, 1982

[Bit] F. Bittner, The universal Euler characteristic for varieties of characteristic zero, Compos. Math. **140** (2004), no. 4, 1011–1032.

[Bor] L. Borisov, Class of the affine line is a zero divisor in the Grothendieck ring, arXiv:1412.6194

[Bou02] D. Bourqui, Fonction zêta des hauteurs des surfaces de Hirzebruch dans le cas fonctionnel, J. of Number Theory **94**, 2002, p. 343–358

[Bou03] D. Bourqui, Fonction zêta des hauteurs des variétés toriques déployées dans le cas fonctionnel, J. Reine Angew. Math. **562**, 2003, p. 171-199.

[Bou09] D. Bourqui, Produit eulérien motivique et courbes rationnelles sur les variétés toriques Comp. Math. **45**, No. 6, 2009, p. 1360-1400

[Bou10] D. Bourqui, Fonctions L d'Artin et nombre de Tamagawa motiviques, New York J. Math. 16 (2010), 179–233

[Bou11] D. Bourqui, Fonctions zêta des hauteurs des variétés toriques non déployées, Memoirs of the AMS **211**, 2011.

[BBDJS] C. Brav, V. Bussi, D. Dupont, D. Joyce, B. Szendrői, Symmetries and stabilization for sheaves of vanishing cycles, with an appendix by J. Schürmann, J. Singul. **11** (2015), 85–151

[Bry] J.-L. Brylinski, Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques, *Géométrie et analyse microlocales*, Astérisque No. 140-141 (1986), 3–134.

[CMSSY] S. E. Cappell, L. Maxim, J. Schürmann, J. L. Shaneson, S. Yokura, Characteristic classes of symmetric products of complex quasi-projective varieties, J. Reine Angew. Math. 728 (2017), 35–63

[CLT02] A. Chambert-Loir, Yu. Tschinkel, On the distribution of points of bounded height on equivariant compactifications of vector groups, Invent. Math. **148** (2002), 421-452

[CLT12] A. Chambert-Loir, Yu. Tschinkel, Integral points of bounded height on partial equivariant compactifications of vector groups, Duke Math. Journal **161** (15), p. 2799–2836 (2012).

[ChL] A. Chambert-Loir, F. Loeser, Motivic height zeta functions, Amer. J. Math. **138** (1), 1-59 (2016)

[CNS] A. Chambert-Loir, J. Nicaise, J. Sebag, *Motivic integration*, Prog. in Math., Birkhäuser, to appear.

[ClL08] R. Cluckers, F. Loeser, Constructible motivic functions and motivic integration, Invent. Math., **173** (1), pp. 23–121 (2008)

[ClL10] R. Cluckers, F. Loeser, Constructible exponential functions, motivic Fourier transform and transfer principle, Annals of Math. **171**, no. 2, 1011-1065 (2010)

[CS] P. Colmez, J.-P. Serre (editors) (2001), *Correspondance Grothendieck- Serre*, Documents Mathématiques (Paris), 2, Société Mathématique de France, Paris.

[Del73] P. Deligne, Comparaison avec la théorie transcendante, in SGA 7, Groupes de Monodromie en Géométrie Algébrique, Part II, Lect. Notes in Math. **340**, pp.116-164, Springer, Berlin, 1973.

[Del77] P. Deligne, Théorèmes de finitude en cohomologie ℓ-adique, in SGA 4 1/2, Cohomologie Etale, Lect. Notes in Math. **569**, pp.233-251, Springer, 1977

[Del] P. Deligne, Théorie de Hodge II, Publ. Math. IHÉS **40** (1971) , p. 5-57

[DL98] J. Denef, F. Loeser, Motivic Igusa zeta functions , J. Algebraic Geom. **7** (1998), no. 3, 505-537.

[DL99a] J. Denef, F. Loeser, Germs of arcs on singular algebraic varieties and motivic integration, Invent. Math., 135 (1), pp. 201–232 (1999)

[DL99b] J. Denef, F. Loeser, Motivic exponential integrals and a motivic Thom-Sebastiani theorem, Duke Math. Journal **99** (1999), no. 2, 285-309

[DL01] J. Denef, F. Loeser, Geometry on arc spaces of algebraic varieties, European Congress of Mathematics, Vol. I (Barcelona 2000), Progr. Math. vol. 201, Birkhäuser, Basel, 2001, pp. 327-348

[DL02] J. Denef, F. Loeser, Lefschetz numbers of iterates of the monodromy and truncated arcs, Topology 41, 1031-1040 (2002)

[Ekedahl] T. Ekedahl, The Grothendieck group of algebraic stacks, arXiv:0903.3143v2, 2009.

[FMT] J. Franke, Yu. I. Manin, Yu. Tschinkel, Rational points of bounded height on Fano varieties, Invent. Math. 95 (1989), 421-435

[GLM] G. Guibert, F. Loeser, M. Merle, Iterated vanishing cycles, convolution, and a motivic analogue of a conjecture of Steenbrink, Duke Math. Journal 132 (3), p. 409-457 (2006).

[GZLM] S. M. Gusein-Zade, I. Luengo, A Melle, A power structure over the Grothendieck ring of varieties, Mathematical Research Letters 11, 49-57 (2004)

[HK06] E. Hrushovski, D.Kazhdan, Integration in valued fields, in *Algebraic geometry and number theory*, Progr. Math. 253, pp. 261–405, Birkhäuser Boston, Boston, MA (2006)

[HK09] E. Hrushovski, D. Kazhdan, Motivic Poisson summation, Mosc. Math. Journal 9 (3), p.569-623 (2009)

[HT] B. Hassett, Yu. Tschinkel, Geometry of equivariant compactifications of $\mathbf{G}_a^n$, Internat. Math. Res. Notices **22** (1999), p. 1211–1230.

[Kapr] M. Kapranov, The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups, arXiv:math/0001005 (2000)

[KS] M. Kashiwara and P. Schapira, Sheaves on Manifolds, Grund. der math. Wiss. 292, Springer, 1990

[Kol] J. Kollár, *Lectures on resolution of singularities*, Annals of Mathematics Studies, vol. **166**, Princeton University Press, Princeton, NJ, 2007

[LY] K. F. Lai and K. M. Yeung, Rational points in flag varieties over function fields, J. Number Theory 95 (2002), 142–149.

[LL] M. Larsen, V.A. Lunts, Motivic measures and stable birational geometry, Mosc. Math. J., 3 (1), pp. 85–95 (2003)

[Loo] E. Looijenga, Motivic measures, Astérisque **276** (2002), 267-297, Séminaire Bourbaki 1999/2000, no 874

[LS16a] V.A. Lunts, O.M. Schnürer, Matrix factorizations and motivic measures, J. Noncommut. Geom. **10** (2016), no. 3, 981–1042.

[LS16b] V.A. Lunts, O.M. Schnürer, Motivic vanishing cycles as a motivic measure, Pure Appl. Math. Q. **12** (2016), no. 1, 33–74.

[LW] S. Lang and A. Weil, Number of points of varieties in finite fields, Amer. J. Math. **76** (1954), 819–827.

[Mas] D. Massey, The Sebastiani-Thom isomorphism in the derived category, Comp. Math. **125**, 353-362, 2001

[MSS] L. Maxim, M. Saito, J. Schürmann, Symmetric products of mixed Hodge modules, J. Math. Pures Appl. **96** (2011), no. 5, 462–483.

[Milne] J. S. Milne, Abelian varieties, in *Arithmetic Geometry*, G. Cornell, J. H. Silverman eds., Springer, 1986

[MFK] D. Mumford, J. Fogarty, F. Kirwan, *Geometric invariant theory*, Third edition, Ergebnisse der Mathematik und ihrer Grenzgebiete (2), **34**, Springer-Verlag, Berlin, 1994

[Mus] M. Mustață, *Zeta functions in algebraic geometry*, Lecture notes available at `http://www.math.lsa.umich.edu/~mmustata/zeta_book.pdf`

[Pey95] E. Peyre, Hauteurs et mesures de Tamagawa sur les variétés de Fano, Duke Math. J. 79 (1995), No 1, 101-218

[Pey12] E. Peyre, Points de hauteur bornée sur les variétés de drapeaux en caractéristique finie, Acta Arith. 152 (2012), 185—216

[PS] C.A.M. Peters, J.H.M. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge: A Series of Modern Surveys in Mathematics **52**, Springer-Verlag, Berlin, 2008.

[S88] M. Saito, Modules de Hodge polarisables, Publ. Res. Inst. Math. Sci. **24**, no. 6, 849-995 (1990)

[S89] M. Saito, Introduction to mixed Hodge modules, pp. 145–162 in Actes du Colloque de Théorie de Hodge (Luminy, 1987), Astérisque Vol. 179–180 (1989).

[S90] M. Saito, Mixed Hodge Modules, Publ. Res. Inst. Math. Sci. **26**, no. 2, 221-333 (1990)

[S] M. Saito, Thom-Sebastiani Theorem for Hodge Modules, preprint

[Sch] S. H. Schanuel, Heights in number fields, Bull. Soc. Math. France **107** (1979), 433–449.

[Schn] Ch. Schnell, An overview of Morihiko Saito's theory of mixed Hodge modules, `https://www.math.stonybrook.edu/~cschnell/pdf/notes/sanya.pdf`

[ST] M. Sebastiani, R. Thom, Un résultat sur la monodromie, Invent. Math. **13** (1971), 90-96.

[ShTTB03] J. Shalika, Yu. Tschinkel, R. Takloo-Bighash, Rational points on compactifications of semi-simple groups of rank 1, *Arithmetic of higher-dimensional algebraic varieties*, 205-233, Progress in Math. 226, Birkhäuser, 2003.

[ShTTB07] J. Shalika, Yu. Tschinkel, R. Takloo-Bighash, Rational points on compactifications of semi-simple groups, Journ. of the AMS, 20, 1135-1186, 2007.

[ShT04] J. Shalika, Yu. Tschinkel, Height zeta functions of equivariant compactifications of the Heisenberg group, *Contributions to Automorphic Forms, Geometry, and Number Theory*, (H. Hida, D. Ramakrishnan, F. Shahidi eds.), 743-771, Johns Hopkins University Press, 2004.

[ShT16] J. Shalika, Yu. Tschinkel, Height zeta functions of equivariant compactifications of unipotent groups, Comm. Pure and Applied Math., 69, no. 4, 693–733, 2016.

[TTB] R. Takloo-Bighash, S. Tanimoto, Distribution of rational points of bounded height on equivariant compactifications of PGL_2 I, Res. number theory (2016) 2: 6.

[TT] S. Tanimoto, Yu. Tschinkel, Height zeta functions of equivariant compactifications of semi-direct products of algebraic groups, *Zeta functions in algebra and geometry*, 119–157, Contemp. Math., 566, Amer. Math. Soc., Providence, RI, 2012.

[Verdier] J.-L. Verdier, Stratifications de Whitney et théorème de Bertini-Sard, Invent. Math. 36 (1976), 295–312.

Index

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Titre : Produits eulériens motiviques

Mots-clefs : Anneaux de Grothendieck des variétés, hauteurs, compactifications équivariantes d'espaces affines, problème de Manin, théorie de Hodge, cycles évanescents.

Résumé : L'objectif de cette thèse est l'étude de la fonction zêta des hauteurs motivique associée à un problème de comptage de courbes sur les compactifications équivariantes d'espaces affines, résolvant au chapitre 6 l'analogue motivique de la conjecture de Manin pour celles-ci.
La fonction zêta des hauteurs provenant du problème de comptage considéré est récrite convenablement à l'aide d'une formule de Poisson motivique démontrée au cinquième chapitre, qui généralise celle de Hrushovski-Kazhdan. Chaque terme est alors décomposé sous la forme d'un produit eulérien motivique, dont la définition et les propriétés sont établies au chapitre 3. La convergence de ces produits eulériens doit être comprise pour une topologie des poids que nous introduisons au quatrième chapitre et qui repose d'une part sur la théorie des modules de Hodge de Saito, et d'autre part sur une mesure motivique sur l'anneau de Grothendieck des variétés avec exponentielles, construite dans le chapitre 2 à l'aide de la notion de cycles évanescents motiviques.
On en déduit ainsi une description de l'asymptotique d'une proportion positive des coefficients du polynôme de Hodge-Deligne des espaces de modules des courbes sur la compactification équivariante donnée, lorsque le degré tend vers l'infini.

Title : Motivic Euler products

Key words : Grothendieck rings of varieties, heights, equivariant compactifications of vector groups, Manin's problem, Hodge theory, vanishing cycles.

Abstract : The goal of this thesis is the study of the motivic height zeta function associated to the problem of counting curves on equivariant compactifications of vector groups, solving in chapter 6 the motivic analogue of Manin's conjecture for such varieties.
The motivic height zeta function coming from this counting problem is rewritten in a convenient way using a Poisson summation formula proved in chapter 5, and which generalises Hrushovski and Kazhdan's motivic Poisson formula. Each term is then expressed as a motivic Euler product, the definition and properties of the latter being established in chapter 3. The convergence of these Euler products must be understood for a weight topology which we introduce in the fourth chapter and which relies both on Saito's theory of mixed Hodge modules and on a motivic measure on the Grothendieck ring of varieties with exponentials, constructed in chapter 2 using the notion of motivic vanishing cycles.
We deduce from this a description of the asymptotic of a positive proportion of the coefficients of the Hodge-Deligne polynomial of the moduli spaces of curves on the given equivariant compactification, when the degree goes to infinity.

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