ON A QUASILINEAR ZAKHAROV SYSTEM DESCRIBING LASER-PLASMA INTERACTIONS

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Abstract. In this paper, starting from the bi-fluid Euler-Maxwell system, we derive a complete set of Zakharov-type equations describing laser-plasma interactions. This system involves a quasilinear part which is not hyperbolic and exhibits some elliptic zones. This difficulty is overcome by making a change of unknowns that are strongly related to the dispersive part. This change of variable is a symmetrization of the quasilinear part and is the key of this paper. This shows that the Cauchy problem is locally well-posed.

1. Introduction

1.1. Physical context. The construction of always-more-powerful lasers allows new experiments where hot plasma is created in which laser beams can propagate. The main goal is to simulate nuclear fusion by inertial confinement in a laboratory. Of course, one wants also to simulate numerically these experiences (it is much cheaper than true experiences!). We therefore need some precise and reliable models of laser-plasma interactions. The kinetic-type models are the more precise ones but the cost in term of computations is exorbitant and no physically relevant situation for nuclear fusion can be simulated in this context. Therefore, we use a bi-fluid model for the plasma: we couple two compressible Euler systems with Maxwell equations. Even under this form, it is not possible to perform direct computations because of high-frequency motions and the small wavelength involved in the problem. At the beginning of the 70’s, Zakharov and his collaborators introduced the so-called Zakharov’s equation [20] in order to describe the electronic plasma waves. Basically, the slowly varying envelope of the electric field $E = \nabla \psi$ is coupled to the low-frequency variation of the density of the ions $\delta n$ by the
following equations written in a dimensionless form:

\[
\begin{align*}
\begin{cases}
i\partial_t \nabla \psi + \Delta (\nabla \psi) &= \nabla \Delta^{-1} \text{div}(\delta n \nabla \psi), \\
\partial_t^2 \delta n - \Delta \delta n &= \Delta(|\nabla \psi|^2).
\end{cases}
\end{align*}
\tag{1.1}
\]

Of course, variations of this systems exist (see [17] for example). For laser propagation, one uses the paraxial approximation and the Zakharov system reads

\[
\begin{align*}
\begin{cases}
i(\partial_t + \partial_y) E + \Delta_\perp E &= n E, \\
\partial_t^2 - \Delta_\perp n &= \Delta_\perp(|E|^2),
\end{cases}
\end{align*}
\tag{1.2}
\]

where \(\Delta_\perp = \partial_x^2 + \partial_z^2\). (See [14] or [17] for a symmetric use of this kind of model for numerical simulation.) It is, however, not clear if system (1.2) is well-posed or not, and no results are available at the moment.

Physically, the complete situation is the following one: the laser enters the plasma. Part of it is backscattered through a Raman-type process and a Brillouin-type process. The Raman and Brillouin parts and the laser combine together to create an electron-plasma wave. These four waves interact in order to create a low-frequency variation of density of the ions which has itself an influence on the four preceding waves. This situation can be described by two systems of the form (1.2) and one of the form (1.1) coupled by quasilinear terms. Such a system is used in [15] for numerical simulations.

**Notation:** As usual, we denote by \(L^p(\mathbb{R}^d)\) the Lebesgue space

\[
L^p(\mathbb{R}^d) = \left\{ u \in S'(\mathbb{R}^d) : ||u||_p < +\infty \right\},
\]

where

\[
||u||_p = \left( \int_{\mathbb{R}^d} |u(x)|^p dx \right)^{\frac{1}{p}} \text{ if } 1 \leq p < +\infty
\]

and

\[
||u||_\infty = \text{ess.sup} \left\{ |u(x)| : x \in \mathbb{R}^d \right\}.
\]

We define the Sobolev space \(H^s(\mathbb{R}^d)\) as follows:

\[
H^s(\mathbb{R}^d) = \left\{ u \in S'(\mathbb{R}^d) : ||u||^2_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < +\infty \right\},
\]

where \(\hat{u}(\xi)\) is the Fourier transform of \(u\). Let \(C(I, E)\) be the space of continuous functions from an interval \(I\) of \(\mathbb{R}\) to a Banach space \(E\). For \(1 \leq j \leq d\), we set \(\partial_j u = \frac{\partial u}{\partial x_j}\). For \(k \in \mathbb{N}^d, k = (k_1, \ldots, k_d)\), we denote \(\partial^k u = \partial_{x_1}^{k_1} \cdots \partial_{x_d}^{k_d} u\). Different positive constants might be denoted by the same letter \(C\). We also denote by \(\text{Re}(u)\) and \(\text{Im}(u)\) the real part and the imaginary part of \(u\).
1.2. Presentation of the problem. The system used in [15] is

\[(\partial_t + v_0 \partial_y - \frac{ic^2}{2\omega_0} \Delta_\perp) A_0 = -\frac{2i\pi e^2}{m\omega_0} n A_0 + \frac{ie}{4m\omega_0} (\nabla \cdot E) A_Re^{-i\theta},\]

\[(\partial_t + v_R \partial_y - \frac{ic^2}{2\omega_R} \Delta_\perp) A_R = -\frac{2i\pi e^2}{m\omega_R} n A_R + \frac{ie}{4m\omega_R} (\nabla \cdot E^*) A_0 e^{i\theta},\]

\[(2i\omega_{pe} \partial_t + 3v_{th}^2 \Delta) E = \frac{4\pi e^2}{m} \nabla \Delta^{-1} \text{div}(nE) + \frac{e}{2mc^2} \omega_{pe}^2 \nabla (A_0 A_R^* e^{i\theta}),\]

\[(\partial_t^2 - c_s^2 \Delta)n = \frac{Z}{16\pi m_i} \Delta \left( \frac{\omega_{pe}^2}{c^2} (|A_0|^2 + |A_R|^2) + |E|^2 \right),\]

where \(\theta = k_1 y - \omega_1 t, t \in \mathbb{R}, y \in \mathbb{R}, \) and \(\Delta_\perp = \partial_x^2 + \partial_y^2\). For simplicity, if \(A\) and \(B\) are two vectors of \(\mathbb{R}^3\), the inner product \(A \cdot B\) is denoted by \(AB\). In this system, \(A_0\) denotes the envelope of the incident laser field, \(A_R\) is the scattered light, \(E\) is the electronic-plasma wave and \(n\) the variation of density of ions. Furthermore, the vector fields \(A_0\), \(A_R\), and \(E\) are such that \(A_0, A_R, E : \mathbb{R}^3 \rightarrow \mathbb{C}^3\). At this point, we do not give the orders of the different constants. We will use them carefully in the next section.

The authors observe some numerical instability explained by the fact that the paraxial approximation is done on \(A_0\) and \(A_R\) and not on \(E\). This appears through the presence of the operators

\[\partial_t + v_0 \partial_y - \frac{ic^2}{2\omega_0} \Delta_\perp \text{ and } \partial_t + v_R \partial_y - \frac{ic^2}{2\omega_R} \Delta_\perp\]

on \(A_0\) and \(A_R\) while the evolution operator on \(E\) is the complete dispersive one \((2i\omega_{pe} \partial_t + 3v_{th}^2 \Delta)\). As mentioned before, no existence results are known for (1.2). It is probably ill-posed. See however [19] for variations around this kind of system. In order to obtain a system having better properties, we will derive a new set of equations starting from the bi-fluid Euler-Maxwell system. This will be done in Section 2. Our system involves four Schrödinger equations coupled by quasilinear terms and a wave equation. It reads

\[i\left(\partial_t + \frac{k_0 c^2}{\omega_0} \partial_y\right) A_0 + \frac{c^2}{2\omega_0} \Delta A_0 - \frac{k_0^2 c^4}{2\omega_0^2} \partial_y \partial_y A_0 = \frac{\omega_{pe}^2}{2\omega_0} \langle \delta n \rangle (A_0 + e^{-2ik_0 y} A_B) - \frac{e}{2mc\omega_0} (\nabla \cdot E_0) A_Re^{i(k_1 y - \omega_1 t)}, \quad (1.3)\]

\[i\left(\partial_t - \frac{k_0 c^2}{\omega_0} \partial_y\right) A_B + \frac{c^2}{2\omega_0} \Delta A_B - \frac{k_0^2 c^4}{2\omega_0^2} \partial_y \partial_y A_B = \frac{\omega_{pe}^2}{2\omega_0} \langle \delta n \rangle (A_0 + e^{-2ik_0 y} A_B) - \frac{e}{2mc\omega_0} (\nabla \cdot E_0) A_Re^{i((2k_0 - k_1) y - \omega_1 t)}, \quad (1.4)\]
where \((\ref{3})\), \((\ref{4})\). This will play a fundamental role in the existence theory. How-

ever, \((\ref{5})\), \((\ref{6})\). We are namely in a situation where we want to describe a three-waves inter-

action. It is easy to see that

\[
\delta n = \frac{1}{4\pi n_0 m_i} \Delta \left( |E_0|^2 + \frac{\omega_{pe}^2}{c^2} \left( |A_0 + e^{-2ik_0y}A_B|^2 + |A_R|^2 \right) \right).
\]

\begin{equation}
(\partial_t^2 - c_s^2 \Delta) \langle \delta n \rangle = \frac{1}{4\pi n_0 m_i} \Delta \left( |E_0|^2 + \frac{\omega_{pe}^2}{c^2} \left( |A_0 + e^{-2ik_0y}A_B|^2 + |A_R|^2 \right) \right),
\end{equation}

\。(1.7) Here \(A_0\) is the incident laser field, \(A_B\) is the Brillouin component, \(A_R\) is the Raman field, \(E_0\) the electronic-plasma field and \(\langle \delta n \rangle\) the low-frequency variation of the density of the ions. The coefficients are described in Section 2. The complete electric field is recovered by

\[
E = \frac{i\omega_0}{c} A_0(t, X)e^{i(k_0y - \omega_0 t)} + \frac{i\omega_R}{c} A_R(t, X)e^{i(k_Ry - \omega_R t)}
\]

\[
+ \frac{i\omega_0}{c} A_B(t, X)e^{-i(k_0y + \omega_0 t)} + E_0(t, X)e^{-i\omega_{pe} t} + c.c.,
\]

\。(1.8) where \((k_1, \omega_1)\) are defined by

\[
k_0 = k_R + k_1, \quad \omega_0 = \omega_R + \omega_{pe} + \omega_1.
\]

We are namely in a situation where we want to describe a three-waves inter-

action. It is easy to see that \(k_R\) has to be nonpositive if \(k_0\) is nonnegative. Therefore, the velocity of \(A_R\) is nonpositive. System \((\ref{1})\)–\((\ref{7})\) differs from that used in \([15]\) by the presence of \(A_B\) and the complete dispersion in \((\ref{1})\), \((\ref{2})\), \((\ref{3})\). This will play a fundamental role in the existence theory. However, \(A_B\) can be eliminated as shown in Section 3.2, and the final system is, using \(A_C = A_0 + e^{-2ik_1y}A_B\),

\[
i \left( \partial_t A_C + \frac{k_0 c^2}{\omega_0} \partial_y A_C \right) + \frac{c^2}{2\omega_0} \Delta A_C - \frac{k_0^2 c^4}{2\omega_0^3} \partial_y^2 A_C = \frac{\omega_{pe}^2}{2\omega_0} \langle \delta n \rangle A_C
\]

\[
- \frac{e}{2m_e \omega_0} (\nabla \cdot E_0) A_R e^{-i(k_1y - \omega_1 t)},
\]

\。(1.9)
\(- \frac{e}{2m_e \omega_R} (\nabla \cdot E_0^*) AC e^{i(k_1 y - \omega_1 t)} \),

\[
-i \partial_t E_0 + \frac{v_{th}^2}{2 \omega_{pe}} \Delta E_0 = \frac{\omega_{pe}}{2} \nabla \Delta^{-1} \text{div} (\langle \delta n \rangle E_0) + \frac{e \omega_{pe}}{2 e^2 m_e} \nabla \left( A_R^* AC e^{i(k_1 y - \omega_1 t)} \right),
\]

\[
(\partial_t^2 - c_s^2 \Delta) \langle \delta n \rangle = \frac{1}{4 \pi n_0 m_i} \Delta \left( |E_0|^2 + \frac{\omega_{pe}^2}{c_s^2} (|AC|^2 + |AR|^2) \right).
\]

1.3. **Statement of the results.** The local-in-time Cauchy problem for the usual Zakharov equations (1.1) is now well understood in the context of regular solutions (see [1], [12], [16], and [18] for local models; see [3] for the nonlocal case (1.1)). For weak solutions, one can see [7]. For finite-time blow-up see [8, 9]. Here, we are interested in proving local existence in time for strong solutions. We want to apply Ozawa-Tsutsumi’s method (see [12]).

The main problem is to deal with the quasilinear part of the system:

\[
i \partial_t AC = - (\nabla \cdot E_0^*) A_R,
\]

\[
i \partial_t A_R = - (\nabla \cdot E_0^*) AC,
\]

\[
i \partial_t E_0 = \nabla (A_R^* AC).
\]

We omit here the exponentials, which play no role regarding the local-existence method. Even if any regular solution of (1.13)–(1.15) satisfies the conservation law

\[
\frac{d}{dt} \int (2|AC|^2 + |AR|^2 + |E_0|^2) = 0,
\]

the system is **not** symmetric, nor is it hyperbolic! It is therefore ill-posed (see Section 3.3). However, when one adds the dispersion part, one can show that

\[
i \partial_t AC + \alpha \Delta AC = - (\nabla \cdot E_0^*) A_R,
\]

\[
i \partial_t A_R + \beta \Delta A_R = - (\nabla \cdot E_0^*) AC,
\]

\[
i \partial_t E_0 + \gamma \Delta E_0 = \nabla (A_R^* AC),
\]

is locally well-posed. The dispersion seems thus to play a fundamental role (see Section 4). The Cauchy problem for quasilinear Schrödinger equations has been studied by many authors [4], [10], [11]. We want here a proof that can be adapted to Zakharov-type models and that involves usual energy estimates. In fact, we show that the dispersion plays the same role as dissipation for nonhyperbolic diphasic model (see [13]). The way we proceed is
as follows: we add new unknowns to the system, and we use the dispersive
terms to symmetrize the quasilinear part. This is the key of this paper.

Our main result reads (see Theorem 5.1)

**Theorem.** Let \((a_C, a_R, e) \in (H^{s+2}(\mathbb{R}^d))^{3d}, n_0 \in H^{s+1}(\mathbb{R}^d), \) and \(n_1 \in H^s(\mathbb{R}^d)\) with \(s > \frac{d}{2} + 3\). There exists \(T^* > 0\) and a unique maximal solution \((A_C, A_R, E, \delta n)\) to (1.9)–(1.12) such that

\[(A_C, A_R, E) \in C([0, T^*); H^{s+2})^{3d}, \quad \delta n \in C([0, T^*); H^{s+1}) \cap C^1([0, T^*); H^s)\]

satisfying

\[(A_C, A_R, E, \delta n)(0) = (a_C, a_R, e, n_0), \quad \partial_t \delta n(0) = n_1.\]

It is clear that the same theorem can be proved for system (1.3)–(1.7).

The paper is organized as follows. In Section 2, we carefully derive our new model. In Section 3, we discuss the algebraic structure of the system: we introduce a dimensionless form, eliminate the Brillouin component, and study the lack of hyperbolicity of the quasilinear part. In Section 4, we solve the Cauchy problem for (1.16)–(1.18) using the dispersive part. In Section 5, we apply the same method to (1.9)–(1.12). In Section 6, we give some complements and open problems.

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### 2. The Nonlinear Coupled Model

In this part, we derive the nonlinear coupled model describing the interaction of a laser beam in a plasma with the Raman and Brillouin backscattered fields. In Section 2.1, we introduce the starting point: the bi-fluid Euler-Maxwell system. We recall the polarization conditions for the transverse and longitudinal unknowns. In Section 2.2, we obtain the amplitude equations for the different electric fields. In Section 2.3, we couple these amplitude equations with the linearized hydrodynamical equations for the ions.

**2.1. The bi-fluid Euler-Maxwell system.** The Euler-Maxwell system describes the evolution of the electromagnetic field in a plasma. The plasma is modeled by two fluids (one for the ions, one for the electrons). Therefore, we couple Maxwell system with two Euler systems. This coupling reads

\[\begin{align*}
\partial_t B + e \nabla \times E &= 0, \\
\partial_t E - e \nabla \times B &= 4\pi e ((n_0 + n_e)v_e - (n_0 + n_i)v_i),
\end{align*}\]
\[(n_0 + n_e) (\partial_t v_e + v_e \cdot \nabla v_e) = -\frac{\gamma_e T_e}{m_e} \nabla n_e - \frac{e(n_0 + n_e)}{m_e} (E + \frac{1}{c} v_e \times B), \tag{2.3}\]

\[(n_0 + n_i) (\partial_t v_i + v_i \cdot \nabla v_i) = -\frac{\gamma_i T_i}{m_i} \nabla n_i + \frac{e(n_0 + n_i)}{m_i} (E + \frac{1}{c} v_i \times B), \tag{2.4}\]

\[\partial_t n_e + \nabla \cdot ((n_0 + n_e) v_e) = 0, \tag{2.5}\]

\[\partial_t n_i + \nabla \cdot ((n_0 + n_i) v_i) = 0. \tag{2.6}\]

The unknowns are as follows:

- \(E\) and \(B\) are respectively the electric and magnetic field.
- \(v_e\) and \(v_i\) denote respectively the electron’s and ion’s velocity.
- \(n_0\) is the mean density of the plasma.
- \(n_e\) and \(n_i\) are the variation of density respectively of electrons and ions with respect to the mean density \(n_0\).

The constants are as follows:

- \(c\) is the velocity of light in a vacuum; \(e\) is the elementary electric charge.
- \(m_e\) and \(m_i\) are respectively the electron’s and ion’s mass.
- \(T_e\) and \(T_i\) are respectively the electronic and ionic temperature and \(\gamma_e\) and \(\gamma_i\) the thermodynamic coefficient.

For a precise description of this kind of model, see classical textbooks [5]. One of the main points is that the mass of the electrons is very small compared to the mass of the ions: \(m_e \ll m_i\). Since the Lorentz force is the same for the ions and the electrons, the velocity of the ions will be negligible with respect to the velocity of the electrons. The consequence is that we neglect the contribution of the ions in equation (2.2).

Now, we recall the polarizations for the longitudinal and transverse components of the fields in the linear regime. We begin with the linear theory of system (2.1)–(2.6). The linearized version around \((E, B, v_e, v_i, n_e, n_i) = (0, 0, 0, 0, 0, 0)\) is (neglecting the contribution of the ions in (2.2))

\[\partial_t B + c \nabla \times E = 0, \tag{2.7}\]

\[\partial_t E - c \nabla \times B = 4\pi e n_0 v_e, \tag{2.8}\]

\[n_0 \partial_t v_e = -\frac{\gamma_e T_e}{m_e} \nabla n_e - \frac{e n_0}{m_e} E, \tag{2.9}\]

\[\partial_t n_e + n_0 \nabla \cdot v_e = 0. \tag{2.10}\]

Now writing \(B = B_\parallel + B_\perp\) with \(\nabla \times B_\parallel = 0\) and \(\nabla \cdot B_\perp = 0\) and using similar notation for \(E\) and \(v_e\), we get

\[\partial_t B_\parallel = 0, \tag{2.11}\]
\[ \partial_t E_\parallel = 4\pi e n_0 v_e \parallel, \quad (2.12) \]
\[ \partial_t v_e \parallel = -\frac{\gamma_e T_e}{m_e n_0} \nabla n_e - \frac{e}{m_e} E_\parallel, \quad (2.13) \]
\[ \partial_t n_e + n_0 \nabla \cdot v_e \parallel = 0, \quad (2.14) \]

and
\[ \partial_t B_\perp + e \nabla \times E_\perp = 0, \quad (2.15) \]
\[ \partial_t E_\perp - e \nabla \times B_\perp = 4\pi e n_0 v_{e\perp}, \quad (2.16) \]
\[ \partial_t v_{e\perp} = -\frac{e}{m_e} E_\perp. \quad (2.17) \]

Systems (2.11)–(2.14) and (2.15)–(2.17) are two decoupled systems that can be treated separately. We begin with the longitudinal part (2.11)–(2.14). Computing \( \partial_t \frac{\partial}{\partial t} (2.13) \) and using (2.12) and (2.14) leads to
\[ \frac{\partial^2}{\partial t^2} v_e \parallel - \frac{\partial}{\partial t} \left( \gamma_e T_e \nabla \nabla \cdot v_e \parallel - \frac{4\pi e^2 n_0}{m_e} v_e \parallel \right) = 0, \quad (2.18) \]
That is,
\[ [\partial^2_t - v_{th}^2 \Delta + \omega_{pe}^2] v_e \parallel = 0, \quad (2.19) \]
since \( \nabla \times v_e \parallel = 0 \) and where \( \omega_{pe} \) is the electronic plasma pulsation
\[ \omega_{pe} = \sqrt{\frac{4\pi e^2 n_0}{m_e}}, \]
and \( v_{th} \) is the thermal velocity of electrons
\[ v_{th} = \sqrt{\frac{\gamma_e T_e}{m_e}}. \]
The dispersion relation associated to (2.18) is therefore
\[ \omega^2 = \omega_{pe}^2 + k^2 v_{th}^2. \quad (2.20) \]
For the transverse part, we compute \( \partial_t (2.16) \) and we use (2.15) and (2.17) to obtain
\[ \frac{\partial^2}{\partial t^2} E_\perp - c^2 \Delta E_\perp + \omega_{pe}^2 E_\perp = 0, \quad (2.21) \]
since \( \nabla \cdot E_\perp = 0 \). The dispersion relation associated to (2.20) is
\[ \omega^2 = \omega_{pe}^2 + k^2 c^2. \quad (2.22) \]
For practical applications, the thermal velocity \( v_{th} \) is always one order of magnitude (at least) smaller than \( c \). Therefore, the status of equations (2.18) and (2.20) are quite different. One has to think of the solution of
(2.20) under the form $e^{i(kx-\omega t)}E(t, x)$ with $\partial_t E \ll \omega E$ and $\partial_x E \ll kE$, while that of (2.18) will be $e^{-i\omega_{pe}t}v_e$ with $\partial_t v_e \ll \omega_{pe} v_e$.

We are then able to write the polarization conditions on the fields. As usual, we introduce the variation of density $\delta n_e = \frac{n_e}{n_0}$.

- Longitudinal part: the linear equations for the longitudinal part are (2.11)–(2.14). Using the time envelope approximation, one sets $(\bar{B}_\parallel, \bar{E}_\parallel, \bar{v}_e\parallel, \bar{\delta}n_e) = e^{-i\omega_{pe}t}(\bar{B}_\parallel, \bar{E}_\parallel, \bar{v}_e\parallel, \bar{\delta}n_e)$ with

$$\partial_t (\bar{B}_\parallel, \bar{E}_\parallel, \bar{v}_e\parallel, \bar{\delta}n_e) \ll \omega_{pe}(\bar{B}_\parallel, \bar{E}_\parallel, \bar{v}_e\parallel, \bar{\delta}n_e).$$

One obtains

$$\bar{B}_\parallel = 0,$$  

$$\bar{v}_e\parallel = -\frac{i\omega_{pe}}{4\pi en_0} \bar{E}_\parallel,$$  

$$\bar{\delta}n_e = -\frac{1}{4\pi en_0} \nabla \cdot \bar{E}_\parallel.$$  

- Transverse part: in this case, in the physical literature, it is usual to work with the vector potential $A$ such that $B = \nabla \times A$ and the scalar potential $\psi$ in the Lorentz jauge $\partial_t \psi = c\nabla \cdot A$. The linearized equations become

$$\partial_t \psi = c\nabla \cdot A,$$

$$\partial_t A + cE = c\nabla \psi,$$

$$\partial_t E - c\nabla \times \nabla \times A = 4\pi en_0v_e,$$

$$\partial_t v_e = -\gamma_e T_e \nabla \delta n_e - \frac{e}{m_e} E,$$

$$\partial_t \delta n_e + \nabla \cdot v_e = 0.$$

The transverse part is

$$\partial_t A_\perp + cE_\perp = 0,$$  

$$\partial_t E_\perp - c\nabla \times \nabla \times A_\perp = 4\pi en_0v_{e\perp},$$  

$$\partial_t v_{e\perp} = -\frac{e}{m_e} E_\perp.$$  

Looking for the solution in the form $(A_\perp, E_\perp, v_{e\perp}) = e^{i(kx-\omega t)}(\bar{A}_\perp, \bar{E}_\perp, \bar{v}_{e\perp})$ with

$$\partial_t (\bar{A}_\perp, \bar{E}_\perp, \bar{v}_{e\perp}) \ll \omega (\bar{A}_\perp, \bar{E}_\perp, \bar{v}_{e\perp}),$$

$$\partial_x (\bar{A}_\perp, \bar{E}_\perp, \bar{v}_{e\perp}) \ll k (\bar{A}_\perp, \bar{E}_\perp, \bar{v}_{e\perp}).$$
one obtains
\[
\vec{E}_\perp = \frac{i\omega}{c} \vec{A}_\perp, \quad (2.28)
\]
\[
\vec{v}_\perp = \frac{e}{\epsilon_0 m_e} \vec{A}_\perp. \quad (2.29)
\]

2.2. Amplitude equations for the electric fields. We now present a weakly nonlinear theory of the laser-plasma interaction. We will generalize the equations used in [15]. The electromagnetic transverse part is decomposed into three parts:

i) the incident wave with vector potential \( A_0 e^{i(k_0 x - \omega_0 t) + c.c.} \)

ii) a Raman backscattered wave with vector potential \( A_R e^{i(k_R x - \omega_R t) + c.c.} \)

iii) a Brillouin backscattered wave with vector potential \( A_B e^{i(-k_0 x - \omega_0 t) + c.c.} \)

These three waves create a longitudinal electronic plasma wave at the pulsation \( \omega_{pe} \). The corresponding electric field is \( E_0 e^{-i\omega_{pe} t} + c.c. \). These four electric fields combine in order to create a low-frequency density modulation \( \langle \delta n \rangle \). Using the polarization condition (2.28) for the transverse part, we look for the electric field in the form
\[
E = \frac{i\omega_0}{c} A_0(t, X) e^{i(K_0 \cdot X - \omega_0 t)} + \frac{i\omega_R}{c} A_R(t, X) e^{i(K_R \cdot X - \omega_R t)}
\]
\[
+ \frac{i\omega_0}{c} A_B(t, X) e^{-i(K_0 \cdot X + \omega_0 t)} + E_0(t, X) e^{-i\omega_{pe} t} + c.c., \quad (2.30)
\]
with \( X = (x, y, z) \). The aim of this section is to find the envelope equations satisfied by \( A_0, A_R, A_B, \) and \( E_0 \). We introduce \( (K_1, \omega_1) \) satisfying
\[
K_0 = K_R + K_1, \quad \omega_0 = \omega_R + \omega_{pe} + \omega_1. \quad (2.31)
\]
Note that \( (K_R, \omega_R) \) and \( (K_0, \omega_0) \) satisfy the same dispersion relation (2.21). In order to match the notation used in [15], we take \( K_0 = k_0(0,1,0) \) and \( K_R = k_R(0,1,0) \). Then (2.30) becomes
\[
E = \frac{i\omega_0}{c} A_0(t, X) e^{i(k_0 y - \omega_0 t)} + \frac{i\omega_R}{c} A_R(t, X) e^{i(k_R y - \omega_R t)}
\]
\[
+ \frac{i\omega_0}{c} A_B(t, X) e^{-i(k_0 y + \omega_0 t)} + E_0(t, X) e^{-i\omega_{pe} t} + c.c., \quad (2.32)
\]
and (2.31)
\[
k_0 = k_R + k_1, \quad \omega_0 = \omega_R + \omega_{pe} + \omega_1.
\]
We begin with the nonlinear Maxwell equations with the vector potential:
\[
\partial_t \psi = c \nabla \cdot A,
\]
\[
\partial_t A + cE = c \nabla \psi,
\]
\[ \partial_t E - e \nabla \times \nabla \times A = 4\pi e((n_0 + n_e)v_e - (n_0 + n_i)v_i). \]

Combining together these equations and neglecting the effects of the ions, we obtain
\[ \partial^2_t A - c^2 \Delta A = -4\pi e n_0 (1 + \delta n_e) v_e. \] (2.33)

Using the form (2.30), the relations (2.31), the dispersion relation (2.21), the polarization condition (2.24) and (2.29), and collecting the factor of \( e^{i(k_0 y - \omega_0 t)} \) in the right-hand-side of (2.33), we get
\[ \partial^2_t A_0 - 2i \omega_0 \partial_t A_0 - c^2 \Delta A_0 - 2ik_0 c^2 \partial_y A_0 \]
\[ = -\omega^2_{pe}(\delta n)(A_0 + e^{-2ik_0 y} A_B) + \frac{e}{m_e} (\nabla \cdot E_0) A_R e^{-i(k_1 y - \omega_1 t)}. \]

We now relax the usual time envelope approximation: at the first order, one has \(-2i \omega_0 \partial_t A_0 - 2ik_0 c^2 \partial_y A_0 = 0\); that is,
\[ \partial_t A_0 = -\frac{k_0 c^2}{\omega_0} \partial_y A_0 \quad \text{and} \quad \partial^2_t A_0 = \frac{k_0 c^4}{\omega_0^2} \partial^2_y A_0. \]

We therefore get
\[ i(\partial_t A_0 + \frac{k_0 c^2}{\omega_0} \partial_y A_0) + \frac{c^2}{2\omega_0} \Delta A_0 - \frac{k_0 c^4}{2\omega_0^2} \partial^2_y A_0 \]
\[ = \frac{\omega^2_{pe}}{2\omega^2_0} (\delta n)(A_0 + e^{-2ik_0 y} A_B) - \frac{e}{2m_e \omega_0} (\nabla \cdot E_0) A_R e^{-i(k_1 y - \omega_1 t)}. \] (2.34)

We make the same computation for the harmonic \( e^{i(kR y - \omega_R t)} \) to obtain
\[ i(\partial_t A_R + \frac{k_R c^2}{\omega_R} \partial_y A_R) + \frac{c^2}{2\omega_R} \Delta A_R - \frac{k_R c^4}{2\omega_R^2} \partial^2_y A_R \]
\[ = \frac{\omega^2_{pe}}{2\omega^2_R} (\delta n) A_R - \frac{e}{2m_e \omega_R} (\nabla \cdot E^*_0)(A_0 + e^{-2ik_0 y} A_B) e^{i(k_1 y - \omega_1 t)}. \] (2.35)

The equation for the Brillouin component \( e^{-i(k_0 y + \omega_0 t)} \) is similar:
\[ i(\partial_t A_B - \frac{k_0 c^2}{\omega_0} \partial_y A_B) + \frac{c^2}{2\omega_0} \Delta A_B - \frac{k_0 c^4}{2\omega_0^2} \partial^2_y A_B \]
\[ = \frac{\omega^2_{pe}}{2\omega_0} (\delta n)(A_B + e^{2ik_0 y} A_0) - \frac{e}{2m_e \omega_0} (\nabla \cdot E_0) A_R e^{i((2k_0 - k_1)y - \omega_1 t)}. \] (2.36)

For the electronic-plasma part, we go back to the Maxwell equations in \((B, E)\) (2.1)–(2.2) and we get
\[ \partial^2_t E + c^2 \nabla \times \nabla \times E = 4\pi e \partial_t ((n_0 + n_e)v_e). \] (2.37)
We use the Euler equations (2.3) and (2.5) in order to determine the right-hand side of (2.37) keeping only at most quadratic terms:

\[
\partial_t^2 E + c^2 \nabla \times \nabla \times E = 4\pi e (\nabla \cdot (n_0 v_e \nabla v_e - \frac{\gamma_e T_e}{m_e} n_e \nabla n_e) - \frac{e(n_0 + n_e)}{m_e} E - \frac{en_0}{cm_e} v_e \times B - n_0 v_e \nabla \cdot v_e).
\]

We write \( E = E_0 e^{-i\omega_p t} \), and we replace \( \delta n_e \) by \(-\frac{1}{4\pi e n_0} \nabla \cdot E_0 \) thanks to the polarization condition (2.24) and get

\[
\partial_t^2 E_0 - 2i\omega_p e \partial_t E_0 + c^2 \nabla \times \nabla \times E - v_{th}^2 \nabla \cdot E_0
\]

\[
= -2e\omega_p e n_0 \int_0^{2\pi} (v_e \cdot \nabla v_e + \frac{e}{m_e} \delta n_e E + \frac{e}{cm_e} v_e \times B + v_e \nabla \cdot v_e) e^{i\omega_p t} dt.
\]

The right-hand side is computed using the polarization relations (2.28)–(2.29). One gets

\[
v_e \cdot \nabla v_e + \frac{e}{cm_e} v_e \times B + v_e \nabla \cdot v_e
\]

\[
= e^2 c^2 m_e^2 (A \cdot \nabla A + A \nabla \cdot A) + \frac{e^2}{c^2 m_e^2} A \times (\nabla \times A) = \frac{e^2}{c^2 m_e^2} (\nabla |A|^2 + A \nabla \cdot A),
\]

since for all \( v \)

\[
v \cdot \nabla v = \nabla \frac{|v|^2}{2} + (\nabla \times v) \times v.
\]

Moreover, at first order \( \nabla \cdot A = 0 \). One finally gets using the time envelope approximation

\[
i\partial_t E_0 + \frac{v_{th}^2}{2\omega_p} \nabla \nabla \cdot E_0 - \frac{e^2}{2\omega_p} \nabla \times \nabla \times E_0 = \frac{\omega_p}{2} (\nabla \cdot (\delta n) E_0)
\]

\[
+ \frac{e\omega_p}{2c^2 m_e} \nabla (A_R^* (A_0 + e^{-2ik_0 y} A_B) e^{i(k_1 y - \omega t)}).
\]

(2.38)

Now since \( v_{th} \ll c \), we perform the limit \( c \to +\infty \) in this expression. We find \( \nabla \times E_0 = 0 \), and therefore \( E_0 \) is a gradient. We then eliminate the term \( \nabla \times E_0 \) in (2.38) by applying the divergence to it:

\[
i\partial_t \nabla \cdot E_0 + \frac{v_{th}^2}{2\omega_p} \Delta \nabla \cdot E_0 = \frac{\omega_p}{2} \nabla \cdot (\nabla \cdot (\delta n) E_0)
\]

\[
+ \frac{e\omega_p}{2c^2 m_e} \Delta (A_R^* (A_0 + e^{-2ik_0 y} A_B) e^{i(k_1 y - \omega t)}).
\]

Recall now that \( E_0 = \nabla \psi \); applying \( \nabla \Delta^{-1} \) to this equation yields

\[
i\partial_t E_0 + \frac{v_{th}^2}{2\omega_p} \Delta E_0 = \frac{\omega_p}{2} \nabla \Delta^{-1} \text{div} (\delta n) E_0
\]
\[
+ \frac{\omega_{pe}^2}{2e^2m_e} \nabla \left( A_R^*(A_0 + e^{-2ik_0y}A_B)e^{i(k_1y - \omega_1t)} \right). \quad (2.39)
\]

See [6] for the justification of such an approximation in the context of non-linear Schrödinger equations. We still have to obtain an equation for \( \langle \delta n \rangle \).

2.3. **Acoustic approximation for the low-frequency part.** Since \( \nabla \cdot E = -c(n_e - n_i) \) and since the electric field has a high frequency at the first order, one gets that the electron’s low-frequency part \( \langle \delta n_e \rangle \) equals the ion’s low-frequency part \( \langle \delta n_i \rangle \). We look for an equation involving \( \langle \delta n_i \rangle \). The linearization of (2.4)–(2.6) yields

\[
\partial_t v_i = -\frac{\gamma_i T_i}{m_i} \nabla \langle \delta n_i \rangle + \frac{e}{m_i} \langle E \rangle, \\
\partial_t \langle \delta n_i \rangle + \nabla \cdot v_i = 0,
\]

where \( \langle E \rangle \) is the low-frequency part of the electric field. One gets

\[
\partial_t^2 \langle \delta n_i \rangle - \frac{\gamma_i T_i}{m_i} \Delta \langle \delta n_i \rangle = -\frac{e}{m_i} \nabla \cdot \langle E \rangle. \tag{2.40}
\]

In order to express \( \langle E \rangle \), we divide (2.3) by \( n_0 + n_e \) and we take the mean-value component

\[
\langle v_e \cdot \nabla v_e \rangle = -v_{th}^2 \nabla \langle n_e \rangle - \frac{e}{m_e} \langle E \rangle - \frac{e}{m_e c} \langle v_e \times B \rangle; 
\]

that is,

\[
\frac{e}{m_e} \langle E \rangle = -v_{th}^2 \nabla \langle n_i \rangle - \langle v_e \cdot \nabla v_e \rangle - \frac{e}{m_e c} \langle v_e \times B \rangle. \tag{2.41}
\]

The contribution of the longitudinal part to the right-hand side of (2.41) is only in the term \( \langle v_e \cdot \nabla v_e \rangle \) which gives

\[
\langle \nabla |v_e|^2 \rangle = \frac{\omega_{pe}^2}{16\pi^2e^2n_0^2} \nabla |E_0|^2,
\]

thanks to the polarization condition (2.23) and since \( \nabla \times E_0 = 0 \) at the first order. The contribution of the transverse part is

\[
\langle -v_e \cdot \nabla v_e - \frac{e}{m_e c} v_e \times B \rangle = -\frac{e^2}{e^2 m_e^2} (A \cdot \nabla A + A \times (\nabla \times A)), \\
= -\frac{e^2}{e^2 m_e^2} (|A|^2),
\]

thanks to the polarization condition (2.29) and \( B = \nabla \times A \). We obtain

\[
\langle v_e \cdot \nabla v_e + \frac{e}{m_e c} v_e \times B \rangle = -\frac{e^2}{e^2 m_e^2} \nabla (|A_0 + e^{-2ik_0y}A_B|^2 + |A_R|^2).
\]
Finally, the low-frequency component of $E$ is
\[
\frac{e}{m_e} \langle E \rangle = -v_{th}^2 \nabla \langle \delta n_i \rangle - \frac{\omega_{pe}^2}{16\pi^2e^2m_e^2} \nabla |E_0|^2
- \frac{e^2}{c^2m_e^2} \nabla (|A_0 + e^{-2i\omega_0 y} A_B|^2 + |A_R|^2).
\]

Plugging this expression in (2.40) gives
\[
(\partial_t^2 - c_s^2 \Delta) \langle \delta n \rangle = \frac{1}{4\pi n_0 m_i} \Delta \left( |E_0|^2 + \frac{\omega_{pe}^2}{c^2} (|A_0 + e^{-2i\omega_0 y} A_B|^2 + |A_R|^2) \right),
\]
where $c_s^2 = \frac{2\gamma_i T_i}{m_i} + \frac{2\gamma_e T_e}{m_e}$ is the acoustic velocity.

Equations (2.34), (2.35), (2.36), (2.39), and (2.42) form a closed set of equations describing the laser-plasma interactions. In the next section, we investigate the algebraic properties of this system.

3. Structural properties of the system

This system is strongly quasilinear, and we investigate its structure. In order to make more comprehensible computations, we first introduce a dimensionless form (3.1)–(3.5). Then we show that the quasilinear part in the equations of $A_0$, $A_R$, $A_B$, and $E_0$ is not symmetric. Then we prove that the Brillouin component $A_B$ can be eliminated. The system becomes (3.6)–(3.9). We finish by discussing a resonance relation on $(k_1, \omega_1)$ and by some remarks.

3.1. Dimensionless form. We use $\frac{1}{\omega_0}$ as time scale and $\frac{1}{k_0}$ as space scale and introduce
\[
\tilde{A}_i = \sqrt{\omega_i \omega_{pe}} \frac{A_i}{\alpha} \quad \text{with} \quad i = 0, R, \text{ or } B,
\]
\[
\tilde{E} = \frac{\omega_{pe}}{\alpha} E \quad \text{with} \quad \alpha = \frac{2m_e \omega_0 \sqrt{\omega_0 \omega_R \omega_{pe}}}{ek_0},
\]
\[
\tilde{k}_1 = \frac{k_1}{k_0}, \quad \tilde{\omega}_1 = \frac{\omega_1}{\omega_0},
\]
and dropping the tildes, we get the following system:
\[
\left( i(\partial_t + \frac{k_0^2 c^2}{\omega_0^2} \partial_y) + \frac{c^2 k_0^2}{2\omega_0^2} \Delta - \frac{k_0^4 c^4}{2\omega_0^4} \partial_y^2 \right) A_0
= \frac{\omega_{pe}^2}{2\omega_0^2} \langle \delta n \rangle (A_0 + e^{-2i\omega_0 y} A_B) - (\nabla \cdot E_0) A_R e^{-i(k_1 y - \omega_1 t)},
\]
one obtains after a straightforward computation

\[ (i\partial_t + \frac{k_R k_0 c^2}{\omega_R \omega_0} \partial_y) + \frac{c^2 k_0^2}{2\omega_R \omega_0} \Delta - \frac{k_0^2 k_R^2 c^4}{2\omega_0^2 \omega_y^2} \partial_y^2 \) \ A_R \]

(3.2)

\[ = \frac{\omega_{pe}^2}{2\omega_R \omega_0} \langle \delta n \rangle \ A_R - (\nabla \cdot E_0^* ) (A_0 + e^{-2iy} A_B) e^{i(k_1 y - \omega_1 t)} , \]

\[ (i\partial_t - \frac{k_0^2 c^2}{\omega_0^2} \partial_y) + \frac{c^2 k_0^2}{2\omega_0^2} \Delta - \frac{k_0^4 c^4}{2\omega_0^2} \partial_y^2 \) \ A_B \]

(3.3)

\[ = \frac{\omega_{pe}^2}{2\omega_0^2} \langle \delta n \rangle (A_B + e^{2iy} A_0) - (\nabla \cdot E_0) A_R e^{i((k_1 y - \omega_1 t)} , \]

\[ i\partial_t E_0 + \frac{\nu_{th} k_0^2}{2\omega_{pe} \omega_0} \Delta E_0 \]

(3.4)

\[ = \frac{\omega_{pe}}{2\omega_0} \nabla \Delta^{-1} \text{div}(\langle \delta n \rangle E_0) + \nabla \left( A_R^* (A_0 + e^{-2iy} A_B) e^{i(k_1 y - \omega_1 t)} \right) , \]

\[ \partial_t^2 \langle \delta n \rangle - \frac{c^2 k_0^2}{\omega_0^2} \Delta \langle \delta n \rangle \]

(3.5)

\[ = \frac{4m_e}{m_i} \frac{\omega_0 \omega_R}{\omega_{pe}^2} \Delta \left( \frac{\omega_{pe}}{\omega_0} |A_0 + e^{-2iy} A_B|^2 + |E_0|^2 + \frac{\omega_{pe}}{\omega_R} |A_R|^2 \right) . \]

Note that \( \frac{m_e}{m_i} \) is small, as well as \( \frac{c^2 k_0^2}{\omega_0^2} \). All other coefficients are of order 1 except \( \frac{\nu_{th} k_0^2}{2\omega_{pe} \omega_0} \), which is small.

3.2. Elimination of the Brillouin component. In this section, we prove that we can in fact decrease the number of unknowns. We introduce

\( A_C = A_0 + e^{-2iy} A_B \).

Let us compute

\[ (i\partial_t + \frac{k_R k_0 c^2}{\omega_R \omega_0} \partial_y) + \frac{c^2 k_0^2}{2\omega_0^2} \Delta - \frac{k_0^2 k_R^2 c^4}{2\omega_R^2 \omega_y^2} \partial_y^2 \) \ A_C ; \]

one obtains after a straightforward computation

\[ (i\partial_t + \frac{k_0^2 c^2}{\omega_0^2} \partial_y) + \frac{c^2 k_0^2}{2\omega_0^2} \Delta - \frac{k_0^4 c^4}{2\omega_0^2} \partial_y^2 \) \ A_C \]

(3.6)

\[ = \frac{\omega_{pe}^2}{2\omega_0^2} \langle \delta n \rangle A_C - (\nabla \cdot E_0) A_R e^{-i(k_1 y - \omega_1 t)} . \]

Of course (3.2), (3.4), and (3.5) become

\[ (i\partial_t + \frac{k_R k_0 c^2}{\omega_R \omega_0} \partial_y) + \frac{c^2 k_0^2}{2\omega_R \omega_0} \Delta - \frac{k_0^2 k_R^2 c^4}{2\omega_R^2 \omega_y^2} \partial_y^2 \) \ A_R \]

(3.7)
\[ M. \text{Colin and T. Colin} \]

\[ \frac{\omega_{pe}^2}{2\omega_R \omega_0} \langle \delta n \rangle A_R - (\nabla \cdot E_0^*) A_C e^{i(k_1 y - \omega_1 t)}, \]

\[ i\partial_t E_0 + \frac{v_{th}^2 k_2^2}{2\omega_{pe} \omega_0} \Delta E_0 = \frac{\omega_{pe}}{2\omega_0} \nabla \Delta^{-1} \text{div} (\langle \delta n \rangle E_0) + \nabla \left( A_R^* A_C e^{i(k_1 y - \omega_1 t)} \right), \]

and

\[ \left( \partial_t^2 - \frac{c_2^2 k_2^2}{\omega_0^2} \Delta \right) \langle \delta n \rangle = \frac{4m_e \omega_0 \omega_R}{m_i} \Delta \left( \frac{\omega_{pe}}{2\omega_0} |A_C|^2 + |E_0|^2 + \frac{\omega_{pe}}{\omega_R} |A_R|^2 \right). \]  

(3.8)

(3.6)–(3.9) form a closed set of equations that we solve in the sequel.

3.3. Study of the quasilinear part. In this section, we consider the quasilinear part of the system (3.6)–(3.8) and we omit the exponentials:

\[ \partial_t A_C = i (\nabla \cdot E_0) A_R, \]

(3.10)

\[ \partial_t A_R = i (\nabla \cdot E_0^*) A_C, \]

(3.11)

\[ \partial_t E_0 = -i \nabla (A_R^* A_C). \]

(3.12)

Unfortunately, this system is not obviously symmetrizable, as we show below for the scalar one-dimensional model. Let us write \( A_C = u_1 + iu_2 \), \( A_R = u_3 + iu_4 \), and \( E_0 = u_5 + iu_6 \). System (3.10)–(3.12) reads

\[ \partial_t u_1 = -\partial_x u_6 u_3 - \partial_x u_5 u_4, \]

\[ \partial_t u_2 = \partial_x u_5 u_3 - \partial_x u_6 u_4, \]

\[ \partial_t u_3 = \partial_x u_6 u_1 - \partial_x u_5 u_2, \]

\[ \partial_t u_4 = \partial_x u_5 u_1 + \partial_x u_6 u_2, \]

\[ \partial_t u_5 = \partial_x (u_2 u_3 - u_1 u_4), \]

\[ \partial_t u_6 = -\partial_x (u_1 u_3 + u_2 u_4), \]

and it can be rewritten in the form

\[ \partial_t \mathcal{U} = M(\mathcal{U}) \partial_\mathcal{U}, \]

with \( \mathcal{U} = (u_1, u_2, u_3, u_4, u_5, u_6)^t \) and

\[ M(\mathcal{U}) = \begin{pmatrix} 0 & 0 & 0 & 0 & -u_4 & -u_3 \\ 0 & 0 & 0 & 0 & u_3 & -u_4 \\ 0 & 0 & 0 & 0 & -u_2 & u_1 \\ 0 & 0 & 0 & 0 & u_1 & u_2 \\ -u_4 & u_3 & u_2 & -u_1 & 0 & 0 \\ -u_3 & -u_4 & -u_1 & -u_2 & 0 & 0 \end{pmatrix}. \]
This matrix is obviously not symmetric, and the blocks
\[
M_1 = \begin{pmatrix}
-u_2 & u_1 \\
 u_1 & u_2
\end{pmatrix}
\quad \text{and} \quad
M_2 = \begin{pmatrix}
u_2 & -u_1 \\
-u_1 & -u_2
\end{pmatrix}
\]
are such that \( M_1^T = -M_2 \). Therefore the Cauchy problem for (3.10)–(3.12) is certainly ill-posed, and the Cauchy problem for (3.6), (3.7), (3.8), and (3.9) can not be handled directly by coupling Ozawa-Tsutsumi’s method with hyperbolic-type energy estimates for symmetric system. We will explain in Section 4 how to overcome this difficulty by using the dispersive part.

3.4. The resonance condition (2.31). Let us recall that the numbers \( k_0, k_R, k_1, \omega_0, \omega_R, \) and \( \omega_1 \) satisfy
\[
k_0 = k_1 + k_R, \quad \omega_0 = \omega_R + \omega_{pe} + \omega_1.
\]
We moreover impose (as in [15]) that \( \omega_1 = \omega_{pe}(k_1\lambda_D)^2 \). This corresponds to the fact that
\[
\omega_{pe} + \omega_1 \simeq \sqrt{\omega_{pe}^2 + v_{th}^2 k_1^2}.
\]
Therefore, condition (2.31) is a usual three-wave interaction between two electromagnetic waves and a plasma-electronic wave:
\[
k_0 = k_R + k_1, \quad \sqrt{\omega_{pe}^2 + c^2 k_0^2} = \sqrt{\omega_{pe}^2 + c^2 k_R^2} + \sqrt{\omega_{pe}^2 + v_{th}^2 k_1^2}.
\]
We expect a Raman growth on the \( A_R \) component. It is therefore not surprising that the quasilinear part is not hyperbolic.

3.5. A hierarchy of model. Note that since \( |\omega_0| > |k_0c| \) and \( |\omega_R| > |k_Rc| \), the second-order operators in (3.6) and (3.7) are elliptic. From now on, we will restrict ourselves to the case where this operator is \( \frac{c^2k_0^2}{2\omega_0^2}\Delta \). This is the result that one obtains when making the time envelope approximation. In order to decompose the difficulty, we will consider successively more and more complex models. The first one is the system of quasilinear Schrödinger equations obtained by setting \( (\delta n) = 0 \) in (3.6), (3.7), and (3.8). We will show how to handle the Cauchy problem in Section 4. One could also use the complete equation on \( E_0 \), that is,
\[
i\partial_t E_0 + \frac{v_{th}^2 k_0^2}{2\omega_{pe}\omega_0} \nabla \cdot E_0 - \frac{c^2 k_0^2}{2\omega_{pe}\omega_0} \nabla \times \nabla \times E_0 = \frac{\omega_{pe}}{2\omega_0} (\langle \delta n \rangle E_0) + \nabla (A^*_R A C e^{-i(k_1 y - \omega_1 t)}).
\]
We do not discuss (3.13) in this paper. It can certainly be handled by the same method, but the proof may be more technical.
4. THE CAUCHY PROBLEM FOR THE NONLINEAR SCHRÖDINGER SYSTEM

Let us consider the system obtained from (3.6), (3.7), and (3.8) by setting \( \langle \delta n \rangle = 0 \). It can be rewritten in the form (omitting the exponentials that have no role in this context)

\[
\begin{align*}
(i \partial_t + v_C \partial_y + \alpha \Delta) A_C &= - (\nabla \cdot E_0) A_R, \\
(i \partial_t + v_R \partial_y + \beta \Delta) A_R &= - (\nabla \cdot E_0^*) A_C, \\
(i \partial_t + \gamma \Delta) E_0 &= \nabla (A_R^* A_C),
\end{align*}
\]

where

\[
\begin{align*}
v_C &= \frac{k_R^2 c^2}{\omega_0^2}, & v_R &= \frac{k_R k_0 c^2}{\omega_R \omega_0}, & \alpha &= \frac{k_0^2 c^2}{2 \omega_0^2}, & \beta &= \frac{k_R^2 c^2}{2 \omega_R \omega_0}, & \gamma &= \frac{v_{th} k^2}{2 \omega_p \omega_0}.
\end{align*}
\]

**Theorem 4.1.** Let \((a_C, a_R, e_0) \in H^s(\mathbb{R}^d)\) with \(s > \frac{d}{2} + 3\). There exist \(T^* > 0\) and a unique solution \((A_C, A_R, E_0) \in (C([0,T^*]; H^s(\mathbb{R}^d)))^{3d}\) of (4.1), (4.2), (4.3) satisfying \((A_C, A_R, E_0)(0,X) = (a_C, a_R, e_0)(X)\).

**Proof.** A lot of work has already been done on quasilinear Schrödinger equations. In [11], Kenig, Ponce, and Vega use local smoothing effects in order to obtain local existence. In [4] or [10], the authors use nonlinear changes of unknowns in order to obtain local existence. Here we want a generalization of such methods that can be adapted to Zakharov-type equations. In order to prove Theorem 4.1, we transform system (4.1)–(4.2) into a dispersive perturbation of a symmetric quasilinear system as follows. Let us first explain how the method works. Consider a system of PDE’s of the form (for simplicity, we consider the one-dimensional case)

\[
\partial_t U + B(U) \partial_x U + K \partial_x^2 U = 0,
\]

where \(U : [0,T] \times \mathbb{R} \rightarrow \mathbb{R}^d\), \(K\) is skew-adjoint and one-to-one, and

\[
B(U) = B_1(U) + B_2(U),
\]

where \(B_1(U)\) is symmetric and \(B_2(U)\) is skew-symmetric. Except if \(B_2 = 0\), one cannot handle directly (4.4) by usual energy methods. One introduces \(V = \partial_t U, X = \partial_x U\), and \(W = \partial_x^2 U\). The equations on \(V, X,\) and \(W\) read

\[
\begin{align*}
\partial_t V + B(U) \partial_x V + K \partial_x^2 V &= \text{semilinear terms}, \\
\partial_t X + B(U) \partial_x X + K \partial_x^2 X &= \text{semilinear terms}, \\
\partial_t W + B(U) \partial_x W + K \partial_x^2 W &= \text{semilinear terms}, \\
\partial_t U &= \text{semilinear terms},
\end{align*}
\]
and
\[ \mathcal{V} + K \mathcal{W} = \text{semilinear terms}, \quad (4.9) \]
where the semilinear terms depend on \( \mathcal{U}, \mathcal{V}, \mathcal{X}, \) and \( \mathcal{W} \). Now in (4.5), we replace \( \mathcal{V} \) by \( -K \mathcal{W} + \text{semilinear terms} \) (depending only on \( \mathcal{U} \) and \( \mathcal{X} \)) in \( B_2(\mathcal{U}) \partial_x \mathcal{V} \) and in (4.7) we replace \( \mathcal{W} \) by \( -K^{-1} \mathcal{V} + \text{semilinear terms} \) (depending only on \( \mathcal{U} \) and \( \mathcal{X} \)) in \( B_2(\mathcal{U}) \partial_x \mathcal{W} \). The new system in \( \mathcal{V} \) and \( \mathcal{W} \) reads
\[
\partial_t \mathcal{V} + B_1(\mathcal{U}) \partial_x \mathcal{V} - B_2(\mathcal{U}) K \partial_x \mathcal{W} + K \partial_x^2 \mathcal{V} = \text{semilinear terms}, \quad (4.10) \\
\partial_t \mathcal{W} + B_1(\mathcal{U}) \partial_x \mathcal{W} - B_2(\mathcal{U}) K^{-1} \partial_x \mathcal{V} + K \partial_x^2 \mathcal{W} = \text{semilinear terms}. \quad (4.11) 
\]
Now system (4.10)–(4.11) may be symmetrizable, depending on the properties of \( B_2 \) and \( K \). In our case, it will be possible, and we explain now why. Let us introduce the following notation (for simplicity, we denote \( E = E_0 \)):
\[
B_I^R = \partial_x A_R, \quad B_I^C = \partial_x A_C, \quad F_I = \partial_x E, \\
C_R = \Delta A_R, \quad C_C = \Delta A_C, \quad G = \Delta E, \\
D_R = \partial_t A_R, \quad D_C = \partial_t A_C, \quad H = \partial_t E. 
\]
We write the equations satisfied by these new unknowns. We set
We also introduce the following linear operator:
\[
L = \begin{pmatrix}
  i(\partial_t + v_C \partial_y) + \alpha \Delta \\
  i(\partial_t + v_R \partial_y) + \beta \Delta \\
  i\partial_t + \gamma \Delta
\end{pmatrix}.
\]
The system becomes
\[
L \begin{pmatrix}
  A_C \\
  A_R \\
  E
\end{pmatrix} = f_1(\tilde{\mathcal{U}}), \quad (4.12) \\
L \begin{pmatrix}
  B_C^I \\
  B_R^I \\
  F_I
\end{pmatrix} = f_2^I(\mathcal{U}), \quad (4.13)
\]
where \( \tilde{\mathcal{U}} = (A_C, A_R, E, B_R, B_C, F) \). Here \( f_1 \) is a smooth function and the \( f_2^I \) are nonlocal functions satisfying \( |f_2^I(\mathcal{U})| \leq K |\mathcal{U}|^\sigma_H \) for some \( \sigma, K \), and
Indeed, the \( f^j_2(U) \) are polynomial in \( A_C, A_R, E, B^j_R, B^j_C, F^j \), and \( \partial^2_{x_ix_j}(A_C, A_R, E) \) that can be expressed in terms of \( C_C, C_R, \) and \( G \) by
\[
\partial^2_{x_ix_j}(A_C, A_R, E) = \partial^2_{x_ix_j} \Delta^{-1}(C_C, C_R, G).
\] (4.14)

In order to find the equations satisfied by \( C_C, C_R, \) and \( G \), we need to express the nonlinear term \( \nabla(A_R^*A_C) \) in (4.3) more precisely using
\[
\nabla(AB) = A \cdot \nabla B + B \cdot \nabla A + A \times (\nabla \times B) + B \times (\nabla \times A).
\] (4.15)

We moreover set
\[
m_A(B) := A \cdot \nabla B + A \times (\nabla \times B).
\] (4.16)

The system satisfied by \( (C_C, C_R, G) \) is therefore
\[
L \begin{pmatrix} C_C \\ C_R \\ G \end{pmatrix} = \begin{pmatrix} -A_R \nabla \cdot G \\ -A_C \nabla \cdot G^* \\ m_{A_R}(C_C) + m_{A_C}(C_R^*) \end{pmatrix} + f_3(U),
\] (4.17)

where \( f_3(U) \) is a nonlocal function as \( f^j_2(U) \). In the same way, the system for \( (D_C, D_R, H) \) is
\[
L \begin{pmatrix} D_C \\ D_R \\ H \end{pmatrix} = \begin{pmatrix} -A_R \nabla \cdot H \\ -A_C \nabla \cdot H^* \\ m_{A_R}(D_C) + m_{A_C}(D_R^*) \end{pmatrix} + f_4(U),
\] (4.18)

where \( f_4(U) \) is a nonlocal function as \( f^j_2(U) \). As mentioned before, the quasilinear part of (4.17) and (4.18) is not symmetric, and therefore usual energy estimates are inoperative. We now prove the following proposition.

**Proposition 4.1.** Let \( b \in \mathbb{C}^3 \) be a constant. The adjoint in \( L^2 \) of \( -b \nabla \cdot \) is \( m_{b^*}(\cdot) \).

**Proof.** Let \( V \) and \( W \) be two \( \mathcal{C}^\infty \) vector fields in \( \mathbb{C}^3 \). Let us compute
\[
\int m_{b^*}(V) \cdot W^* = \int (b^* \cdot \nabla V + b^* \times (\nabla \times V)) \cdot W^*
\]
by (4.16),
\[
= \int (b^* \cdot \nabla) V \cdot W^* + (W^* \times b^*) \cdot \nabla V
\]
\[
= \int (b^* \cdot \nabla) V \cdot W^* + \nabla \times (W^* \times b^*) \cdot V
\]
since the operator \( \nabla \times \) is self-adjoint. Now using the relation
\[
\nabla \times (A \times B) = A \nabla \cdot B - B \nabla \cdot A + B \cdot \nabla A - A \cdot \nabla B
\]
and the fact that \( b \) is constant, one gets
\[
\nabla \times (W^* \times b^*) = -b^* \nabla \cdot W^* + b^* \cdot \nabla W^*.
\]
Moreover,
\[
\int (b^* \cdot \nabla) V \cdot W^* = - \int V \cdot (b^* \cdot \nabla) W^*.
\]

It follows that
\[
\int m_{b^*} (V) \cdot W^* = - \int V \cdot (b^* \cdot \nabla) W^* - \int b^* \nabla \cdot W^* \cdot V + \int b^* \nabla W^* \cdot V
\]
\[
= - \int b^* \nabla \cdot W^* \cdot V = - \int b \nabla \cdot W \cdot V^*.
\]

This ends the proof of Proposition 4.

In the quasilinear part of (4.17) or (4.18) the terms \(-A_R \nabla \cdot H\) in the first equation and \(m_{A_*} (D_C)\) in the third one can be rewritten as
\[
\begin{pmatrix}
0 & 0 & -A_R \nabla \cdot \\
0 & 0 & 0 \\
m_{A_*} (\cdot) & 0 & 0
\end{pmatrix}
\begin{pmatrix}
D_C \\
D_R \\
H
\end{pmatrix}
\]

where the operator
\[
\begin{pmatrix}
0 & 0 & -A_R \nabla \cdot \\
0 & 0 & 0 \\
m_{A_*} (\cdot) & 0 & 0
\end{pmatrix}
\]
is skew-adjoint. Therefore usual energy estimates will apply. Unfortunately, this is not the case for the other part. For the terms \(-C \nabla \cdot G^*\), \(m_{A_C} (C_R^*)\), \(-C \nabla \cdot H^*\) and \(m_{A_C} (D_R^*)\) we use equations (4.2)–(4.3):
\[
i D_R + \beta C_R = g_1 (\bar{U}),
\]
(4.19)

and
\[
i \partial_{x_j} D_R + \beta \partial_{x_j} C_R = g_{2j} (U),
\]
(4.20)

where \(g_1\) is a smooth function and the \(g_{2j}\) are functions satisfying the same properties as the \(f_{2j}\). In the same way, one has
\[
i H + \gamma G = g_3 (\bar{U}),
\]
(4.21)

and
\[
i \partial_{x_j} H + \gamma \partial_{x_j} G = g_{3j} (U).
\]
(4.22)

Plugging these relationships into the bad terms of equations (4.17) and (4.18), we obtain
\[
L \begin{pmatrix}
C_C \\
C_R \\
G
\end{pmatrix} = \begin{pmatrix}
-A_R \nabla \cdot G \\
-A_C \nabla \cdot H^* \\
m_{A_*} (C_C) + \frac{i}{\beta} m_{A_C} (D_R^*)
\end{pmatrix} + f_5 (U),
\]
(4.23)
The complete system reads
\[
\begin{pmatrix}
L & L & L & L \\
L & L & L & L
\end{pmatrix}
\begin{pmatrix}
U \\
\tilde{U}
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
0 & M(A_C, A_R, \partial)
\end{pmatrix}
\begin{pmatrix}
U \\
\tilde{U}
\end{pmatrix}
+ F_1(U),
\] (4.25)
where \(\tilde{U} = (A_C, A_R, E, B_C, B_R, F)\), \(V = (C_C, C_R, G, D_C, D_R, H)\), and \(M\) is a 6 \(\times\) 6 matrix of operators whose value is
\[
M(V, W, \partial) = \begin{pmatrix}
0 & 0 & -W \nabla \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{i\gamma}{\beta}m_V(\sigma) \\
0 & 0 & 0 & 0 & 0 & -W \nabla \cdot \\
0 & 0 & i\gamma m_V(\sigma) & 0 & 0 & 0 \\
0 & -i\beta m_V(\sigma) & 0 & m_{W^*}(\cdot) & 0 & 0
\end{pmatrix}
\] (4.26)
where \(\sigma\) is the complex conjugation (for a vector field \(A\) in \(\mathbb{R}^3\), \(\sigma A = A^*\)). A straightforward change of variable enables us to restrict to the case \(\gamma = \beta = 1\). In this case, the operator \(M(V, W, \partial)\) with \(V\) and \(W\) fixed constants is symmetric. Then (4.25) reads
\[
(\partial_t + C(\nabla) + iD\Delta) U = \begin{pmatrix}
0 & 0 \\
0 & iM(A_C, A_R, \partial)
\end{pmatrix} U + iF_1(U),
\] (4.27)
where
\[
C(\nabla) = \sum_{j=1}^{d} C_j \partial x_j,
\]
corresponds to the order-one terms in (4.25) and \(D\) is the diagonal dispersion matrix. Usual energy estimates for the system (4.27) yield Theorem 4.1.

5. The Cauchy problem for the nonlinear Zakharov system

In this section, we consider the following nondimensional form of system (3.6)–(3.9). We have omitted the exponentials which have no role in this context.
\[
(i(\partial_t + v_C \partial_y) + \alpha \Delta) A_C = \frac{b^2}{2} n A_C - (\nabla \cdot E) A_R,
\] (5.1)
\[(i(\partial_t + v_R \partial_y) + \beta \Delta) A_R = \frac{bc}{2} n A_R - (\nabla \cdot E^*) A_C, \quad (5.2)\]
\[(i \partial_t + \gamma \Delta) E = \frac{b}{2} n E + \nabla (A_R^* A_C), \quad (5.3)\]
\[(\partial_t^2 - v_s^2 \Delta) n = a \Delta (|E|^2 + b|A_C|^2 + c|A_R|^2), \quad (5.4)\]

where
\[a = 4 \frac{m_e \omega_m \omega \omega_0 \omega_R}{m_i \omega_{pe}^2}, \quad b = \frac{\omega_{pe}}{\omega_0}, \quad c = \frac{\omega_{pe}}{\omega_R}, \quad v_s = \frac{c_s k_0}{\omega_0}.\]

Here, \((\delta n)\) is denoted by \(n\) for simplicity.

**Theorem 5.1.** Let \((a_C, a_R, e) \in (H^{s+2}(\mathbb{R}^d))^3\), \(n_0 \in H^{s+1}(\mathbb{R}^d)\), and \(n_1 \in H^s(\mathbb{R}^d)\) with \(s > \frac{d}{2} + 3\). There exists \(T^* > 0\) and a unique maximal solution \((A_C, A_R, E, n)\) to (5.1)–(5.4) such that \((A_C, A_R, E, n) \in \mathcal{C}([0, T^*]; H^{s+2})^3\), \(n \in \mathcal{C}([0, T^*]; H^{s+1}) \cap \mathcal{C}^1([0, T^*]; H^s)\) with the initial value
\[(A_C, A_R, E, n)(0) = (a_C, a_R, e, n_0), \quad \partial_t n(0) = n_1.\]

The proof relies on the transformation described in Section 4. In addition to the variables of Section 4, we have to use \(\Delta n\) and \(\partial_t n\). We therefore write the equations satisfied by \(n\) and \(\partial_t n\). Take

\[p = \nabla n, \quad q = \Delta n, \quad r = \partial_t n, \quad \lambda = \nabla \Delta^{-1} \partial_t n, \quad \mu = \nabla \lambda, \quad \nu = \Delta \lambda, \quad \tau = \partial_t \lambda,\]

and set \(\mathcal{N} := (n, p, q, r, \lambda, \mu, \nu, \tau), \quad \tilde{U} := (A_C, A_R, E, B_C, B_R, F, C_C, C_R, G, D_C, D_R, H)\).

The equations for \(\mathcal{N}\) are
\[
\partial_t n = \nabla \cdot \lambda, \\
\partial_t \lambda = \nabla n + h_1(\tilde{U}), \\
\partial_t p_j = \nabla \cdot j, \\
\partial_t j = \nabla p_j + h_2(\tilde{U}), \\
\partial_t q = \nabla \cdot \nu, \\
\partial_t \nu = \nabla q + \nabla(h_3(\tilde{U}, \tilde{U})), \\
\partial_t r = \nabla \cdot \tau, \\
\partial_t \tau = \nabla r + \nabla(h_4(\tilde{U}, \tilde{U})).
\]
where \( h_1, h_2, h_3, \) and \( h_4 \) are smooth functions; \( h_3(\cdot, \cdot) \) and \( h_4(\cdot, \cdot) \) are bilinear. The system is

\[
\partial_t N + A(\nabla)N + F_2(\tilde{U}) + \nabla(F_3(\tilde{U}, \tilde{U})) = 0, \quad (5.5)
\]

where

\[
A(\nabla) = \sum_{j=1}^{d} A_j \partial_{x_j},
\]

the matrices \( A_j \) are symmetric, \( F_2 \) is a smooth function, and \( F_3(\cdot, \cdot) \) a smooth bilinear function. The equations satisfied by \( \tilde{U} \) are

\[
\left( \partial_t + \tilde{C}(\nabla) + iD\Delta \right) \tilde{U} = \begin{pmatrix} 0 & 0 \\ 0 & iM(A_C, A_R, \partial) \end{pmatrix} \tilde{U} + i\tilde{F}_1(\tilde{U}) + \tilde{B}(N, \tilde{U}), \quad (5.6)
\]

where \( \tilde{B}(\cdot, \cdot) \) is a bilinear mapping, \( M(A_C, A_R, \partial) \) is the symmetric operator defined by (4.26), and

\[
\tilde{C} = \sum_{j=1}^{d} \tilde{C}_j \partial_{x_j}
\]

corresponds to the order-one terms in (5.1)–(5.2). Now using the real form of \( \tilde{U} \), one obtains the following system for \( U = (\text{Re}(\tilde{U}), \text{Im}(\tilde{U})) \):

\[
\partial_t U + C(\nabla)U + D\Delta U = \sum_{j=1}^{d} A_j(U) \partial_{x_j} U + F_1(U) + B(N, U), \quad (5.7)
\]

where

\[
C(\nabla) = \sum_{j=1}^{d} C_j \partial_{x_j},
\]

with \( C_j \) being constant symmetric matrices. \( D \) is a skew-symmetric nonsingular matrix, \( B(\cdot, \cdot) \) is a (possibly nonlocal of order 0) bilinear mapping, and for all \( j \), \( A_j(U) \) is symmetric and \( U \mapsto A_j(U) \) is linear. Moreover, system (5.5) becomes

\[
\partial_t N + E(\nabla)N + F_2(U) + \nabla(F_3(U, U)) = 0, \quad (5.8)
\]

where \( F_2 \) and \( F_3 \) are smooth functions. We will now apply Ozawa-Tsutsumi’s method to (5.7). Set \( A = \partial_t U \). The system satisfied by \( A \) is

\[
\partial_t A + C(\nabla)A + D\Delta A = B(\partial_t N, U) + B(N, A) + F_1(U)A + \sum_{j=1}^{d} A_j(U) \partial_{x_j} A + \sum_{j=1}^{d} A_j(A) \partial_{x_j} U, \quad (5.9)
\]
and
\[ A + C(\nabla)U + D\Delta U = \sum_{j=1}^{d} A_j(U) \partial_{x_j} U + F_1(U) + B(N, U). \]  \hspace{1cm} (5.10)

Let us consider an initial datum \( U_0 \in H^{s+2}, \, A_0 \in H^{s} \) given by (5.10) and \( N_0 \in H^{s+1} \). Just as in \([12]\), we first transform (5.10) by adding \(-\alpha D U \) to both sides of (5.10):
\[ A + C(\nabla)U + D(-\alpha + \Delta)U = \sum_{j=1}^{d} A_j(U) \partial_{x_j} U + F_1(U) + B(N, U) - \alpha DU. \]  \hspace{1cm} (5.11)

We now consider a linearized version of (5.9)–(5.11) and (5.8) as follows. Take \( A \in C([0,T]; H^s) \) and \( N \in C([0,T]; H^{s+1}) \cap C^1([0,T]; H^s) \), and construct new functions \( B \) and \( P \) by the following procedure. First construct \( U \in C([0,T]; H^s) \) by
\[ U = U_0 + \int_0^t A(s) ds, \]  \hspace{1cm} (5.12)
and obtain \( V \) by solving
\[ A + C(\nabla) V + D(-\alpha + \Delta) V = \sum_{j=1}^{d} A_j(U) \partial_{x_j} V + F_1(U) + B(N, U) - \alpha DU. \]  \hspace{1cm} (5.13)

We will obtain a solution to (5.13) for \( \alpha \) large enough thanks to the fact that \( A_j(U) \) are symmetric. Now take \((B, P)\) as being the solutions to
\[
\begin{align*}
\partial_t B + C(\nabla)B + D\Delta B &= B(\partial_t N, U) + B(N, B) + F_1(U)B \\
+ \sum_{j=1}^{d} A_j(B) \partial_{x_j} V + \sum_{j=1}^{d} A_j(A) \partial_{x_j} B,
\end{align*}
\]  \hspace{1cm} (5.14)
\[
\partial_t P + E(\nabla)P + F_2(V) + \nabla(F_3(V, V)) = 0.
\]  \hspace{1cm} (5.15)

Note that the symmetry of the matrices \( A_j \) will play a crucial role in the resolution of (5.14). Let \( T : (A, N) \rightarrow (B, P) \). The proof consists in showing that \( T \) is a contraction on a suitable ball of \( C([0,T]; H^s) \). We split the proof into five steps. Take \( R = 2(|U_0|_{H^s} + |N_0|_{H^s}) \), and consider \( B(R) \) the ball of radius \( R \) in \( C([0,T]; H^s(\mathbb{R}^d) \times H^{s+1}(\mathbb{R}^d)) \).

• **Step 1.** Solving the elliptic equation (5.13). Assume that \((A, N) \in B(R)\). Then
\[ |U|_{H^s} \leq |U_0|_{H^s} + TR \leq R, \]
provided that $T$ is small enough. Applying $D^{-1}$ on (5.13) yields

$$D^{-1}C(\nabla)V + (-\alpha + \Delta)V - \sum_{j=1}^{d} D^{-1}A_j(\mathcal{U})\partial_{x_j}V = D^{-1}F,$$

where

$$F = -A + F_1(\mathcal{U}) + B(\mathcal{N}, \mathcal{U}) - \alpha DU.$$

**Lemma 5.1.** The bilinear form

$$a(V_1, V_2) = \int \left( -D^{-1}C(\nabla)\cdot\nabla V_1 + \alpha V_1 \cdot V_2 + \nabla V_1 \cdot \nabla V_2 + \sum_{j=1}^{d} D^{-1}A_j(\mathcal{U})\partial_{x_j}V_1 \cdot V_2 \right)$$

is elliptic for $\alpha$ large enough.

**Proof.** Compute

$$I = \left| \int \left( -D^{-1}C(\nabla)\cdot\nabla V_1 + \alpha V_1 \cdot V_2 + \nabla V_1 \cdot \nabla V_2 + \sum_{j=1}^{d} D^{-1}A_j(\mathcal{U})\partial_{x_j}V_1 \cdot V_2 \right) \right|$$

$$\leq K \left( |V|_{L^2}^2 |\nabla V|_{L^2}^2 + |U|_{\infty} |V|_{L^2} |\nabla V|_{L^2} \right).$$

Recall that $|U|_{\infty} \leq K'' |U|_{H^s} \leq K'' R$, so that

$$I \leq K'' \left( 1 + R^2 \right) |V|_{L^2}^2 + \frac{1}{2} |\nabla V|_{L^2}^2.$$

Take $\alpha = 2K'' \left( 1 + R^2 \right)$. This provides

$$a(V, V) \geq \frac{\alpha}{2} |V|_{L^2}^2 + \frac{1}{2} |\nabla V|_{L^2}^2. \quad \square$$

Then using usual arguments for elliptic equations, one obtains that there exists a unique $V \in H^{s+2}$ solution to (5.13) such that $|V|_{H^{s+2}} \leq C_1(R) |F|_{H^s}$, and

$$|V|_{L^\infty(0,T;H^{s+2})} \leq C_1(R) \left( |A|_{L^\infty(0,T;H^s)} + C(|U|_{L^\infty(0,T;H^s)}) + |\mathcal{N}|_{L^\infty(0,T;H^s)} |U|_{L^\infty(0,T;H^s)} + |U|_{L^\infty(0,T;H^s)} \right),$$

since $|F(U)|_{H^s} \leq C(|U|_{H^s})$, where $x \mapsto C(x)$ is a continuous function. It follows that there exists $C_2(R)$ such that

$$|V|_{L^\infty(0,T;H^{s+2})} \leq C_2(R). \quad (5.16)$$

**Step 2.** Resolution of the dispersive equation (5.14). We first use a long-wave-type regularization of (5.14): take $\mathcal{B}^\varepsilon$ solution to

$$\partial_t (1 - \varepsilon \Delta) \mathcal{B}^\varepsilon + C(\nabla) \mathcal{B}^\varepsilon + D\Delta \mathcal{B}^\varepsilon = B(\partial_t \mathcal{N}, \mathcal{U}) + B(\mathcal{N}, \mathcal{B}^\varepsilon) + F_1'(\mathcal{U}) \mathcal{B}^\varepsilon$$
with $B^\varepsilon(0) = B_0$. There exists a unique solution $B^\varepsilon \in \mathcal{C}([0,T]; H^s)$ to (5.17) since (5.17) is a linear, zero-order (in space) equation. We will now perform energy estimates on (5.17) and we will use several times the following inequality:

$$\forall (u,v) \in L^\infty \cap H^s, \forall (\alpha, \beta) \text{ such that } |\alpha| + |\beta| = s, \quad |\partial^\alpha u \partial^\beta v|_{L^2} \leq C \left( |u|_{L^\infty} |v|_{H^s} + |u|_{H^s} |v|_{L^\infty} \right).$$

(5.18)

See [2]. Note that $\partial^\alpha$ denotes here any spatial derivative of order $\alpha$.

**Proposition 5.1.** There exists $C_5(R) > 0$ such that

$$|B^\varepsilon|_{L^\infty(0,T; H^s)} \leq |B_0|_{H^s} + C_5(R)T.$$  

Proof. $L^2$ estimate: multiply (5.17) by $B^\varepsilon$ and get

$$
\frac{1}{2} \partial_t \int (|B^\varepsilon|^2 + \varepsilon |\nabla B^\varepsilon|^2) \leq C \left( |\partial_t \mathcal{N}|_{L^\infty} |\mathcal{U}|_{L^2} |B^\varepsilon|_{L^2} + |\mathcal{N}|_{L^\infty} |B^\varepsilon|_{L^2}^2 \right)
+ \sum_{j=1}^d |B^\varepsilon|_{L^2}^2 |\partial_x j \mathcal{V}|_{L^\infty} + C \left( |\mathcal{U}|_{L^\infty} |B^\varepsilon|_{L^2}^2 \right) + \sum_{j=1}^d \int A_j(\mathcal{A}) \partial_x j B^\varepsilon \cdot B^\varepsilon.
$$

Now, since $A_j(\mathcal{A})$ is symmetric, one obtains

$$
\sum_{j=1}^d \int A_j(\mathcal{A}) \partial_x j B^\varepsilon \cdot B^\varepsilon = -\frac{1}{2} \sum_{j=1}^d \int A_j(\partial_x j \mathcal{A}) B^\varepsilon \cdot B^\varepsilon,
$$

and this last quantity is controlled by $|\partial_x \mathcal{A}|_{L^\infty} |B^\varepsilon|_{L^2}^2$. Therefore, one has

$$
\int |B^\varepsilon|^2 \leq |B_0^\varepsilon|_{L^2}^2 + TC_5(R).
$$

(5.19)

$H^s$ estimate: one applies $\partial^s$ on (5.17) and takes the $L^2$ inner product with $\partial^s B^\varepsilon$. The only nonstraightforward term is the quasilinear one,

$$
\int A_j(\mathcal{A}) \partial^s \partial_x j B^\varepsilon \cdot \partial^s B^\varepsilon.
$$

Again, the symmetry of $A_j$ gives

$$
\int A_j(\mathcal{A}) \partial^s \partial_x j B^\varepsilon \cdot \partial^s B^\varepsilon = -\frac{1}{2} \int A_j(\partial_x j \mathcal{A}) \partial^s B^\varepsilon \cdot \partial^s B^\varepsilon,
$$

which is controlled by

$$
|\partial_x \mathcal{A}|_{L^\infty} |\partial^s B^\varepsilon|_{L^2}^2 \leq C |\mathcal{A}|_{H^s} |\partial^s B^\varepsilon|_{L^2}^2,
$$
since \( s > \frac{d}{2} + 1 \). We then obtain
\[
\frac{1}{2} \partial_t \left( |B|_{H^s}^2 + \varepsilon |B|_{H^{s+1}}^2 \right) \leq C_3(R) |B|_{H^s}^2 + C_4(R) |B|_{H^{s+1}}^2,
\]
from which Proposition (5.1) follows. \( \square \)

Letting \( \varepsilon \to 0 \) gives a solution \( B \in L^\infty(0, T; H^s) \cap \mathcal{C}([0, T]; H^{s-\eta}) \) \( \forall \eta > 0 \), to (5.14) satisfying
\[
|B|_{L^\infty(0, T; H^s)} \leq |B|_{H^s} + C_5(R) T.
\] (5.20)

**Step 3.** Solving the “wave equation” (5.15). Note that
\[
|\nabla F_2(V, V)|_{L^\infty(0, T; H^{s+1})} \leq C |V|_{L^\infty(0, T; H^{s+1})},
\]
and
\[
|V|_{L^\infty(0, T; H^{s+2})} \leq C_2(R),
\]
thanks to (5.16). It follows that
\[
|P|_{L^\infty(0, T; H^{s+1})} \leq |N_0|_{H^s} + C_6(R) T.
\] (5.21)

Inequalities (5.20) and (5.21) show the following:

**Proposition 5.2.** There exists \( T_0 > 0 \) such that if \( T \leq T_0 \), \( T \) maps \( B(R) \) into itself.

**Step 4.** \( T \) is a contraction in \( L^\infty(0, T; L^2) \) if \( T \) is small enough. It is obtained as usual. The crucial point is again the symmetry of \( A_j(\mathcal{U}) \).

**Proposition 5.3.** There exists \( T_1 \leq T_0 \) such that if \( T \leq T_1 \), \( T \) is a contraction mapping on \( B(R) \) in the \( L^\infty(0, T; L^2) \) norm.

It follows that there exists a unique \((\mathcal{A}, \mathcal{N}) \in B(R)\) such that \( T(\mathcal{A}, \mathcal{N}) = (\mathcal{A}, \mathcal{N}) \), that is, a unique solution of
\[
\partial_t \mathcal{A} + C(\nabla) \mathcal{A} + D \Delta \mathcal{A} = B(\partial_t \mathcal{N}, \mathcal{U}) + B(\mathcal{N}, \mathcal{A}) + \sum_{j=1}^{d} A_j(\mathcal{A}) \partial_x \mathcal{V}
\]
\[
+ \sum_{j=1}^{d} A_j(\mathcal{A}) \partial_x \mathcal{A} + F_1(\mathcal{U}) \mathcal{A},
\]
\[
\partial_t \mathcal{N} + E(\nabla) \mathcal{N} + F_2(\mathcal{V}) + \nabla (F_3(\mathcal{V}, \mathcal{V})) = 0,
\]
where \( \mathcal{U} \) is given by
\[
\mathcal{U} = \mathcal{U}_0 + \int_0^t \mathcal{A}(s) ds,
\]
and $V$ is the solution to the elliptic equation

$$A + C(\nabla)V + D(-\alpha + \Delta)V = \sum_{j=1}^{d} A_j(U) \partial_{x_j} V + F_1(U) + B(N, U) - \alpha D U.$$  \hspace{1cm} (5.22)

- **Step 5.** Going back to the original problem. We still have to prove that $V = U$, which is done by making an estimate on the elliptic equation (5.22) just as in [12]. This ends the proof of Theorem (5.1).

6. SOME COMPLEMENTS

6.1. Invariants. System (3.6), (3.7), (3.8), and (3.9) has some invariants that we will make more precise now. We rewrite this system using the relations of Section 4:

$$i(\partial_t + v_C \partial_y) + \alpha \Delta) A_C = \frac{b^2}{2} \langle \delta n \rangle A_C - (\nabla \cdot E_0) A_R e^{-i\theta},$$  \hspace{1cm} (6.1)

$$i(\partial_t + v_R \partial_y) + \beta \Delta) A_R = \frac{bc}{2} \langle \delta n \rangle A_R - (\nabla \cdot E_0^*) A_C e^{i\theta},$$  \hspace{1cm} (6.2)

$$i \partial_t E_0 + \gamma \Delta E_0 = \frac{b}{2} \nabla \Delta^{-1} \text{div}(\langle \delta n \rangle E_0) + \nabla \left(A_R^* A_C e^{i\theta}\right),$$  \hspace{1cm} (6.3)

$$\partial^2_t (\delta n) - v_s^2 \Delta (\delta n) = a \Delta \left(|E_0|^2 + b |A_C|^2 + c |A_R|^2\right),$$  \hspace{1cm} (6.4)

where $\theta = k_1 y - \omega_1 t$ and

$v_C = \frac{k_0^2 c^2}{\omega_0^2}$, $v_R = \frac{k_R k_0 c^2}{\omega_R \omega_0}$, $\alpha = \frac{c^2 k_0^2}{2 \omega_0^2}$, $\beta = \frac{c^2 k_0^2}{2 \omega_R \omega_0}$,

$\gamma = \frac{v_s^2 k^2}{2 \omega_R \omega_0}$, $a = \frac{4 m_e \omega_0 \omega_R}{m_i \omega_R}$, $b = \frac{\omega_{pe}}{\omega_0}$, $c = \frac{\omega_{pe}}{\omega_R}$, $v_s = \frac{c_s k_0}{\omega_0}$.

**Proposition 6.1.** For any regular solution of (6.1)–(6.4), one has

$$\frac{d}{dt} \int \left(2|A_C|^2 + |A_R|^2 + |E_0|^2\right) = 0.$$

**Proof.** Multiply (6.1) by $2 A_C^*$, (6.2) by $A_R^*$, and (6.3) by $E_0^*$; integrating over $\mathbb{R}^d$, summing the results, and taking the imaginary part gives

$$\frac{1}{2} \partial_t \int \left(2|A_C|^2 + |A_R|^2 + |E_0|^2\right)$$

$$= \text{Im} \int \left( -2\nabla \cdot E_0 A_R A_C e^{-i\theta} - \nabla \cdot E_0^* A_R^* A_C e^{i\theta} + \nabla (A_R^* A_C e^{i\theta}) E_0^* \right)$$

$$= \text{Im} \int \left( 2\nabla \cdot E_0^* A_R^* A_C e^{i\theta} - \nabla \cdot E_0^* A_R^* A_C e^{i\theta} - A_R^* A_C \nabla \cdot E_0^* e^{i\theta} \right) = 0. \quad \Box$$
Note that this does not mean that the quasilinear part is hyperbolic. In fact, as noticed before, it is not! For some particular values of the parameters, we have a Hamiltonian structure:

**Proposition 6.2.** Assume that \( \omega_1 = 0 \) (that is \( \theta = k_1 y \)). Then, for any regular solution to (6.1)–(6.4) such that \( \delta n \in C([0,T]; H^{-1}) \)

\[
\frac{d}{dt} \left( \int \frac{1}{2} \left( \alpha |\nabla A_C|^2 + \beta |\nabla A_R|^2 + \gamma |E_0|^2 \right) + \delta n \left( \frac{b_1^2}{2} |A_C|^2 + \frac{bc}{2} |A_R|^2 \right) \\
+ \frac{b}{2} |E_0|^2 + \frac{b v_s^2}{8a} (|\delta n|^2 + \frac{1}{v_s^2} |\nabla \Delta^{-1} \partial_t (\delta n)|^2) \right) - \text{Re} \int \left( \nabla \cdot E_0^* A_R A_C e^{i\theta} \right) \\
+ \frac{1}{2} \text{Im} \int (v_C \partial_y A_C A_C^* + v_R \partial_y A_R A_R^*) = 0
\]

**Proof.** Multiply (6.1) by \( \partial_t A_C^* \), (6.2) by \( \partial_t A_R^* \), and (6.3) by \( \partial_t E_0^* \); summing the results, taking the real part, and integrating over \( \mathbb{R}^d \) leads to

\[
- \text{Im} \int (v_C \partial_y A_C \partial_t A_C^* + v_R \partial_y A_R \partial_t A_R^*) \\
- \frac{1}{2} \partial_t \int (\alpha |\nabla A_C|^2 + \beta |\nabla A_R|^2 + \gamma |E_0|^2) \\
= \frac{1}{2} \int \delta n \left( \frac{b_1^2}{2} \partial_t |A_C|^2 + \frac{bc}{2} \partial_t |A_R|^2 + \frac{b}{2} \partial_t |E_0|^2 \right) \\
+ \text{Re} \int (\nabla \cdot E_0^* A_R \partial_t A_C^* e^{-i\theta} - \nabla \cdot E_0^* A_C \partial_t A_R^* e^{i\theta}) \\
+ \text{Re} \int (A_R^* A_C e^{i\theta}) \partial_t E_0^*.
\]

(6.5)

We compute each term separately.

\[
\text{Im} \int (v_C \partial_y A_C \partial_t A_C^* + v_R \partial_y A_R \partial_t A_R^*) \\
= -\text{Im} \int (v_C A_C \partial_t \partial_y A_C^* + v_R A_R \partial_t \partial_y A_R^*). \\
= \text{Im} \int (v_C A_C^* \partial_t \partial_y A_C + v_R A_R^* \partial_t \partial_y A_R).
\]

Therefore,

\[
\text{Im} \int (v_C \partial_y A_C \partial_t A_C^* + v_R \partial_y A_R \partial_t A_R^*) \\
= \frac{1}{2} \text{Im} \partial_t \int (v_C \partial_y A_C A_C^* + v_R \partial_y A_R A_R^*).
\]

(6.6)
Moreover,

\[
\int \delta n \left( \frac{b^2}{2} \partial_t |AC|^2 + \frac{bc}{2} \partial_t |AR|^2 + \frac{b}{2} \partial_t |E_0|^2 \right)
\]
\[= \partial_t \int \delta n \left( \frac{b^2}{2} |AC|^2 + \frac{bc}{2} |AR|^2 + \frac{b}{2} |E_0|^2 \right)
\]
\[= - \int \partial_t \delta n \left( \frac{b^2}{2} |AC|^2 + \frac{bc}{2} |AR|^2 + \frac{b}{2} |E_0|^2 \right). \tag{6.7}\]

The last term is

\[
\Re \int (\nabla \cdot E_0 A_R \partial_t A_* e^{-i\theta} - \nabla \cdot E_0^* A_C \partial_t A_R e^{i\theta} + \nabla (A_R^* A_C e^{i\theta}) \partial_t E_0^*)
\]
\[= - \Re \int \partial_t (\nabla \cdot E_0^* A_R^* A_C) e^{i\theta}. \tag{6.8}\]

Now (6.6), (6.7), and (6.8) in (6.5) provide

\[
\partial_t \left( -\text{Im} \frac{1}{2} \int (v_C \partial_y A_C A_C^* + v_R \partial_y A_R A_R^*) \right)
\]
\[= - \frac{1}{2} \int (\alpha |\nabla A_C|^2 + \beta |\nabla A_R|^2 + \gamma |\nabla E_0|^2)
\]
\[- \frac{1}{2} \int \delta n \left( \frac{b^2}{2} |AC|^2 + \frac{bc}{2} |AR|^2 + \frac{b}{2} |E_0|^2 \right)
\]
\[= - \frac{1}{2} \int \partial_t \delta n \left( \frac{b^2}{2} |AC|^2 + \frac{bc}{2} |AR|^2 + \frac{b}{2} |E_0|^2 \right)
\]
\[= - \Re \int \partial_t (\nabla \cdot E_0^* A_R^* A_C) e^{i\theta}. \tag{6.9}\]

We still have to deal with the two terms in the right-hand side of (6.9). For the second one, recalling that \(\theta = k_1 y - \omega_1 t\), we take \(\omega_1 = 0\) (which is not a physical case; see next section) and we get

\[
\Re \int \partial_t (\nabla \cdot E_0^* A_R^* A_C) e^{i\theta} = \partial_t \int \Re (\nabla \cdot E_0^* A_R^* A_C e^{i k_1 y}). \tag{6.10}\]

For the first term, we need to write the first-order equation satisfied by \(\delta n\). Introducing \(V = \frac{1}{v_s} \nabla \Delta^{-1} \partial_t \delta n\) gives

\[
\partial_t \delta n = v_s \nabla \cdot V, \tag{6.11}\]
\[
\partial_t V = v_s \nabla \delta n + \frac{a}{v_s} \nabla(|E_0|^2 + b|AC|^2 + c|AR|^2). \tag{6.12}\]
Multiplying (6.11) by $\delta n$ and (6.12) by $V$ provides
\[
\frac{1}{2} \partial_t \int (|\delta n|^2 + V^2) = -\frac{a}{\nu_s} \int (|E_0|^2 + b|A_C|^2 + c|A_R|^2) \nabla \cdot V,
\]
\[
= -\frac{a}{\nu_s^2} \int (|E_0|^2 + b|A_C|^2 + c|A_R|^2) \partial_t \delta n.
\]
Then
\[
\int \partial_t \delta n \left( \frac{b^2}{2} |A_C|^2 + \frac{bc}{2} |A_R|^2 + \frac{b}{2} |E_0|^2 \right) = \frac{b}{2} \int \partial_t \delta n \left( |b|A_C|^2 + c|A_R|^2 + |E_0|^2 \right) = -\frac{b v_s^2}{4a} \int (|\delta n|^2 + V^2).
\]
Using (6.10) and (6.13) in (6.9) gives
\[
\partial_t \left( \int \left( \frac{1}{2} \alpha |\nabla A_C|^2 + \beta |\nabla A_R|^2 + \gamma |\nabla E_0|^2 + \delta n \left( \frac{b^2}{2} |A_C|^2 + \frac{bc}{2} |A_R|^2 \right) + \frac{b}{2} |E_0|^2 \right) + \frac{b v_s^2}{8a} (|\delta n|^2 + \frac{1}{v_s^2} |\nabla \Delta^{-1} \partial_t \delta n|^2) \right) - \text{Re} \int \left( \nabla \cdot \left( E_0^* A_R^* A_C e^{i\theta} \right) \right) + \frac{1}{2} \text{Im} \int (v_C \partial_y A_C A_C^* + v_R \partial_A R A_R^*) = 0,
\]
which ends the proof of Proposition 6.2. \hfill \Box

At this point, we do not know how to use this conservation law.

6.2. Back to the resonance condition. In order to obtain our model, we imposed the following relation (2.31): $k_0 = k_R + k_1$, $\omega_0 = \omega_R + \omega_{pe} + \omega_1$, where $\omega_0^2 = \omega_{pe}^2 + k_0^2 c^2$, $\omega_R^2 = \omega_{pe}^2 + k_R^2 c^2$. In fact, in the physical literature, one finds $(\omega_{pe} + \omega_1)^2 = \omega_{pe}^2 + v_{th}^2 k_1^2$. This is the classical three-wave resonance.

At the first approximation, this gives (using $\omega_1 \ll \omega_{pe}$)
\[
2 \omega_1 \omega_{pe} \simeq v_{th}^2 k_1^2 \quad \Rightarrow \quad \omega_1 \simeq \frac{v_{th}^2 k_1^2}{2 \omega_{pe}}.
\]

Now rewrite (2.38):
\[
i \partial_t E_0 + \frac{v_{th}^2}{2 \omega_{pe}} \Delta E_0 = \frac{\omega_{pe}}{2} \nabla \Delta^{-1} \text{div}(\langle \delta n \rangle E_0) + \frac{e \omega_{pe}}{2 c^2 m_e} \nabla (A_R^* A_0 e^{i(k_1 y - \omega_1 t)}).
\]

The dispersion relation of the linear part is $\omega = \frac{v_{th}^2 k_1^2}{2 \omega_{pe}}$. Equation (6.14) means that $(k_1, \omega_1)$ satisfies the dispersion relation, and therefore the term $A_R^* A_0 e^{i(k_1 y - \omega_1 t)}$ is resonant and the process will be efficient. More precisely, since $v_{th} = \lambda_D \omega_{pe}$ where $\lambda_D$ is the Debye’s length, the equation on $E_0$ can
be written in one dimension (neglecting $\delta n$ and considering $A_R$ and $A_0$ as given):

$$i\partial_t E_0 + \frac{\lambda^2_0 \omega_{pe}}{2} \partial^2_y E_0 = A e^{i(k_1 y - \omega_1 t)}. \quad (6.15)$$

Let $L$ be a characteristic scale in space, $T$ in time. The dimensionless form of (6.15) reads

$$i\partial_t E_0 + \frac{\lambda^2_0 \omega_{pe} T}{2L^2} \partial^2_y E_0 = A T e^{i(k_1 L y - (\omega_1 T)t)}.$$

Introduce $k_1 L = \frac{k}{\varepsilon}$ and $\omega_1 T = \frac{\omega}{\varepsilon}$, and replace $AT$ by $A$. Since $\omega_1 = \omega_{pe} \lambda^2_0 k_1^2$ and $\omega_1 T = \frac{\omega}{\varepsilon}$ (with $\omega, k = O(1)$), one gets

$$\omega_{pe} \lambda^2_0 T = \frac{\omega_1 T}{k_1^2 L^2} = \frac{\varepsilon^2 \omega}{k^2 \varepsilon} = \frac{\omega}{k^2 \varepsilon} = O(\varepsilon).$$

The dimensionless form of (6.15) is therefore

$$i\partial_t E_0 + \frac{\varepsilon \omega}{k^2} \partial^2_y E_0 = A e^{i(k y - \omega t)}. \quad (6.16)$$

A standard WKB expansion shows that $E_0 = B e^{i(k y - \omega t) \varepsilon} + O(\varepsilon)$, with $i(\partial_t B + \frac{\omega}{k} \partial_y B) = A$. This means a linear growth in time for $B$ at the group velocity $\frac{\omega}{k}$. On the other hand, if the equation is

$$i\partial_t E_0 + \frac{\varepsilon \omega}{k^2} \partial^2_y E_0 = A e^{i(k y - \omega' t) \varepsilon},$$

with $\omega' \neq \omega$, then $E_0$ will stay of size $O(\varepsilon)$. Of course, nonlinear versions of these results can be proved. We postpone the numerical study of this problem to a further work.

6.3. **Open problems.** The existence and stability of solitary waves for these equations is open. It can certainly be done for the coefficient corresponding to Proposition 6.2. It would be interesting also to have some existence proofs for system (6.1)–(6.4) with an existence time and bounds on the solutions which are independent of $k_1$ and $\omega_1$ in order to investigate the limit $k_1 \to +\infty$ and $\omega_1 \to +\infty$ as in Section 6.2. Our proof is clearly not uniform with respect to these parameters.

**References**


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