Orbital stability of standing waves for the nonlinear Schrödinger equation coupled with the Maxwell equation

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Abstract

In this paper, we study the orbital stability of standing waves for the nonlinear Schrödiger equation coupled with the Maxwell equation. Firstly we describe conditions for the existence of minimizers with prescribed charge in terms of a coupling constant e. Secondly we study the existence of ground states for the stationary problem, the uniqueness of ground states for small e and a link between minimizers and ground states. Finally we obtain the orbital stability for the quadratic nonlinearity.

Résumé

Dans cet article, nous étudions la stabilité orbitale des ondes solitaires pour un système couplant une équation de Schrödinger nonlinéaire avec les équations de Maxwell. Nous montrons d'abord que le problème consistant à minimiser l'énergie associée aux équations sous une contrainte L^2 admet une solution pour des valeurs particulières d'un paramètre de couplage e. Ensuite, nous prouvons l'existence et l'unicité pour des petites valeurs de e d'états fondamentaux pour le problème stationnaire associé. Finalement, nous montrons la stabilité orbitale de ces ondes dans le cas où l'équation de Schrödinger possède une nonlinéarité quadratique.

Key words: Schrödinger-Maxwell system, constraint minimization problem, ground states, orbital stability of standing waves.

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1 Introduction and main results

In this paper, we consider stability issues of solitary waves for the following nonlinear Schrödinger equation coupled with Maxwell equation:

$$i\psi_t + \Delta\psi = e\phi\psi + e^2|\mathbf{A}|^2\psi + ie\psi\,\mathrm{div}\mathbf{A} + 2ie\nabla\psi\cdot\mathbf{A} - |\psi|^{p-1}\psi. \tag{1.1}$$

$$\mathbf{A}_{tt} - \Delta\mathbf{A} = e\,\mathrm{Im}(\bar{\psi}\nabla\psi) - e^2|\psi|^2\mathbf{A} - \nabla\phi_t - \nabla\mathrm{div}\mathbf{A}. \tag{1.2}$$

$$-\Delta\phi = \frac{e}{2}|\psi|^2 + \mathrm{div}\mathbf{A}_t. \tag{1.3}$$

where $\psi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$, $\mathbf{A} : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$, $\phi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$, $e \in \mathbb{R}$, $1 < p < 5$ and i denotes the unit complex number, that is, $i^2 = -1$.

System (1.1)-(1.3) describes the interaction of the Schrödinger wave function ψ with the gauge potential $(\mathbf{A}, \phi)$. More precisely, a particle moves in an external electromagnetic field which is represented by the gauge potential $(\mathbf{A}, \phi)$, and the wave function ψ produces a current which acts as a force for the electromagnetic field. For the derivation, see Section 2 below. The constant e describes the strength of the interaction and plays an important role in our analysis. When $e = 0$, the equation (1.1) reduces to the standard nonlinear Schrödinger equation:

$$i\psi_t + \Delta\psi + |\psi|^{p-1}\psi = 0, \tag{1.4}$$

and there is several papers dealing with the orbital stability of standing waves for (1.4), see e.g. [13], [15].

It is known that System (1.1)-(1.3) has a so-called *gauge ambiguity*. Namely if $(\psi, \mathbf{A}, \phi)$ is a solution of (1.1)-(1.3), then $(e^{ie\chi}\psi, \mathbf{A}+\nabla\chi, \phi+\chi_t)$ is also a solution of (1.1)-(1.3) for any smooth function $\chi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$. Thus in order to guarantee the uniqueness of solutions for the initial value problem, we need to choose a suitable gauge condition. In this paper, we impose the Coulomb condition, that is, we look for a solution $\mathbf{A}$ which satisfies

$$\mathrm{div}\mathbf{A} = 0. \tag{1.5}$$

In this setting, (1.3) is reduced to $-\Delta\phi = \frac{e}{2}|\psi|^2$ and hence ϕ can be explicitly expressed by $\phi = \frac{e}{2}(-\Delta)^{-1}|\psi|^2$. Moreover one has $|\,\mathrm{rot}\mathbf{A}|^2 = |\nabla\mathbf{A}|^2$, which is useful for the stability analysis of standing waves. It is known that if

$$\mathrm{div}\mathbf{A}(0,\cdot) = \mathrm{div}\mathbf{A}_t(0,\cdot) = 0,$$

then (1.5) holds for all $t > 0$. (See [18], [33].) Finally from (1.7), we see that (1.1) can be written as

$$i\psi_t + L_A\psi - V(x)\psi + |\psi|^{p-1}\psi, \tag{1.6}$$

where V is the non-local potential: $V(x) = \frac{e^2}{2}(-\Delta)^{-1}|\psi|^2$ and L_A is the *magnetic Schrödinger operator* which is defined by $\mathbf{A} = (A_1, A_2, A_3)$ and

$$L_A\psi := \sum_{j=1}^{3}\left(\frac{\partial}{\partial x_j} - ieA_j(x)\right)^2\psi = \Delta\psi - 2ie\nabla\psi\cdot\mathbf{A} - e^2|\mathbf{A}|^2\psi.$$

The (in)stability of standing waves of (1.6) for *given* divergence-free magnetic potential $\mathbf{A}$ and given potential V has been studied in [14], [22].

For $\omega > 0$ and $(u, \phi) \in H^1(\mathbb{R}^3, \mathbb{C}) \times D^{1,2}(\mathbb{R}^3, \mathbb{R})$, we consider the standing wave for (1.1)-(1.3) of the form:

$$\psi(x,t) = u(x)e^{i\omega t},\ \mathbf{A}(x,t) = \mathbf{0} \text{ and } \phi(x,t) = \phi(x), \tag{1.7}$$

where $D^{1,2}(\mathbb{R}^3)$ denotes the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^3}|\nabla u|^2\,dx$. Plugging (1.7) into (1.1)-(1.3), we obtain the following elliptic system:

$$\begin{cases} -\Delta u + \omega u + e\phi u = |u|^{p-1}u. \\ -\Delta\phi = \frac{e}{2}|u|^2. \end{cases} \tag{1.8}$$

One can see that (u, ϕ) satisfies (1.8) if it is a critical point of the functional $E(u, \phi) : H^1(\mathbb{R}^3, \mathbb{C}) \times D^{1,2}(\mathbb{R}^3, \mathbb{R}) \to \mathbb{R}$ which is defined by

$$E_{e,\omega}(u,\phi) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + \omega|u|^2 + e\phi|u|^2 - |\nabla\phi|^2\right)dx - \frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\,dx.$$

We also note that by the maximum principle, $\phi \geq 0$ for $e \geq 0$ and $\phi \leq 0$ for $e \leq 0$ respectively. Thus by replacing (e, ϕ) by $(-e, -\phi)$ when $e \leq 0$, we may assume that $e \geq 0$ and $\phi \geq 0$.

Since the functional E is strongly indefinite, it is difficult to handle it directly. To overcome this difficulty, we adapt the reduction method as in [2], [35]. To this end, let $S(u) := \frac{1}{2}(-\Delta)^{-1}|u|^2 \in D^{1,2}(\mathbb{R}^3, \mathbb{R})$ and put

$$\begin{aligned} I_{e,\omega}(u) &:= E_{e,\omega}\big(u, eS(u)\big) \\ &= \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + \omega|u|^2\right)dx + \frac{e^2}{4}\int_{\mathbb{R}^3}S(u)|u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\,dx. \end{aligned}$$

Then one can see that if $u \in H^1(\mathbb{R}^3, \mathbb{C})$ is a critical point of $I(u)$, then $\big(u, eS(u)\big)$ is a solution of (1.8). Moreover the Euler-Lagrange equation for $I(u)$ is given by

$$-\Delta u + \omega u + e^2 S(u)u = |u|^{p-1}u. \tag{1.9}$$

We also note that $S(u)$ is explicitly given by

$$S(u) = (-\Delta)^{-1}\left(\frac{1}{2}|u|^2\right) = \frac{1}{4\pi|x|} * \left(\frac{1}{2}|u|^2\right) = \frac{1}{8\pi}\int_{\mathbb{R}^3}\frac{|u(y)|^2}{|x-y|}\,dy.$$

For $\omega > 0$ and $e > 0$, we denote by $m(\omega) = m_e(\omega)$ the ground state energy level for (1.9), that is,

$$m_e(\omega) = \inf\left\{I_{e,\omega}(u)\ ;\ I'_{e,\omega}(u) = 0,\ u \in H^1(\mathbb{R}^3,\mathbb{C}) \setminus \{0\}\right\}.$$

We also define the set of ground states $\mathcal{G}_e(\omega)$ by

$$\mathcal{G}_e(\omega) = \left\{u \in H^1(\mathbb{R}^3,\mathbb{C})\ ;\ I'_{e,\omega}(u) = 0,\ I_{e,\omega}(u) = m_e(\omega)\right\}.$$

The existence of solutions or ground states for (1.9) has been obtained in [2], [10], [23], [25], [35]. However the uniqueness of ground states is not known yet, which is one of our main results in this paper.

Next in order to study the stability of the standing wave, for $\mu > 0$, we consider the following minimization problem:

$$c_e(\mu) := \inf_{u \in B(\mu)} J_e(u), \tag{1.10}$$

where $B(\mu) = \{u \in H^1(\mathbb{R}^3,\mathbb{C}),\ \|u\|^2_{L^2(\mathbb{R}^3)} = \mu\}$ and

$$J_e(u) := \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx + \frac{e^2}{4}\int_{\mathbb{R}^3}S(u)|u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\,dx.$$

Note that the introduction of Problem (1.10) is motivated by the conservations laws derived in Section 2 for the evolution equations (1.1)-(1.3). We also define the set of minimizers $\mathcal{M}_e(\mu)$ by

$$\mathcal{M}_e(\mu) := \left\{u \in H^1(\mathbb{R}^3,\mathbb{C})\ ;\ \ u \in B(\mu),\ J_e(u) = c_e(\mu)\right\}.$$

In this setting, ω appears as a Lagrange multiplier associated with the L^2-constraint $\|u\|^2_{L^2(\mathbb{R}^3)} = \mu$. Moreover it is known that the key of the stability of standing waves is to study a link between $\mathcal{G}_e(\omega)$ and $\mathcal{M}_e(\mu)$, see [13], [15], [16].

Recently there has been a lot of works on the existence of minimizers of (1.10), see [4], [5], [6], [12], [24], [36]. Moreover the stability of standing waves for the *Schrödinger Poisson Slater* equation:

$$i\psi_t + \Delta\psi - e^2 S(\psi)\psi + |\psi|^{p-1}\psi = 0$$

has been studied in [5], [26], [36]. We also refer to [7], [31] for the stability result of the nonlinear Klein-Gordon equation coupled with the Maxwell equation. However there are only few results on the stability issue for (1.1)-(1.3), and this is another motivation of our paper.

Our main results in this paper are the followings.

Theorem 1.1 (Existence of minimizers with prescribed charge).
Let $\mu > 0$ be given and suppose that $1 < p < \frac{7}{3}$.

(i) *If $2 \le p < \frac{7}{3}$, there exists $e^* = e^*(\mu, p) > 0$ such that for $e < e^*$, $c_e(\mu)$ admits a minimizer $u_{e,\mu}$. On the other hand for $e > e^*$, $c_e(\mu)$ has no minimizer. Moreover it follows that $e^*(\mu, 2) = e^*(2) \le \frac{2}{3}$ for any $\mu > 0$.*

(ii) *If $1 < p < 2$, there exists $\tilde{e} = \tilde{e}(\mu, p) > 0$ such that for $e < \tilde{e}$, $c_e(\mu)$ admits a minimizer $u_{e,\mu}$.*

As we will see later, we can obtain quantitative and qualitative estimates of e^*, see (4.5) and (4.7). We also note that up to phase shift, the minimizer $u_{e,\mu}$ can be chosen to be real-valued. In the previous papers on the existence of minimizers of (1.10), e was chosen to be 1 and the condition for the existence has been described by the size μ or C_s, where C_s is a positive constant obtained by replacing $|u|^{p-1}u$ by $C_s|u|^{p-1}u$ in (1.9) and is referred as the *Slater* constant. However from the physical point of view, it is rather natural to describe the condition in terms of e, which is exactly what we did in Theorem 1.1. We also notice that the result $e^*(2) \le \frac{2}{3}$ is consistent with the non-existence result in [24] for the case $e = 1$ and $p = 2$.

The next result states the existence of ground states of (1.9), their uniqueness for sufficiently small e and their characterization for $p = 2$.

Theorem 1.2 (Existence of ground states and asymptotic uniqueness).
Let $\omega > 0$ be given. Suppose that $2 \le p < 5$ and $e < e^*$ if $p = 2$.

(i) *The problem (1.9) has a ground state $u_{e,\omega}$. Moreover if $p = 2$, there exists a unique $\mu(\omega) > 0$ such that u is a ground state of (1.9) if and only if u is a minimizer of $c_e\big(\mu(\omega)\big)$.*

(ii) *There exists $e_0 = e_0(\omega) > 0$ such that for $e < e_0$, the ground state of (1.9) is unique up to phase shift and translation, that is, it holds*

$$\mathcal{G}_e(\omega) = \big\{ e^{i\theta} u_{e,\omega}(\cdot - y) \ ; \ \theta \in [0, 2\pi), \ y \in \mathbb{R}^3 \big\}.$$

Moreover if $p = 2$, e_0 is independent of ω.

(iii) *If $p = 2$, for any $\mu > 0$ and $e < e_0$, the minimizer $u_{e,\mu}$ of (1.10) is unique up to phase shift and translation, that is, it holds*

$$\mathcal{M}_e(\mu) := \big\{ e^{i\theta} u_{e,\omega}(\cdot - y) \ ; \ \theta \in [0, 2\pi), \ y \in \mathbb{R}^3 \big\}.$$

Although the existence of ground states for $2 < p < 5$ has been shown in [2], the other parts in Theorem 1.2 are new. Especially the characterization of the minimizer set $\mathcal{M}_e(\mu)$ enables us to obtain the orbital stability of standing waves of (1.8). We emphasize that our uniqueness result has another interesting consequence. Indeed as we will see below, we cannot say *a priori* that any ground state of (1.9) is radially symmetric. However thanks to the uniqueness, we are able to obtain *a posteriori* the radiality of the ground state, see Remark 5.10 below. Especially as a consequence of Theorem 1.2, one can see that the unique ground state $u_{e,\omega}$ is positive, real-valued up to phase shift and radially symmetric with respect to the origin up to translation. If $p = 2$, the same conclusion holds true for the unique minimizer $u_{e,\mu}$.

Finally to obtain the orbital stability of standing waves, we impose the following initial conditions at $t = 0$:

$$\psi(0,x) = \psi_{(0)}(x),\ \mathbf{A}(0,x) = \mathbf{A}_{(0)}(x),\ \mathbf{A}_t(0,x) = \mathbf{A}_{(1)}(x),$$
$$\mathrm{div}\mathbf{A}_{(0)} = 0,\ \mathrm{div}\mathbf{A}_{(1)} = 0. \tag{1.11}$$

Let us discuss briefly the Cauchy problem associated with System (1.1)-(1.3). In a previous paper (see [18]), we have proved the local existence of solutions for a close set of equations (that is nonlinear Klein-Gordon-Maxwell system) in Sobolev spaces of high regularity. We believe that a similar result holds for System (1.1)-(1.3), assuming additional regularity conditions on the initial values $(\psi_{(0)}, \mathbf{A}_{(0)}, \mathbf{A}_{(1)})$. We postpone this question to a future work. We also refer to [3], [33] for results on the linear Schrödinger-Maxwell equation. The stability result is the following one.

Theorem 1.3 (Stability of standing waves)**.**

Let $\omega > 0$ be given. Suppose $p = 2$, $e < e_0$ and $u_{e,\omega}$ is the unique real-valued ground state of (1.9)*. Then the standing wave*

$$(\psi_{e,\omega}, \mathbf{A}_{e,\omega}, \phi_{e,\omega}) := \Big(u_{e,\omega}e^{i\omega t},\ \mathbf{0},\ \frac{e}{2}(-\Delta)^{-1}|u_{e,\omega}|^2\Big)$$

of (1.1)-(1.3) *is stable in the following sense: For every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if an initial value $(\psi_{(0)}, \mathbf{A}_{(0)}, \mathbf{A}_{(1)})$ satisfies* (1.11) *and*

$$\|\psi_{(0)} - u_{e,\omega}\|_{H^1} + \|\nabla \mathbf{A}_{(0)}\|_{L^2} + \|\mathbf{A}_{(1)}\|_{L^2} < \delta,$$

then the corresponding solution $(\psi, \mathbf{A}, \phi)$ of (1.1)-(1.3) *satisfies*

$$\sup_{t>0}\Bigg\{ \inf_{y\in\mathbb{R}^3} \Big(\big\|\nabla|\psi(t,\cdot)| - \nabla u_{e,\omega}(\cdot+y)\big\|_{L^2} + \Big\|\phi(t,\cdot) - \frac{e}{2}(-\Delta)^{-1}|u_{e,\omega}(\cdot+y)|^2\Big\|_{D^{1,2}}\Big)$$
$$+ \inf_{y\in\mathbb{R}^3}\inf_{\theta\in[0,2\pi)} \|\psi(t,\cdot) - e^{i\theta}u_{e,\omega}(\cdot+y)\|_{L^2} + \|\nabla\mathbf{A}(t,\cdot)\|_{L^2} + \|\mathbf{A}_t(t,\cdot)\|_{L^2}\Bigg\} < \varepsilon.$$

In the case $1 < p < \frac{7}{3}$, $p \neq 2$ and $e < \min\{e^*, \tilde{e}\}$, we are able to show the *stability of the minimizer set* $\mathcal{M}_e(\mu)$. For this result, we refer to Remark 6.2. A similar stability result for (1.1)-(1.3) has been already obtained in [8] by developing an abstract existence theory of solitons. Our approach is the standard variational one performed for example in [13], [15], [26], [34]. Our result shows that the standing wave is stable for small $e > 0$. But in physical point of view, it is also important to study the (non)existence of stable standing waves for large e. (See [27] for the nonlinear Klein-Gordon case.) Unfortunately our non-existence result of minimizers for large e does not imply the instability of standing waves. Thus the stability analysis for large e still remains open.

When $e = 0$, the problem (1.9) reduces to

$$-\Delta u + \omega u = |u|^{p-1}u \quad \text{in } \mathbb{R}^3. \tag{1.12}$$

We denote by $u_{0,\omega}$ a ground state of (1.12), and do the same for $I_{0,\omega}(u)$, $J_0(u)$, $m_0(\omega)$, $\mathcal{G}_0(\omega)$, $c_0(\mu)$ and $\mathcal{M}_0(\mu)$. It is well-known that the ground state $u_{0,\omega}$ is real-valued up to phase shift, positive, radially symmetric, unique up to translation and non-degenerate, that is,

$$\mathrm{Ker}(\mathcal{L}_{0,\omega}) = \mathrm{span}\left\{ \frac{\partial u_{0,\omega}}{\partial x_1}, \frac{\partial u_{0,\omega}}{\partial x_2}, \frac{\partial u_{0,\omega}}{\partial x_3} \right\}, \tag{1.13}$$

$$\mathcal{L}_{0,\omega} := -\Delta + \omega - p u_{0,\omega}^{p-1} \ : \ H^2(\mathbb{R}^3, \mathbb{R}) \to L^2(\mathbb{R}^3, \mathbb{R}).$$

Moreover one has a precise equivalence between $\mathcal{G}_0(\omega)$ and $\mathcal{M}_0(\mu)$. (See [13, Corollary 8.3.8] and Lemma 5.3 below.) This equivalence, the uniqueness of $u_{0,\omega}$ and a scaling property of (1.12) enable us to characterize $\mathcal{M}_0(\mu)$ as

$$\mathcal{M}_0(\mu) = \left\{ e^{i\theta} u_{0,\omega}(\cdot - y) \ ; \ \theta \in [0, 2\pi), \ y \in \mathbb{R}^3 \right\},$$

from which we derive the orbital stability of the standing wave. Our goal is to show that the same result holds for small $e > 0$. We can also see that the unique ground state of (1.9) is non-degenerate for small e, see Remark 5.10. Although this property is not used in the proof of Theorem 1.3, we believe that our non-degeneracy result will be useful in further stability analysis.

This paper is organized as follows. In Section 2, we briefly introduce the derivation of System (1.1)-(1.3). We prepare auxiliary results in Section 3. In Section 4, we prove Theorem 1.1. We consider the existence of ground states of (1.9) in Subsection 5.1 and show their uniqueness in Subsection 5.2. Finally in Section 6, we prove Theorem 1.3.

2 Derivation and conservation laws

In this section, we briefly introduce the derivation of System (1.1)-(1.3). We also prepare some conservation laws which we will use later on. For a while, let us consider the nonlinear Schrödinger equation:

$$i\psi_t + \Delta\psi = |\psi|^{p-1}\psi$$

and the corresponding Lagrangian:

$$\mathcal{L}_0 = \frac{1}{2}\big(\operatorname{Im}(\psi\bar{\psi}_t) - |\nabla\psi|^2\big) + \frac{1}{p+1}|\psi|^{p+1}. \tag{2.1}$$

If ψ is an electrically charged field, it must interacts with the Maxwell field. Let $\mathbf{E}$ and $\mathbf{H}$ be the electric and the magnetic fields respectively, and assume that they are described by the Gauge potential $(\phi, \mathbf{A})$, $\mathbf{A} = (A_1, A_2, A_3)$ as follows:

$$\mathbf{E} = \nabla\phi + \mathbf{A}_t \quad \text{and} \quad \mathbf{H} = \operatorname{rot}\mathbf{A}.$$

By the gauge invariance of the combined theory, the interaction between ψ and $(\phi, \mathbf{A})$ is given by exchanging the usual derivatives ∂_α with the *gauge covariant derivative*:

$$\mathcal{D}_\alpha = \partial_\alpha - ie\mathbf{A}_\alpha,\ \ \mathbf{A}_\alpha = (-\phi, A_1, A_2, A_3),$$

where $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$, $\alpha = 0, 1, 2, 3$ and $x_0 = t$. Thus, from (2.1), one can derive the following Lagrangian:

$$\mathcal{L}_0 = \frac{1}{2}\Big(\operatorname{Im}(\psi\bar{\psi}_t) - e\phi|\psi|^2 - \big|\nabla\psi - ie\mathbf{A}\psi\big|^2\Big) + \frac{1}{p+1}|\psi|^{p+1}.$$

Moreover since the Lagrangian corresponding to $\mathbf{E}$ and $\mathbf{H}$ is described by

$$\mathcal{L}_1 = \frac{1}{2}\left(|\mathbf{E}|^2 - |\mathbf{H}|^2\right) = \frac{1}{2}\left(|\nabla\phi + \mathbf{A}_t|^2 - |\operatorname{rot}\mathbf{A}|^2\right),$$

the total Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ is given by

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\Big(\operatorname{Im}(\psi\bar{\psi}_t) - e\phi|\psi|^2 - \big|\nabla\psi - ie\mathbf{A}\psi\big|^2\Big) + \frac{1}{p+1}|\psi|^{p+1}\\ &+ \frac{1}{2}\big|\nabla\phi + \mathbf{A}_t\big|^2 - \frac{1}{2}\big|\operatorname{rot}\mathbf{A}\big|^2.\end{aligned} \tag{2.2}$$

In this context, Equations (1.1)-(1.3) are no more than the Euler-Lagrange equations for $(\psi, \mathbf{A}, \phi)$ obtained from $\mathcal{L}$. For more details and physical backgrounds, see [20]. We also refer to [7], [18], [27] for related results on the nonlinear Klein-Gordon equation coupled with the Maxwell equations.

Next we introduce some conservation laws associated with (1.1)-(1.3). The first one corresponds to the conservation of charge. It is obtained by a straightforward computation on Equation (1.1) (multiply by $\bar{\psi}$, integrate over $\mathbb{R}^3$ and take the imaginary part)

$$\frac{d}{dt}\int_{\mathbb{R}^3}|\psi|^2\,dx = 0 \quad \text{(charge)}. \tag{2.3}$$

The second represents the conservation of energy. We multiply (1.1) by $\bar{\psi}_t$, integrate the resulting equation over $\mathbb{R}^3$ and take the real part. One obtains

$$\begin{aligned}\int_{\mathbb{R}^3}\Big\{\frac{\partial}{\partial t}\left(\frac{1}{2}|\nabla\psi|^2-\frac{1}{p+1}|\psi|^{p+1}\right)+e\phi\,\mathrm{Re}(\psi\bar{\psi}_t)+e^2|\mathbf{A}|^2\,\mathrm{Re}(\psi\bar{\psi}_t)\\ -\,\mathrm{Im}\Big(2e\bar{\psi}_t\nabla\psi\cdot\mathbf{A}+e\psi\bar{\psi}_t\mathrm{div}\,\mathbf{A}\Big)\Big\}\,dx=0.\end{aligned} \tag{2.4}$$

In the same spirit, we multiply $\mathbf{A}_t$ by (1.2) to get

$$\int_{\mathbb{R}^3}\Big\{\frac{\partial}{\partial t}\left(\frac{1}{2}|\mathbf{A}_t|^2+\frac{1}{2}|\,\mathrm{rot}\mathbf{A}|^2\right)-e\,\mathrm{Im}(\bar{\psi}\nabla\psi)\cdot\mathbf{A}_t+e^2|\psi|^2\mathbf{A}\cdot\mathbf{A}_t+\nabla\phi_t\cdot\mathbf{A}_t\Big\}\,dx=0, \tag{2.5}$$

where we used the fact

$$\mathrm{rot}\mathbf{A}\cdot\,\mathrm{rot}\mathbf{A}_t=\,\mathrm{div}(\mathbf{A}_t\times\,\mathrm{rot}\mathbf{A})+\mathbf{A}_t\cdot\nabla(\,\mathrm{div}\mathbf{A})-\mathbf{A}_t\cdot\Delta\mathbf{A}.$$

Finally we multiply ϕ_t by (1.3) to derive

$$\int_{\mathbb{R}^3}\Big\{\frac{\partial}{\partial t}\left(-\frac{1}{2}|\nabla\phi|^2\right)+\frac{e}{2}\phi_t|\psi|^2+\phi_t\,\mathrm{div}\mathbf{A}_t\Big\}\,dx=0. \tag{2.6}$$

Summing (2.4)-(2.6) up and using integration by parts, one has

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3}\Big\{\frac{1}{2}|\nabla\psi|^2+\frac{e}{2}\phi|\psi|^2-\frac{1}{p+1}|\psi|^{p+1}-\frac{1}{2}|\nabla\phi|^2+\frac{1}{2}|\,\mathrm{rot}\mathbf{A}|^2+\frac{1}{2}|\mathbf{A}_t|^2\\ +\frac{e^2}{2}|\mathbf{A}|^2|\psi|^2-e\,\mathrm{Im}(\bar{\psi}\nabla\psi)\cdot\mathbf{A}+\nabla\phi\cdot\mathbf{A}_t+\phi\,\mathrm{div}\mathbf{A}_t\Big\}\,dx=0.\end{aligned}$$

Moreover, using the facts that

$$\begin{aligned}\nabla\phi\cdot\mathbf{A}_t+\phi\,\mathrm{div}\mathbf{A}_t&=\,\mathrm{div}(\phi\mathbf{A}_t),\\ |\,\mathrm{rot}\mathbf{A}|^2&=\,\mathrm{div}(\mathbf{A}\times\,\mathrm{rot}\mathbf{A})+\mathbf{A}\cdot\nabla(\,\mathrm{div}\mathbf{A})-\mathbf{A}\cdot\Delta\mathbf{A},\end{aligned}$$

we get the identity:

$$\begin{aligned}0=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\Big\{|\nabla\psi-ie\mathbf{A}\psi|^2-\frac{2}{p+1}|\psi|^{p+1}+e\phi|\psi|^2-|\nabla\phi|^2\\ +|\mathbf{A}_t|^2+|\nabla\mathbf{A}|^2+\mathbf{A}\cdot\nabla(\,\mathrm{div}\mathbf{A})\Big\}\,dx.\end{aligned}$$

Furthermore, from (1.3) and by the Coulomb condition (1.5), it follows that

$$\int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx = \frac{e}{2} \int_{\mathbb{R}^3} \phi |\psi|^2 \, dx.$$

Thus from $\phi = \frac{e}{2}(-\Delta)^{-1}|\psi|^2 = eS(\psi)$, we obtain the following energy conservation law:

$$\frac{d}{dt}\mathcal{E}(\psi, \mathbf{A})(t) = 0 \qquad \text{(energy)},$$

where

$$\begin{aligned} \mathcal{E}(\psi, \mathbf{A}) := \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla \psi - ie\mathbf{A}\psi|^2 + |\nabla \mathbf{A}|^2 + |\mathbf{A}_t|^2 \right) dx \\ + \frac{e^2}{4} \int_{\mathbb{R}^3} S(\psi)|\psi|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^3} |\psi|^{p+1} \, dx. \end{aligned} \tag{2.7}$$

Note that the conservation laws (2.3) and (2.7) will play a crucial role in our stability result given in Theorem 1.3.

Remark 2.1. *We can also derive* (2.7) *from the Noether theorem applied to the Lagrangian* $\mathcal{L}$ *of* (2.2)*. See [8] for this topics.*

3 Preparatory results

The aim of this section is to prepare several lemmas which we will use later on.

Lemma 3.1. *Let* $\omega \in \mathbb{R}$, $e \geq 0$ *and* $1 < p < 5$*. Suppose that* $u \in H^1(\mathbb{R}^3, \mathbb{C})$ *is a weak solution of* (1.9)*. Then* u *satisfies the following identities:*

$$\begin{aligned} 0 = N_{e,\omega}(u) &:= \int_{\mathbb{R}^3} \left(|\nabla u|^2 + \omega |u|^2 + e^2 S(u)|u|^2 - |u|^{p+1} \right) dx, \\ 0 = P_{e,\omega}(u) &:= \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla u|^2 + \frac{3\omega}{2}|u|^2 + \frac{5e^2}{4} S(u)|u|^2 - \frac{3}{p+1}|u|^{p+1} \right) dx. \end{aligned}$$

Proof. We notice that the relation $N_{e,\omega}(u) = 0$ is the Nehari identity whereas $P_{e,\omega}(u) = 0$ is the Pohozaev identity. For the proof, we refer to [19]. □

A straightforward consequence of Lemma 3.1 is given in the Lemma 3.2 below. It gives new identities which will be used to study the sign of the Lagrange multiplier associated with Problem (1.10), to determine a link between $\mathcal{G}_e(\omega)$ and $\mathcal{M}_e(\mu)$ in the case $p = 2$ and to provide several uniform estimates for the ground states.

Lemma 3.2. *Let $\omega \in \mathbb{R}$, $e \geq 0$ and $1 < p < 5$. Suppose that u is a weak solution of* (1.9)*. Then it holds*

$$(5p-7)J_e(u) = 2(p-2)\int_{\mathbb{R}^3} |\nabla u|^2\,dx - \frac{(3p-5)\omega}{2}\int_{\mathbb{R}^3} |u|^2\,dx, \tag{3.1}$$

$$(5p-7)I_{e,\omega}(u) = 2(p-2)\int_{\mathbb{R}^3} |\nabla u|^2\,dx + (p-1)\omega\int_{\mathbb{R}^3} |u|^2\,dx, \tag{3.2}$$

$$I_{e,\omega}(u) = \frac{1}{3}\int_{\mathbb{R}^3} |\nabla u|^2\,dx - \frac{e^2}{6}\int_{\mathbb{R}^3} S(u)|u|^2\,dx. \tag{3.3}$$

Proof. Combine the definition of J_e and $I_{e,\omega}$ with the two identities $N_{e,\omega}(u) = 0$, $P_{e,\omega}(u) = 0$. □

Next we prepare basic properties of $S(u)$. To this aim, we put

$$A(u) = \int_{\mathbb{R}^3} S(u)|u|^2\,dx = \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}\,dx\,dy.$$

Lemma 3.3. *For $u \in H^1(\mathbb{R}^3, \mathbb{C})$, $S(u)$ satisfies the following properties.*

(i) $S(u)(x) \geq 0$ *for all* $x \in \mathbb{R}^3$.

(ii) *Let $\lambda > 0$, $a \in \mathbb{R}$, $b \in \mathbb{R}$ and put $u_\lambda(x) = \lambda^a u(\lambda^b x)$. Then it holds*

$$S(u_\lambda)(x) = \lambda^{2a-2b} S(u)(\lambda^b x), \quad A(u_\lambda) = \lambda^{4a-5b} A(u).$$

(iii) *There exists $C > 0$ such that for all u, $\varphi \in H^1(\mathbb{R}^3, \mathbb{C})$,*

$$\|S(u)\|_{L^6} \leq C\|\nabla S(u)\|_{L^2} \leq C\|u\|^2_{L^{\frac{12}{5}}} \leq C\|u\|^2_{H^1},$$

$$A(u) \leq C\|u\|^4_{L^{\frac{12}{5}}} \leq C\|\nabla u\|_{L^2}\|u\|^3_{L^2} \leq C\|u\|^4_{H^1},$$

$$\int_{\mathbb{R}^3} S(u)u\varphi\,dx \leq C\|u\|^3_{H^1}\|\varphi\|_{H^1}.$$

(iv) *If $u_n \to u$ in $L^{\frac{12}{5}}(\mathbb{R}^3, \mathbb{C})$, then $A(u_n) \to A(u)$.*

Proof. See e.g. [35]. □

To end this section, we present a Gagliardo-Nirenberg type inequality.

Lemma 3.4. *Suppose $2 \leq p \leq \frac{7}{3}$. Then it holds*

(i) *There exists $C = C(p) > 0$ such that*

$$\|u\|^{p+1}_{L^{p+1}} \leq CA(u)^{\frac{7-3p}{2}}\|\nabla u\|^{3p-5}_{L^2}\|u\|^{4(p-2)}_{L^2} \text{ for all } u \in H^1(\mathbb{R}^3, \mathbb{C}).$$

(ii) *Denote by* $C^* = C^*(p) > 0$ *the quantity*

$$C^* = \sup_{u \in H^1(\mathbb{R}^3, \mathbb{C}),\, u \neq 0} \frac{\|u\|_{L^{p+1}}^{p+1}}{A(u)^{\frac{7-3p}{2}} \|\nabla u\|_{L^2}^{3p-5} \|u\|_{L^2}^{4(p-2)}}.$$

Then C^* *is well-defined, that is* $C^* < +\infty$*. Moreover, for any* $\tilde{C} < C^*$ *and* $\mu > 0$*, there exists* $\tilde{u} \in H^1(\mathbb{R}^3, \mathbb{C})$ *such that* $\|\tilde{u}\|_{L^2}^2 = \mu$ *and*

$$\|\tilde{u}\|_{L^{p+1}}^{p+1} > \tilde{C} \mu^{2(p-2)} A(\tilde{u})^{\frac{7-3p}{2}} \|\nabla \tilde{u}\|_{L^2}^{3p-5}.$$

(iii) *If* $p = 2$*, then it follows that* $C^* \leq \sqrt{2}$*.*

Proof. The proof of (i) can be done by applying the Gagliardo-Nirenberg inequality, see e.g. [12]. Moreover from (i), C^* is well-defined. Next by the definition of C^*, for any $\tilde{C} < C^*$, there exists $u_0 \in H^1(\mathbb{R}^3, \mathbb{C})$ such that

$$\|u_0\|_{L^{p+1}}^{p+1} > \tilde{C} A(u_0)^{\frac{7-3p}{2}} \|\nabla u_0\|_{L^2}^{3p-5} \|u_0\|_{L^2}^{4(p-2)}.$$

By the definition of $A(u)$, it follows that $A(\lambda u) = \lambda^4 A(u)$ for $\lambda > 0$. Then putting $\tilde{u} = \frac{\sqrt{\mu}}{\|u_0\|_{L^2}} u_0$, we can see that (ii) holds. Finally by the definition of $S(u)$, one has

$$-\Delta S = \frac{1}{2} |u|^2. \tag{3.4}$$

Multiplying (3.4) by $|u|$ and integrating over $\mathbb{R}^3$, we get

$$\frac{1}{2} \|u\|_{L^3}^3 \leq \|\nabla S\|_{L^2} \|\nabla u\|_{L^2}. \tag{3.5}$$

Next we multiply (3.4) by $S(u)$ to derive

$$\|\nabla S\|_{L^2}^2 = \frac{1}{2} \int_{\mathbb{R}^3} S(u) |u|^2 \, dx = \frac{1}{2} A(u). \tag{3.6}$$

Thus from (3.5) and (3.6), we obtain

$$\|u\|_{L^3}^3 \leq \sqrt{2} A(u)^{\frac{1}{2}} \|\nabla u\|_{L^2},$$

which provides $C^*(2) \leq \sqrt{2}$. This completes the proof of (iii). □

4 Existence of minimizers with prescribed charge

In this section, we study the minimization problem (1.10) and prove Theorem 1.1.

Lemma 4.1. *Suppose* $1 < p < \frac{7}{3}$*,* $e > 0$ *and* $\mu > 0$*. Then it holds*

(i) $J_e(u)$ *is bounded from below on* $\{u \in H^1(\mathbb{R}^3, \mathbb{C}) \ ; \ \|u\|_{L^2}^2 = \mu\}$*.*

(ii) $c_e(\mu) \le 0$*, non-increasing with respect to* μ *and satisfies the weak sub-additive condition:*

$$c_e(\mu) \le c_e(\mu') + c_e(\mu - \mu') \quad \textit{for all } \mu > 0 \textit{ and } \mu' \in (0, \mu], \tag{4.1}$$

Proof. By the Gagliardo-Nirenberg inequality and Lemma 3.3 (i), one has

$$J_e(u) \ge \frac{1}{2}\|\nabla u\|_{L^2}^2 - C\|\nabla u\|_{L^2}^{\frac{3}{2}(p-1)}\|u\|_{L^2}^{\frac{5-p}{2}}.$$

Since $\frac{3}{2}(p-1) < 2$, the Young inequality gives

$$J_e(u) \ge \frac{1}{4}\|\nabla u\|_{L^2}^2 - C\mu^{\frac{5-p}{7-3p}} \ge -C\mu^{\frac{5-p}{7-3p}}, \tag{4.2}$$

which provides (i).

Next, we fix $u \in H^1(\mathbb{R}^3, \mathbb{C})$ with $\|u\|_{L^2}^2 = \mu$ and for $\lambda > 0$, we define $\hat{u}(x) := \lambda^{\frac{3}{2}} u(\lambda x)$. Then it follows that $\|\hat{u}\|_{L^2}^2 = \mu$ for any $\lambda > 0$. Moreover applying Lemma 3.3 (ii) with $a = \frac{3}{2}$ and $b = 1$, one has

$$J_e(\hat{u}) = \frac{\lambda^2}{2}\|\nabla u\|_{L^2}^2 + \frac{\lambda e^2}{4}A(u) - \frac{\lambda^{\frac{3(p-1)}{2}}}{p+1}\|u\|_{L^{p+1}}^{p+1}. \tag{4.3}$$

Thus we get

$$c_e(\mu) \le \limsup_{\lambda \to 0+} J(\hat{u}) = 0 \quad \text{for all } \mu > 0.$$

Finally since $c_e(\mu)$ is invariant by translation, one can show that $c_e(\mu)$ satisfies the weak sub-additivity condition. (For the proof, we refer to [29], P. 113 or [12].) Thus from (4.1) and $c_e(\mu - \mu') \le 0$, one gets $c_e(\mu) \le c_e(\mu')$ for $\mu' < \mu$. This completes the proof of (ii). □

Lemma 4.2. *Let* $\mu > 0$ *be given.*

(i) *If* $2 \le p < \frac{7}{3}$*, then there exists* $e^* = e^*(\mu, p) > 0$ *such that* $c_e(\mu) < 0$ *for* $0 < e < e^*$*. Moreover if* $p = 2$*,* e^* *is independent of* μ *and* $e^*(2) \le \frac{2}{3}$*.*

(ii) *If* $1 < p < 2$, *then* $c_e(\mu) < 0$ *for all* $e > 0$.

Proof. First we show (i). To this end, let us consider $u \in H^1(\mathbb{R}^3, \mathbb{C})$ with $\|u\|_{L^2}^2 = \mu$ and use the same scaled function $\hat{u}$ as in the proof of Lemma 4.1. Following (4.3), we define $f(\lambda)$ by

$$f(\lambda) := \frac{1}{\lambda} J(\hat{u}) = \frac{\lambda}{2} \|\nabla u\|_{L^2}^2 + \frac{e^2}{4} A(u) - \frac{\lambda^{\frac{3p-5}{2}}}{p+1} \|u\|_{L^{p+1}}^{p+1} \quad \text{for } \lambda \geq 0.$$

By a direct calculation, it is easy to see that $f(\lambda)$ achieves its maximum at

$$\bar{\lambda} = \left(\frac{3p-5}{p+1} \frac{\|u\|_{L^{p+1}}^{p+1}}{\|\nabla u\|_{L^2}^2} \right)^{\frac{2}{7-3p}},$$

and takes its maximum value $\max\limits_{\lambda \geq 0} f(\lambda) = f(\bar{\lambda})$ which is given by

$$\max_{\lambda \geq 0} f(\lambda) = -\frac{7-3p}{2} \left(\frac{(3p-5)^{\frac{3p-5}{2}}}{p+1} \right)^{\frac{2}{7-3p}} \|\nabla u\|_{L^2}^{\frac{10-6p}{7-3p}} \|u\|_{L^{p+1}}^{\frac{2(p+1)}{7-3p}} + \frac{e^2}{4} A(u).$$

It is clear that if $\max\limits_{\lambda \geq 0} f(\lambda) < 0$, then $c_e(\mu) < 0$, that is one has to find a function $u \in H^1(\mathbb{R}^3, \mathbb{C})$ with $\|u\|_{L^2}^2 = \mu$ such that

$$\|u\|_{L^{p+1}}^{p+1} > \frac{(p+1) e^{7-3p}}{\big(2(7-3p)\big)^{\frac{7-3p}{2}} (3p-5)^{\frac{3p-5}{2}} \mu^{2(p-2)}} \mu^{2(p-2)} A(u)^{\frac{7-3p}{2}} \|\nabla u\|_{L^2}^{5-3p}. \tag{4.4}$$

Now let $C^* = C^*(p)$ be the constant defined in Lemma 3.4 (ii) and put

$$e^* = e^*(\mu, p) = \frac{\sqrt{2}(7-3p)^{\frac{1}{2}} (3p-5)^{\frac{3p-5}{2(7-3p)}}}{(p+1)^{\frac{1}{7-3p}}} \mu^{\frac{2(p-2)}{7-3p}} (C^*)^{\frac{1}{7-3p}}. \tag{4.5}$$

Then for $e < e^*$, one can see that the coefficient of $\mu^{2(p-2)} A(u)^{\frac{7-3p}{2}} \|\nabla u\|_{L^2}^{5-3p}$ in (4.4) is less than C^*. Thus by Lemma 3.4 (ii), there exists $\tilde{u} \in H^1(\mathbb{R}^3, \mathbb{C})$ with $\|\tilde{u}\|_{L^2}^2 = \mu$ such that (4.4) holds, proving the first part of (i). Moreover, when $p = 2$, it follows that

$$e^*(\mu, 2) = \frac{\sqrt{2}}{3} C^*(2) \quad \text{for any } \mu > 0.$$

By Lemma 3.4 (iii), we obtain $e^*(2) \leq \frac{2}{3}$ and hence the proof of (i) is complete.

Finally, we prove (ii). To this aim, let $u \in H^1(\mathbb{R}^3, \mathbb{C})$ be fixed and consider $u_\lambda(x) := \lambda^2 u(\lambda x)$ for $\lambda > 0$. Using Lemma 3.3 (ii) with $a = 2$ and $b = 1$, one has $\|u_\lambda\|_{L^2}^2 = \lambda \|u\|_{L^2}^2$ and

$$J_e(u_\lambda) = \frac{\lambda^3}{2}\|\nabla u\|_{L^2}^2 + \frac{\lambda^3 e^2}{4} A(u) - \frac{\lambda^{2p-1}}{p+1}\|u\|_{L^{p+1}}^{p+1}. \tag{4.6}$$

Since $2p - 1 < 3$, it follows that $J_e(u_\lambda) < 0$ for sufficiently small $\lambda > 0$. This implies that there exists $\mu_0 > 0$ such that $c_e(\mu) < 0$ for $0 < \mu \le \mu_0$. Take $\mu \in (\mu_0, 2\mu_0]$ and apply Lemma 4.1 (ii) to get

$$c_e(\mu) \le c_e(\mu_0) + c_e(\mu - \mu_0) \le c_e(\mu_0) < 0,$$

since $\mu - \mu_0 \le \mu_0$. This means that $c_e(\mu) < 0$ for $\mu \in (\mu_0, 2\mu_0]$. Continuing this procedure, we obtain $c_e(\mu) < 0$ for all $\mu > 0$. □

Remark 4.3. *By the proof of Lemma 4.2 (i) and by the definition of e^*, one can see that e^* is characterized as*

$$e^*(\mu, p) = \sup\{e > 0 \ ; \ c_e(\mu) < 0\} \quad \textit{for } 2 \le p < \frac{7}{3}. \tag{4.7}$$

Lemma 4.4. *Let $\mu > 0$ be given. Assume that $c_e(\mu) < 0$, or $c_e(\mu) = 0$ and that there exists $u \in H^1(\mathbb{R}^3, \mathbb{C})$ such that $c_e(\mu) = J_e(u)$.*

(i) *If $2 \le p < \frac{7}{3}$, then it holds*

$$c_e(\lambda\mu) < \lambda c_e(\mu) \quad \textit{for any } \lambda > 1. \tag{4.8}$$

In particular, $c_e(\mu)$ satisfies the sub-additivity condition:

$$c_e(\mu) < c_e(\mu') + c_e(\mu - \mu') \quad \textit{for all } \mu' \in (0, \mu). \tag{4.9}$$

(ii) *If $1 < p < 2$, then there exists $\tilde{e} = \tilde{e}(\mu) > 0$ such that (4.9) holds for $0 < e < \tilde{e}$.*

Proof. For the proof of (i), we use again the scaled function $u_\lambda(x) = \lambda^2 u(\lambda x)$. From (4.6), since $\|u_\lambda\|_{L^2}^2 = \lambda\|u\|_{L^2}^2$, one has

$$J_e(u_\lambda) = \lambda^3 J_e(u) + \frac{\lambda^3 - \lambda^{2p-1}}{p+1}\|u\|_{L^{p+1}}^{p+1}.$$

Since $\lambda^3 - \lambda^{2p-1} < 0$ for $2 \le p < \frac{7}{3}$ and $\lambda > 1$, we get $J_e(u_\lambda) < \lambda^3 J_e(u)$. In the case $c_e(\mu) < 0$, choosing $u \in H^1(\mathbb{R}^3, \mathbb{C})$ so that $\|u\|_{L^2}^2 = \mu$ and taking infimums in both sides, we have

$$c(\lambda\mu) \le \lambda^3 c(\mu) < \lambda c(\mu).$$

In the case $c_e(\mu) = 0$, we choose u as a minimizer of $c_e(\mu)$. Then the same statement holds. Moreover the second assertion follows from (4.8). (See [29], Lemma II.1, p. 120.)

To prove (ii), we argue as in [12]. (See also [6] for another proof.) First for each $\mu > 0$, we observe that $c_e(\mu) \to c_0(\mu) < 0$ as $e \to 0$ and

$$c_0(\mu) < c_0(\mu') + c_0(\mu - \mu') \quad \text{for all } \mu' \in (0, \mu). \tag{4.10}$$

(See [30], Section I.1 for the proof of (4.10).) We suppose by contradiction that (4.9) does not hold for any small $e > 0$. Then from (4.1), there exists $\mu_0 > 0$ such that for any $e > 0$,

$$c_{e_0}(\mu_0) = c_{e_0}(\mu_0') + c_{e_0}(\mu_0 - \mu_0')$$

holds for some $e_0 \in (0, e)$ and $\mu_0' \in (0, \mu_0)$. Taking $e = \frac{1}{n}$, there exist $e_n \to 0$ and $\mu_n \in (0, \mu_0)$ such that

$$c_n(\mu_0) = c_n(\mu_n) + c_n(\mu_0 - \mu_n), \tag{4.11}$$

where we write $c_n(\mu_0) = c_{e_n}(\mu_0)$ for simplicity. Moreover replacing μ_n by $\mu_0 - \mu_n$ if necessary, we may assume that $\frac{\mu_0}{2} \leq \mu_n < \mu_0$. Next we claim that $\mu_n \to \mu_0$ as $n \to \infty$. If this is not the case, one has $\mu_n \to \mu^* \in \left[\frac{\mu_0}{2}, \mu_0\right)$. Then passing to the limit in (4.11), we get

$$c_0(\mu_0) = c_0(\mu^*) + c_0(\mu_0 - \mu^*),$$

which contradicts (4.10). Thus it follows that $\mu_n \to \mu_0$ as claimed. Moreover choosing μ_n smaller, we may assume that

$$\mu_n = \inf \left\{ \mu \in \left[\frac{\mu_0}{2}, \mu_0\right) \ ; \ c_n(\mu_0) = c_n(\mu) + c_n(\mu_0 - \mu) \right\}. \tag{4.12}$$

From (4.11), it follows that

$$\frac{c_n(\mu_0) - c_n(\mu_n)}{\mu_0 - \mu_n} = \frac{c_n(\mu_0 - \mu_n)}{\mu_0 - \mu_n}. \tag{4.13}$$

We next claim that

$$\lim_{n \to \infty} \frac{c_n(\mu_0 - \mu_n)}{\mu_0 - \mu_n} = 0. \tag{4.14}$$

To this end, we fix $u \in H^1(\mathbb{R}^3, \mathbb{C})$ with $\|u\|_{L^2}^2 = \mu_0$. Putting $\tilde{\lambda}_n = \frac{\mu_0 - \mu_n}{\mu_0} < 1$ and $\tilde{u}_n(x) = \tilde{\lambda}_n^2 u(\tilde{\lambda}_n x)$, one has from (4.6) that $\|\tilde{u}_n\|_{L^2}^2 = \mu_0 - \mu_n$ and

$$c_n(\mu_0 - \mu_n) \leq J_n(\tilde{u}_n) = \tilde{\lambda}_n^3 J_n(u) + \frac{\tilde{\lambda}_n^3 - \tilde{\lambda}_n^{2p-1}}{p+1} \|u\|_{L^{p+1}}^{p+1},$$

where we write $J_n(u) = J_{e_n}(u)$. Since $1 < p < 2$ and $\tilde{\lambda}_n < 1$, it follows that $\tilde{\lambda}_n^3 - \tilde{\lambda}_n^{2p-1} < 0$ and hence $c_n(\mu_0 - \mu_n) \leq \tilde{\lambda}_n^3 J_n(u)$. This implies that

$$\left|\frac{c_n(\mu_0 - \mu_n)}{\mu_0 - \mu_n}\right| \leq \frac{(\mu_0 - \mu_n)^2}{\mu_0^3}|J_n(u)| \to 0 \quad \text{as } n \to \infty$$

and hence (4.14) holds. Our next goal is to show that

$$\lim_{n\to\infty} \frac{c_n(\mu_0) - c_n(\mu_n)}{\mu_0 - \mu_n} = -\frac{\omega_0}{2} \tag{4.15}$$

for some $\omega_0 > 0$. If (4.15) is true, then passing to the limit in (4.13), we conclude, from (4.14) and (4.15), that $0 \leq -\frac{\omega_0}{2}$. This is a contradiction and hence (4.7) must hold for small $e > 0$. To prove (4.15), we first show that

$$c_n(\mu_n) < c_n(\nu) + c_n(\mu_n - \nu) \quad \text{for all } \nu \in (0, \mu_n). \tag{4.16}$$

Suppose that (4.16) does not hold. Then there exists $\nu_n \in \left[\frac{\mu_n}{2}, \mu_n\right)$ such that

$$c_n(\mu_n) = c_n(\nu_n) + c_n(\mu_n - \nu_n). \tag{4.17}$$

Then from (4.1), (4.11) and (4.17), one has

$$\begin{aligned} c_n(\nu_n) + c_n(\mu_0 - \nu_n) &\geq c_n(\mu_0) = c_n(\mu_n) + c_n(\mu_0 - \mu_n) \\ &= c_n(\nu_n) + c_n(\mu_n - \nu_n) + c_n(\mu_0 - \mu_n) \\ &\geq c_n(\nu_n) + c_n(\mu_0 - \nu_n). \end{aligned}$$

This implies that

$$c_n(\mu_0) = c_n(\nu_n) + c_n(\mu_0 - \nu_n). \tag{4.18}$$

Next we claim that $\frac{\mu_0}{4} \leq \nu_n < \frac{\mu_0}{2}$. Indeed, we already know that $\nu_n \geq \frac{\mu_n}{2} \geq \frac{\mu_0}{4}$. Moreover if $\nu \geq \frac{\mu_0}{2}$, then from (4.18) and by the definition of μ_n in (4.12), it follows that $\nu_n \geq \mu_n$. This contradicts the fact that $\nu_n < \mu_n$ and hence the claim holds. Now, up to a subsequence, we may assume that $\nu_n \to \nu_0 \in \left[\frac{\mu_0}{4}, \frac{\mu_0}{2}\right]$. Thus passing to the limit in (4.17), we get $c_0(\mu_0) = c_0(\nu_0) + c_0(\mu_0 - \nu_0)$, contradicting (4.10). Thus (4.16) must holds. Now from (4.16) and by Lemma 4.5 below, there exists a minimizer u_n of $c_n(\mu_n)$, that is, $\|u_n\|_{L^2}^2 = \mu_n$ and

$$c_n(\mu_n) = J_n(u_n) = \frac{1}{2}\|\nabla u_n\|_{L^2}^2 + \frac{e_n^2}{4}A(u_n) - \frac{1}{p+1}\|u_n\|_{L^{p+1}}^{p+1}.$$

We put $\lambda_n = \frac{\mu_0}{\mu_n}$ and $\hat{u}_n(x) = \lambda_n^2 u_n(\lambda_n x)$. Then one has $\|\hat{u}_n\|_{L^2}^2 = \mu_0$ and

$$J_n(\hat{u}_n) = J_0(\hat{u}_n) + \frac{e_n^2}{4}A(\hat{u}_n).$$

Since $c_n(\mu_n) \to c_0(\mu_0)$, we have from (4.2) that $\|\hat{u}_n\|_{H^1}$ is bounded. Thus from Lemma 3.3 (iii), we get $\frac{e_n^2}{4}A(\hat{u}_n) \to 0$ and hence $\{\hat{u}_n\}$ is a minimizing sequence of $c_0(\mu_0)$. Then by the result in [30, Theorem I.2, P. 227], there exist $\{y_n\} \subset \mathbb{R}^3$ and $u_0 \in H^1(\mathbb{R}^3, \mathbb{C})$ such that $\hat{u}_n(\cdot - y_n) \to u_0$ in $H^1(\mathbb{R}^3, \mathbb{C})$ and $c_0(\mu_0) = J_0(u_0)$. But since $\lambda_n \to 1$, we also have $u_n - \hat{u}_n \to 0$ in $H^1(\mathbb{R}^3, \mathbb{C})$. This implies that $u_n(\cdot - y_n) \to u_0$ in $H^1(\mathbb{R}^3, \mathbb{C})$. Moreover since u_0 is a minimizer of $c_0(\mu_0)$, there exists a Lagrange multiplier ω_0 such that

$$-\Delta u_0 + \omega_0 u_0 = |u_0|^{p-1} u_0.$$

Applying Lemma 3.1, we deduce that $\omega_0 > 0$ and

$$\|\nabla u_0\|_{L^2}^2 = \frac{3(p-1)}{5-p}\omega_0\mu_0, \quad \|u_0\|_{L^{p+1}}^{p+1} = \frac{2(p+1)}{5-p}\omega_0\mu_0. \tag{4.19}$$

We are now ready to prove (4.15). Recalling that $J_n(u_n) = c_n(\mu_n)$, we derive from (4.6),

$$c_n(\mu_0) \le J_n(\hat{u}_n) = \frac{\lambda_n^3}{2}\|\nabla u_n\|_{L^2}^2 + \frac{\lambda_n^3 e_n^2}{4}A(u_n) - \frac{\lambda_n^{2p-1}}{p+1}\|u_n\|_{L^{p+1}}^{p+1}.$$

Thus we get

$$\begin{aligned} c_n(\mu_0) - c_n(\mu_n) &\le J_n(\hat{u}_n) - J(u_n) \\ &= \frac{\lambda_n^3 - 1}{2}\|\nabla u_n\|_{L^2}^2 + \frac{(\lambda_n^3 - 1)e_n^2}{4}A(u_n) - \frac{\lambda_n^{2p-1} - 1}{p+1}\|u_n\|_{L^{p+1}}^{p+1}. \end{aligned}$$

Since $\lambda_n = \frac{\mu_0}{\mu_n}$, we have by the mean value theorem that

$$\begin{aligned} \frac{c_n(\mu_0) - c_n(\mu_n)}{\mu_0 - \mu_n} &\le \frac{\mu_0^2 + \mu_0\mu_n + \mu_n^2}{2\mu_n^3}\|\nabla u_n\|_{L^2}^2 + \frac{(\mu_0^2 + \mu_0\mu_n + \mu_n^2)e_n^2}{4\mu_n^3}A(u_n) \\ &\quad - \frac{(2p-1)\big(\kappa_n\mu_0 + (1-\kappa_n)\mu_n\big)^{2p-2}}{(p+1)\mu_n^{2p-1}}\|u_n\|_{L^{p+1}}^{p+1} \end{aligned}$$

for some $\kappa_n \in (0,1)$. Using the convergences $\mu_n \to \mu_0$, $u_n(\cdot - y_n) \to u_0$ in $H^1(\mathbb{R}^3, \mathbb{C})$ and the relation (4.19), we obtain

$$\lim_{n\to\infty} \frac{c_n(\mu_0) - c_n(\mu_n)}{\mu_0 - \mu_n} \le \frac{3}{2\mu_0}\|\nabla u_0\|_{L^2}^2 - \frac{2p-1}{(p+1)\mu_0}\|u_0\|_{L^{p+1}}^{p+1} = -\frac{\omega_0}{2}.$$

We conclude that (4.15) holds and hence the proof is complete. □

Let us remark that, in the case $1 < p < 2$, the constant $\tilde{e}$ is given through a contradiction argument. Thus, we are not able to obtain further information on $\tilde{e}$.

The next lemma deals with the compactness of any minimizing sequence for Problem (1.10).

Lemma 4.5. *Suppose* $1 < p < \frac{7}{3}$ *and* $\mu > 0$. *Assume that* $c_e(\mu) < 0$ *and* $c_e(\mu)$ *satisfies* (4.9). *Let* $\{u_j\} \subset H^1(\mathbb{R}^3, \mathbb{C})$ *be a sequence satisfying* $\|u_j\|_{L^2}^2 \to \mu$ *and* $J_e(u_j) \to c_e(\mu)$.

Then there exist a subsequence of $\{u_j\}$ *which is still denoted by the same, a sequence* $\{y_j\} \subset \mathbb{R}^3$ *and* $u = u_{e,\mu} \in H^1(\mathbb{R}^3, \mathbb{C})$ *such that* $u_j(\cdot - y_j) \to u$ *in* $H^1(\mathbb{R}^3, \mathbb{C})$ *and* $J_e(u) = c_e(\mu)$.

Proof. Although the result is rather standard, we give the proof for the sake of completeness. First we observe from (4.2) that $\|u_j\|_{H^1}$ is bounded. Moreover by replacing u_j by $\frac{\sqrt{\mu}}{\|u_j\|_{L^2}^2} u_j$, we may assume that $\{u_j\}$ is a minimizing sequence of $c_e(\mu)$.

Now we apply the concentration compactness principle [29, Lemma I.1, p. 115] to the sequence $\rho_j(x) = |u_j(x)|^2$. It is well-known that the behavior of the sequence $(\rho_j)_{j\in\mathbb{N}}$ is governed by the three possibilities: Compactness, Vanishing and Dichotomy. Our goal is to show that Compactness occurs.

If Vanishing occurs, there exists a subsequence of $\{\rho_j\}$, still denoted by $\{\rho_j\}$, such that

$$\lim_{j\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} \rho_j(x)\, dx = 0 \quad \text{for all } R > 0.$$

Here $B_R(y)$ describes a ball of radius R with the center at $y \in \mathbb{R}^3$. Then by [30, Lemma I.1, P. 231], it follows that $u_j \to 0$ in $L^q(\mathbb{R}^3)$ for any $q \in (2, 6)$. On the other hand since $\{u_j\}$ is a minimizing sequence for $c_e(\mu)$, one has

$$\begin{aligned} c_e(\mu) + o(1) = J_e(u_j) &= \frac{1}{2}\|\nabla u_j\|_{L^2}^2 + \frac{e^2}{4}A(u_j) - \frac{1}{p+1}\|u_j\|_{L^{p+1}}^{p+1} \\ &\geq -\frac{1}{p+1}\|u_j\|_{L^{p+1}}^{p+1}. \end{aligned}$$

Passing a limit $j \to \infty$, we get $0 > c_e(\mu) \geq 0$. This is a contradiction, which rules out Vanishing.

Next we assume that Dichotomy occurs. Then by a standard argument (see [30, Section I.2] or [13, Proposition 1.7.6, P. 23]), there exist $\mu' \in (0, \mu)$ and $\{u_{j,1}\}$, $\{u_{j,2}\} \subset H^1(\mathbb{R}^3, \mathbb{C})$ such that

$$\|u_{j,1}\|_{L^2}^2 \to \mu', \quad \|u_{j,2}\|_{L^2}^2 \to \mu - \mu',$$

$$\operatorname{supp}(u_{j,1}) \cap \operatorname{supp}(u_{j,2}) = \emptyset, \ \delta_j := \operatorname{dist}\big(\operatorname{supp}(u_{j,1}), \operatorname{supp}(u_{j,2})\big) \to \infty, \tag{4.20}$$

$$\|u_j - u_{j,1} - u_{j,2}\|_{L^q} \to 0 \quad \text{for all } 2 \le q < 6, \tag{4.21}$$

$$\liminf_{j\to\infty} \int_{\mathbb{R}^3} \big(|\nabla u_j|^2 - |\nabla u_{j,1}|^2 - |\nabla u_{j,2}|^2\big)\, dx \ge 0. \tag{4.22}$$

Moreover replacing $u_{j,1}$, $u_{j,2}$ by $\frac{\sqrt{\mu'}}{\|u_{j,1}\|_{L^2}^2} u_{j,1}$, $\frac{\sqrt{\mu-\mu'}}{\|u_{j,2}\|_{L^2}^2} u_{j,2}$ respectively, we may assume that $\|u_{j,1}\|_{L^2}^2 = \mu'$, $\|u_{j,2}\|_{L^2}^2 = \mu - \mu'$ and (4.20)-(4.21) hold. Now from (4.20), one has

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_{j,1}(x)|^2 |u_{j,2}(y)|^2}{|x-y|}\, dx\, dy &= \int_{\operatorname{supp}(u_{j,2})}\int_{\operatorname{supp}(u_{j,1})} \frac{|u_{j,1}(x)|^2 |u_{j,2}(y)|^2}{|x-y|}\, dx\, dy \\ &\le \frac{1}{\delta_j} \|u_{j,1}\|_{L^2}^2 \|u_{j,2}\|_{L^2}^2 \to 0 \quad \text{as } j \to \infty.\end{aligned}$$

Using (4.21), a direct computation furnishes

$$\begin{aligned}&A(u_j) - A(u_{j,1}) - A(u_{j,2}) \\ &= \int_{\mathbb{R}^3} \big(S(u_j) + S(u_{j,1}) + S(u_{j,2})\big)\big(|u_j|^2 - |u_{j,1}|^2 - |u_{j,2}|^2\big)\, dx + o(1) \\ &\le C\big(\|u_j\|_{L^6} + \|u_{j,1}\|_{L^6} + \|u_{j,2}\|_{L^6}\big)\|u_j - u_{j,1} - u_{j,2}\|_{L^{\frac{12}{5}}} + o(1) \to 0.\end{aligned}$$

Thus from (4.21) and (4.22), we obtain

$$\begin{aligned}c_e(\mu) &= \liminf_{j\to\infty} J_e(u_j) \\ &\ge \liminf_{j\to\infty} J_e(u_{j,1}) + \liminf_{j\to\infty} J_e(u_{j,2}) \\ &\ge c_e(\mu') + c_e(\mu - \mu'),\end{aligned}$$

which contradicts (4.9). Thus Dichotomy does not occur.

The only remaining possibility is Compactness, that is, there exists $\{y_j\} \subset \mathbb{R}^3$ such that for all $\varepsilon > 0$, there exists $R_\varepsilon > 0$ satisfying

$$\int_{B_{R_\varepsilon}} \rho_j(x)\, dx \ge \mu - \varepsilon. \tag{4.23}$$

Since $\|u_j\|_{H^1}$ is bounded, there exists $u \in H^1(\mathbb{R}^3, \mathbb{C})$ such that up to a subsequence, $u_j(\cdot - y_j) \rightharpoonup u$ in $H^1(\mathbb{R}^3, \mathbb{C})$. Then from (4.23), it follows that $u_j(\cdot - y_j) \to u$ in $L^q(\mathbb{R}^3, \mathbb{C})$ for any $2 \le q < 6$. Thus by the weak lower semi-continuity of $\|\nabla \cdot\|_{L^2}$ and by Lemma 3.3 (iv), we get

$$c_e(\mu) = \liminf_{j\to\infty} J_e\big(u_j(\cdot - y_j)\big) \ge J_e(u) \ge c_e(\mu).$$

This implies that $J_e(u) = c_e(\mu)$ and $\|\nabla u_j(\cdot - y_j)\|_{L^2} \to \|\nabla u\|_{L^2}$. Thus we obtain $u_j(\cdot - y_j) \to u$ in $H^1(\mathbb{R}^3, \mathbb{C})$ and hence the proof is complete. □

Now suppose that $u \in H^1(\mathbb{R}^3, \mathbb{C})$ is a minimizer of Problem (4.9), that is $J_e(u) = c_e(\mu)$ and $\|u\|_{L^2} = \mu$. Up to a phase shift, we may assume that u is real-valued. Indeed by the well-known pointwise inequality $\big|\nabla|u|\big| \le |\nabla u|$ (see also Lemma 6.1 below), one can see that $|u|$ is also a minimizer and hence u can be chosen to be real-valued.

By the method of Lagrange multiplier, there exists a Lagrange multiplier $\omega = \omega(\mu) \in \mathbb{R}$ such that u satisfies (1.9) with $\omega(\mu)$. As for the sign of ω, we have the following lemma.

Lemma 4.6. *Let $2 \le p < \frac{7}{3}$ and suppose that $c_e(\mu) < 0$. Then the corresponding Lagrange multiplier $\omega = \omega_e(\mu)$ associated with any minimizer u of Problem* (1.10) *is positive.*

Proof. Since $c_e(\mu) < 0$, one can see that the claim follows by (3.1). □

We notice that the same conclusion holds for $1 < p \le \frac{7}{5}$, but we don't use the result for this case later. We finish this section by showing the following result on the non-existence of minimizers for $2 \le p < \frac{7}{3}$ and $e > e^*$. This type of results has been already obtained in [24].

Lemma 4.7. *Suppose $2 \le p < \frac{7}{3}$ and let $e^* = e^*(p, \mu)$ be the constant defined in Lemma 4.2 (i). Then for $e > e^*$, the minimization problem* (1.10) *does not admit any minimizers.*

Proof. We suppose by contradiction that there exists $\hat{e} > e^*$ such that $c_{\hat{e}}(\mu)$ has a minimizer for some $\mu > 0$. By Lemma 4.1 (ii) and the characterization of e^* in (4.7), it follows that $c_{\hat{e}}(\mu) = 0$. Then from (4.8), we get $c_e(\mu) < 0$ for any $e > \hat{e}$. This contradicts (4.7) and hence the proof is complete. □

Proof of Theorem 1.1. It is a straightforward consequence of Lemmas 4.1, 4.2, 4.4, 4.5 and 4.7. □

5 Existence of ground states and their asymptotic uniqueness

5.1 Existence of ground states and their characterizations

In this subsection, we study the existence of ground states of Equation (1.9). Now by the same reason as for minimizers of $c_e(\mu)$, up to a phase shift, we may assume that any ground states of (1.9) are real-valued. Thus we can restrict ourselves to $H^1(\mathbb{R}^3, \mathbb{R})$ to consider the existence of ground states.

Moreover by the maximum principle, the ground states can be chosen to be positive.

In the case $2 < p < 5$, the following result has been obtained in [2], [35].

Proposition 5.1. *Let $\omega > 0$ be given. Suppose that $2 < p < 5$ and $e > 0$.*

(i) *For any $u \in H^1(\mathbb{R}^3)$ with $u \neq 0$, there exists a unique $\bar{\lambda} = \bar{\lambda}(u) > 0$ such that $u_\lambda(x) := \lambda^2 u(\lambda x)$ satisfies*

$$G_{e,\omega}(u_{\bar{\lambda}}) = 0 \quad \textit{and} \quad I_{e,\omega}(u_{\bar{\lambda}}) = \max_{\lambda \geq 0} I_{e,\omega}(u_\lambda),$$

where $G_{e,\omega}(u) = 2N_{e,\omega}(u) - P_{e,\omega}(u)$ and $N_{e,\omega}(u)$, $P_{e,\omega}(u)$ are functionals defined in Lemma 3.1.

(ii) *The ground state energy $m_e(\omega)$ is characterized by*

$$m_e(\omega) = \inf_{u \neq 0, G_{e,\omega}(u)=0} I_{e,\omega}(u) = \inf_{\gamma \in \Gamma_{e,\omega}} \max_{\lambda \in [0,1]} I_{e,\omega}\big(\gamma(\lambda)\big) = \inf_{u \neq 0} \max_{\lambda \geq 0} I_{e,\omega}(u_\lambda),$$

$$\Gamma_{e,\omega} = \Big\{ \gamma \in C\big([0,1], H^1(\mathbb{R}^3)\big) \ ; \ \gamma(0) = 0, \ I_{e,\omega}\big(\gamma(1)\big) < 0 \Big\}.$$

(iii) *The problem* (1.9) *has a ground state $u_{e,\omega} \in H^1(\mathbb{R}^3)$.*

For the case $p = 2$, the proof of Proposition 5.1 (i) fails (see [35] Lemma 3.3 and the proof of Step 6, p. 665), and hence the existence of a ground state cannot be obtained by the minimization on $\{u \in H^1(\mathbb{R}^3) \setminus \{0\} \ ; \ G(u) = 0\}$. However when $p = 2$, we can handle good relations between $J(u)$ and $I(u)$, and nice scaling property which furnish the existence of a ground state. This is the aim of the next lemma.

Lemma 5.2. *Let $\omega > 0$ be given. Suppose that $p = 2$ and $e < e^*$, where e^* is the constant defined in Lemma 4.2. Then the problem* (1.9) *has a ground state $u_{e,\omega} \in H^1(\mathbb{R}^3)$.*

Proof. Let $\mu > 0$ and $e < e^*$ be fixed. Then by Theorem 1.1 and Lemma 4.6, there exist $\tilde{\omega} = \tilde{\omega}(\mu) > 0$ and $\tilde{u} \in H^1(\mathbb{R}^3)$ such that $\|\tilde{u}\|_{L^2}^2 = \mu$ and

$$-\Delta \tilde{u} + \tilde{\omega} \tilde{u} + e^2 S(\tilde{u}) \tilde{u} = |\tilde{u}| \tilde{u}. \tag{5.1}$$

First we claim that $\tilde{u}$ is a ground state of (5.1). Indeed, let $v \in H^1(\mathbb{R}^3)$ be a nontrivial solution of (5.1). We define $\tilde{\mu} = \frac{\mu}{\|v\|_{L^2}^2}$ and $\tilde{v}(x) = \tilde{\mu}^2 v(\tilde{\mu} x)$ so that $\|\tilde{v}\|_{L^2}^2 = \mu$. Taking $p = 2$ in (3.1) and using the characterization of $\tilde{u}$ (see the proof of Theorem 1.1) and Lemma 3.3 (ii), one has

$$-\frac{\tilde{\omega}\mu}{6} = J_e(\tilde{u}) \leq J_e(\tilde{v}) = \tilde{\mu}^3 J_e(v) = -\frac{\tilde{\omega}}{6} \|v\|_{L^2}^2 \tilde{\mu}^3 = -\frac{\tilde{\omega}\mu}{6} \tilde{\mu}^2.$$

This shows that $\tilde{\mu} \leq 1$ and hence $\mu \leq \frac{\mu}{\tilde{\mu}} = \|v\|_{L^2}^2$. Then from (3.2), we get

$$I_{e,\tilde{\omega}}(\tilde{u}) = \frac{\tilde{\omega}\mu}{3} \leq \frac{\tilde{\mu}}{3}\|v\|_{L^2}^2 = I_{e,\tilde{\omega}}(v),$$

yielding that $\tilde{u}$ is a ground state of (5.1) as claimed.

Next we define $\tilde{\lambda} = \sqrt{\frac{\omega}{\tilde{\omega}}}$ and $u_{e,\omega}(x) = \tilde{\lambda}^2\tilde{u}(\tilde{\lambda}x)$. Then using Lemma 3.3 (ii), a direct computation shows that $u_{e,\omega}$ is a solution of (1.9). Furthermore, let $u \in H^1(\mathbb{R}^3)$ be any nontrivial solution of (1.9) and define $\hat{u}(x) = \frac{1}{\tilde{\lambda}^2}u(\frac{x}{\tilde{\lambda}})$. Again, $\hat{u}$ is a solution of (5.1) and hence one has from (3.2) that

$$\frac{\tilde{\omega}}{3\tilde{\lambda}}\|u_{e,\omega}\|_{L^2}^2 = \frac{\tilde{\omega}}{3}\|\tilde{u}\|_{L^2}^2 = I_{e,\tilde{\omega}}(\tilde{u}) \leq I_{e,\tilde{\omega}}(\hat{u}) = \frac{\tilde{\omega}}{3}\|\hat{u}\|_{L^2}^2 = \frac{\tilde{\omega}}{3\tilde{\lambda}}\|u\|_{L^2}^2.$$

This implies that $\|u_{e,\omega}\|_{L^2} \leq \|u\|_{L^2}$. Using (3.2) again, we obtain

$$I_{e,\omega}(u_{e,\omega}) = \frac{\omega}{3}\|u_{e,\omega}\|_{L^2}^2 \leq \frac{\omega}{3}\|u\|_{L^2}^2 = I_{e,\omega}(u),$$

from which we conclude that $u_{e,\omega}$ is a ground state of (1.9). □

Finally we give a characterization of the ground state $u_{e,\omega}$ for the case $p = 2$, which is similar to the one obtained for the case $e = 0$ (see [13, Corollary 8.3.8, p. 277]).

Lemma 5.3. *Let $\omega > 0$ be given. Suppose that $p = 2$ and $e < e^*$.*

(i) *There exists a unique $\mu = \mu(\omega) > 0$ such that $\|u\|_{L^2}^2 = \mu(\omega)$ for any ground state of* (1.9)*.*

(ii) *u_ω is a ground state of* (1.9) *if and only if u_ω is a minimizer of $c_e\big(\mu(\omega)\big)$. Especially it follows that*

$$J_e(u_\omega) = c_e\big(\mu(\omega)\big) = \inf_{\|u\|_{L^2}^2 = \mu(\omega)} J_e(u).$$

Proof. For the proof of (i), let u_1, u_2 be two ground states of (1.9). Then from (3.2), one has

$$\frac{\omega}{3}\|u_1\|_{L^2}^2 = I_{e,\omega}(u_1) = I_{e,\omega}(u_2) = \frac{\omega}{3}\|u_2\|_{L^2}^2,$$

which proves that the claim holds.

To prove (ii), we argue as in [13], [17]. First, we show that if $u_{e,\omega}$ is a ground state of (1.9), then $u_{e,\omega}$ is a minimizer of $c_e\big(\mu(\omega)\big)$. To this aim, suppose by contradiction that there exists $v \in H^1(\mathbb{R}^3)$ such that $\|v\|_{L^2}^2 = \mu(\omega)$ and

$$c_e\big(\mu(\omega)\big) = J(v) < J(u_{e,\omega}). \tag{5.2}$$

Then by Lemma 4.6, there exists $\tilde{\omega} > 0$ such that v satisfies

$$-\Delta v + \tilde{\omega} v + e^2 S(v) v = |v| v. \tag{5.3}$$

From (3.1), (5.2) and (i), one has

$$-\frac{\tilde{\omega}}{6}\mu(\omega) = J_e(v) < J_e(u_\omega) = -\frac{\omega}{6}\mu(\omega),$$

which yields that $\omega < \tilde{\omega}$. Moreover, if $\tilde{\lambda} = \sqrt{\frac{\omega}{\tilde{\omega}}}$ and $\tilde{u}(x) = \tilde{\lambda}^2 v(\tilde{\lambda} x)$, we have already seen that $\tilde{u}$ is a solution of (1.9). Thus from (3.2), one gets

$$\frac{\omega}{3}\mu(\omega) = I_{e,\omega}(u_{e,\omega}) \leq I_{e,\omega}(\tilde{u}) = \frac{\omega}{3}\|\tilde{u}\|_{L^2}^2 = \frac{\omega\tilde{\lambda}}{3}\|v\|_{L^2}^2 = \frac{\omega\tilde{\lambda}}{3}\mu(\omega).$$

This implies that $1 \leq \tilde{\lambda}$, contradicting to $\omega < \tilde{\omega}$. Thus $u_{e,\omega}$ is a minimizer of $c\big(\mu(\omega)\big)$ as claimed.

Finally we show that if $u_{e,\omega}$ is a minimizer of $c\big(\mu(\omega)\big)$, then $u_{e,\omega}$ is a ground state of (1.9). Indeed by the argument above, there exists $\tilde{\omega} > 0$ such that $u_{e,\omega}$ satisfies (5.3). Then arguing similarly as in the proof of Lemma 5.2, one can show that $u_{e,\omega}$ is a ground state of (5.3). Thus it remains to prove that $\tilde{\omega} = \omega$. To this end, let u be a ground state of (1.9). Then from (3.1) and (i), it follows that

$$-\frac{\tilde{\omega}}{6}\mu(\omega) = J_e(u_{e,\omega}) \leq J_e(u) = -\frac{\omega}{6}\mu(\omega),$$

which shows that $\omega \leq \tilde{\omega}$. Furthermore, for $\tilde{\lambda} = \sqrt{\frac{\omega}{\tilde{\omega}}}$, if $\hat{u}(x) = \tilde{\lambda}^2 u_\omega(\tilde{\lambda} x)$, then from (3.2), we obtain

$$\frac{\omega}{3}\mu(\omega) = I_{e,\omega}(u) \leq I_{e,\omega}(\hat{u}) = \frac{\omega}{3}\|\hat{u}\|_{L^2}^2 = \frac{\omega\tilde{\lambda}}{3}\mu(\omega).$$

This implies that $\tilde{\omega} \leq \omega$ and hence $\tilde{\omega} = \omega$. This completes the proof. □

5.2 Asymptotic uniqueness of ground states

In this subsection, we prove that the ground state of (1.9) is unique when $2 \leq p < 5$ and e is sufficiently small. To this end, we have to prepare several uniform estimates on ground states with respect to e. Throughout this subsection, we fix $\omega > 0$ and suppose that $u_e \in H^1(\mathbb{R}^3)$ is a ground state of (1.9). As already observed, we may assume that u_e is real-valued and positive.

Lemma 5.4. *Assume that* $2 \le p < 5$ *and* $e < e^*$ *if* $p = 2$*. If* $0 \le e_1 < e_2 < \min\{1, e^*\}$*, then* $m_{e_1} < m_{e_2}$*. Moreover, one has* $\lim_{e \to 0} m_e = m_0$*.*

Proof. Let u_1, u_2 be two ground states of (1.9) with $e = e_1$, e_2 respectively. We have to separate the case $p = 2$ and the case $2 < p < 5$.

• Case $p = 2$: We put $\mu_1 = \|u_1\|_{L^2}^2$, $\mu_2 = \|u_2\|_{L^2}^2$ and claim that $\mu_1 < \mu_2$. Denoting $\hat{\lambda} = \frac{\mu_1}{\mu_2}$ and $v(x) = \hat{\lambda}^2 u_2(\hat{\lambda} x)$, we have $\|v\|_{L^2}^2 = \mu_1$. Then by Lemma 5.3 (ii), one obtains

$$
\begin{aligned}
-\frac{\omega\mu_1}{6} = J_{e_1}(u_1) = c_{e_1}(\mu_1) \le J_{e_1}(v) &= \hat{\lambda}^3 J_{e_1}(u_2) \\
&= \hat{\lambda}^3 \left(J_{e_2}(u_2) + \frac{e_1^2 - e_2^2}{4} A(u_2) \right) \\
&= -\frac{\omega\mu_2}{6}\hat{\lambda}^3 + \frac{e_1^2 - e_2^2}{4}\hat{\lambda}^3 A(u_2).
\end{aligned}
$$

Thus it follows that

$$
-\frac{\omega\mu_1}{6} \le -\frac{\omega\mu_1}{6}\hat{\lambda}^2 + \frac{e_1^2 - e_2^2}{4}\hat{\lambda}^3 A(u_2) < -\frac{\omega\mu_1}{6}\hat{\lambda}^2,
$$

which implies that $\hat{\lambda} < 1$ and hence $\mu_1 < \mu_2$. From (3.2), we obtain

$$
m_{e_1} = I_{e_1,\omega}(u_1) = \frac{\omega\mu_1}{3} < \frac{\omega\mu_2}{3} = I_{e_2,\omega}(u_2) = m_{e_2},
$$

yielding that the first assertion holds.

In order to prove the second claim, we introduce $\mu_e = \|u_e\|_{L^2}^2$, $\mu_0 = \|u_0\|_{L^2}^2$, $\tilde{\lambda} = \frac{\mu_e}{\mu_0}$ and $\tilde{u}(x) = \tilde{\lambda}^2 u_0(\tilde{\lambda} x)$. Then one has $\|\tilde{u}\|_{L^2}^2 = \mu_e$ and hence

$$
\begin{aligned}
-\frac{\omega\mu_e}{6} &\le J_e(\tilde{u}) = \tilde{\lambda}^3 \left(J_0(u_0) + \frac{e^2}{4} A(u_0) \right) \\
&= -\frac{\omega\mu_0}{6}\tilde{\lambda}^3 + \frac{e^2}{4}\tilde{\lambda}^3 A(u_0) = -\frac{\omega\mu_e^3}{6\mu_0^2} + \frac{e^2\mu_e^3}{4\mu_0^3} A(u_0).
\end{aligned}
$$

Since $\mu_e < \min\{\mu_1, \mu_{e^*}\}$ for $e < \min\{1, e^*\}$, we obtain

$$
\mu_e^2 \le \mu_0^2 + \frac{3e^2\mu_e^2}{2\omega\mu_0} A(u_0) \le \mu_0^2 + Ce^2 \to \mu_0^2 \quad \text{as } e \to 0.
$$

This implies that $\limsup_{e \to 0} \mu_e \le \mu_0$. Then from (3.2), it follows that

$$
m_0 \le \liminf_{e \to 0} m_e = \liminf_{e \to 0} \frac{\omega\mu_e}{3} \le \limsup_{e \to 0} \frac{\omega\mu_e}{3} \le \frac{\omega\mu_0}{3} = m_0,
$$

which provides that $\lim_{e\to 0} m_e = m_0$.

• Case $2 < p < 5$: First we observe that $I_{e_1,\omega}(u) < I_{e_2,\omega}(u)$ for every $u \in H^1(\mathbb{R}^3)$. Let u_1, u_2 be as above. Then by Proposition 5.1 (i), there exists a unique $\lambda_2 > 0$ such that $u_{2,\lambda}(x) = \lambda^2 u_2(\lambda x)$ satisfies

$$G_{e_2,\omega}\big(u_{2,\lambda_2}\big) = 0 \text{ and } I_{e_2,\omega}\big(u_{2,\lambda_2}\big) = \max_{\lambda\geq 0} I_{e_2,\omega}\big(u_{2,\lambda}\big).$$

However since u_2 is a solution of (1.9) with $e = e_2$, it follows that $G_{e_2,\omega}(u_2) = 0$, showing that $\lambda_2 = 1$. By Proposition 5.1 (ii), one gets

$$\begin{aligned} m_{e_1} &= \inf_{u\neq 0}\max_{\lambda\geq 0} I_{e_1,\omega}(u_\lambda) \leq \max_{\lambda\geq 0} I_{e_1,\omega}\big(u_{2,\lambda}\big) \\ &< \max_{\lambda\geq 0} I_{e_2,\omega}\big(u_{2,\lambda}\big) = I_{e_2,\omega}(u_2) = m_{e_2}, \end{aligned}$$

from which we deduce that $m_{e_1} < m_{e_2}$ and $\liminf_{e\to 0+} m_e \geq m_0$.

Finally we observe that $I_{0,\omega}\big((u_0)_{\lambda_0}\big) < 0$ for sufficiently large $\lambda_0 > 0$. Putting $\gamma_0(\lambda) = (\lambda\lambda_0)^2 u_0(\lambda\lambda_0 x)$ for $\lambda \in [0,1]$, we get $\gamma_0(0) = 0$, $I_{0,\omega}\big(\gamma_0(1)\big) < 0$ and $\max_{\lambda\in[0,1]} I_{0,\omega}\big(\gamma_0(\lambda)\big) = m_0$. Moreover we have

$$\begin{aligned} I_{e,\omega}\big(\gamma_0(\lambda)\big) &= I_{0,\omega}\big(\gamma_0(\lambda)\big) + \frac{e^2}{4}(\lambda\lambda_0)^3 A(u_0) \\ &\leq I_{0,\omega}\big(\gamma_0(\lambda)\big) + \frac{e^2}{4}\lambda_0^3 A(u_0) \quad \text{for all } \lambda \in [0,1]. \end{aligned}$$

Thus for sufficiently small $e > 0$, it follows that $I_{e,\omega}\big(\gamma_0(1)\big) < 0$ and hence $\gamma_0 \in \Gamma_{e,\omega}$. By Proposition 5.1 (ii), we get

$$\begin{aligned} m_e &= \inf_{\gamma\in\Gamma_e}\max_{\lambda\in[0,1]} I_{e,\omega}\big(\gamma(\lambda)\big) \leq \max_{\lambda\in[0,1]} I_{e,\omega}\big(\gamma_0(\lambda)\big) \\ &\leq \max_{\lambda\in[0,1]} I_{0,\omega}\big(\gamma_0(\lambda)\big) + \frac{e^2}{4}\lambda_0^3 A(u_0) = m_0 + \frac{e^2}{4}\lambda_0^3 A(u_0), \end{aligned}$$

from which we obtain $\limsup_{e\to 0} m_e \leq m_0$ and hence $\lim_{e\to 0} m_e = m_0$. □

Lemma 5.5. *Assume that $2 \leq p < 5$ and $e < e^*$ if $p = 2$. Then $\|u_e\|_{H^1}$ is uniformly bounded for $0 < e < \min\{1, e^*\}$ and*

$$\lim_{e\to 0} \|\nabla u_e\|_{L^2}^2 = 3m_0.$$

Proof. By Lemma 5.3, we know that there exists a constant $C > 0$ such that $m_e \leq m_0 + C$ for $e < \min\{1, e^*\}$. Moreover from (3.2), we also have

$$\begin{aligned} (5p-7)m_e &= 2(p-2)\|\nabla u_e\|_{L^2}^2 + (p-1)\omega\|u_e\|_{L^2}^2 \\ &\geq (p-1)\omega\|u_e\|_{L^2}^2 \quad \text{for all } 2 \leq p < 5. \end{aligned}$$

This implies that $\|u_e\|_{L^2} \leq C$ for $e < \min\{1, e^*\}$. Next by Lemma 3.3 (iii), one has

$$A(u_e) \leq C\|\nabla u_e\|_{L^2}\|u_e\|_{L^2}^3 \leq C\|\nabla u_e\|_{L^2}.$$

From (3.3) and by the Young inequality, we get

$$\begin{aligned}\|\nabla u_e\|_{L^2}^2 &= 3m_e + \frac{e^2}{2}A(u_e) \leq 3m_0 + 3C + \frac{e^2}{2}C\|\nabla u_e\|_{L^2},\\ &\leq 3m_0 + 3C + \frac{1}{2}\|\nabla u_e\|_{L^2}^2 + \frac{e^4}{8}C^2,\end{aligned}$$

which ensures that $\|\nabla u_e\|_{L^2} \leq C$. Moreover by Lemma 5.3, we obtain

$$\lim_{e\to 0}\|\nabla u_e\|_{L^2}^2 = \lim_{e\to 0}\left\{3m_e + \frac{e^2}{2}A(u_e)\right\} = 3m_0.$$

This competes the proof. □

The next result concerns the uniform decay at infinity of the ground state. When one deals with radially symmetric functions, the uniform boundedness of $\|u_e\|_{H^1}$ and the radial lemma (see [9, Lemma A.II, p. 339]) ensures that this particular behavior occurs. However, in our case with the hypothesis $2 \leq p < 5$, it seems out of reach to prove such symmetry. The main reason is that, on one hand, for given $u \in H^1(\mathbb{R}^3)$ and if we adopt the Schwarz symmetrization u^*, the kinetic energy $\|\nabla u\|_{L^2}$ decreases, that is, $\|\nabla u^*\|_{L^2} \leq \|\nabla u\|_{L^2}$. On the other hand, the nonlocal energy $A(u)$ increases, causing a competition between two terms. In this direction, we refer to [21], [32] for radial results on minimizers of (1.10) when $1 < p < 2$. For the case $2 \leq p < 5$, we apply the following result due to [11].

Proposition 5.6. *Let $V : \mathbb{R}^N \to \mathbb{R}_+$, $f \in L^{\frac{r}{r-2}}(\mathbb{R}^N)$ and $u \in H_V^1(\mathbb{R}^N, \mathbb{R})$, where*

$$H_V^1(\mathbb{R}^N) = \left\{u \in W_{loc}^{1,1}(\mathbb{R}^N)\ ;\ \|u\|_{H_V^1}^2 := \int_{\mathbb{R}^N}\left(|\nabla u|^2 + V(x)|u|^2\right)dx < \infty\right\}.$$

Assume that the embedding $H_V^1 \hookrightarrow L^r(\mathbb{R}^N)$ is continuous and u satisfies

$$-\Delta u + V(x)u = f(x)u \quad \text{in } \mathbb{R}^N. \tag{5.4}$$

If there exist $\beta > 0$ and $k > 0$ such that

$$\liminf_{|x|\to\infty} V(x)|x|^{2-2\beta} > k^2,$$

then there exists $C > 0$ depending on β, k, $\|f\|_{L^{\frac{r}{r-2}}}$ and $\|u\|_{H_V^1}$ such that

$$u(x) \leq Ce^{-k(1+|x|)^\beta} \quad \text{for all } x \in \mathbb{R}^N.$$

Proof. The claim follows from [11, Theorem 8, p.279]. Moreover checking the proof carefully, it transpires that the constant C depends only on β, k, $\|f\|_{L^{\frac{r}{r-2}}}$ and $\|u\|_{H^1_V}$. □

An important consequence of Proposition 5.6 is given in the next lemma and concerns the uniform decay estimate of u_e.

Lemma 5.7. *Assume that $2 \leq p < 5$ and $e < e^*$ if $p = 2$. Then for any $k < \sqrt{\omega}$, there exists $C > 0$ independent of $e > 0$ such that*

$$u_e(x) \leq Ce^{-k(1+|x|)} \quad \textit{for all } x \in \mathbb{R}^N.$$

Proof. We argue as in [10, Proof of Theorem 6]. To apply Proposition 5.6, we let $r = p+1$, $V(x) = \omega + e^2 S(u_e)(x)$ and $f(x) = |u_e(x)|^{p-1}$ so that u_e satisfies (5.4) and $H^1_V \hookrightarrow L^r$. Moreover by the definition of S, we also have

$$\begin{aligned} S(u_e)(x) &= \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|u_e(y)|^2}{|x-y|}\,dy \geq \frac{1}{4\pi}\int_{|y|\leq 1} \frac{|u_e(y)|^2}{|x-y|}\,dy \\ &\geq \frac{1}{4\pi}\int_{|y|\leq 1} \frac{|u_e(y)|^2}{1+|x|}\,dy = \frac{1}{4\pi(1+|x|)}\int_{|y|\leq 1} |u_e(y)|^2\,dy. \end{aligned}$$

Thus one gets $\liminf_{|x|\to\infty} V(x) \geq \omega$. Applying Proposition 5.6 with $\beta = 1$ and any $k \in (0, \sqrt{\omega})$, we obtain the desired decay estimate. Moreover, by Lemma 5.5, it follows that $\|u_e\|_{H^1}$ and $\|f\|_{L^{\frac{r}{r-2}}} = \|u_e\|_{L^{p+1}}$ are uniformly bounded, yielding that C is independent of e. □

Lemma 5.8. *Assume that $2 \leq p < 5$ and $e < e^*$ if $p = 2$. Then, up to a subsequence, one has*

$$u_e \to u_0 \textit{ in } H^1(\mathbb{R}^3) \quad \textit{as } e \to 0,$$

where u_0 is the unique ground state of (1.12).

Proof. By Lemma 5.5, we may assume that $u_e \rightharpoonup u_0$ in $H^1(\mathbb{R}^3)$ and $u_e \to u_0$ in $L^q_{loc}(\mathbb{R}^3)$ for $2 \leq q < 6$ as $e \to 0$ for some $u_0 \in H^1(\mathbb{R}^3)$. Moreover, since u_e is a ground state of (1.9), one has

$$0 = I'_{e,\omega}(u_e)\varphi = \int_{\mathbb{R}^3} \nabla u_e \cdot \nabla\varphi + \omega u_e\varphi + e^2 S(u_e)u_e\varphi - |u_e|^{p-1}u_e\varphi\,dx,$$

for all $\varphi \in H^1(\mathbb{R}^3)$. Recalling the uniform boundedness of $\|u_e\|_{H^1}$ and using Lemma 3.3 (iii), it is standard to show that the weak limit u_0 satisfies

$$0 = I'_{0,\omega}(u_0)\varphi = \int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla\varphi + \omega u_0\varphi - |u_0|^{p-1}u_0\varphi\,dx,$$

from which we deduce that u_0 is a weak solution of (1.12). Thus by Lemmas 3.1 and 3.2, we get

$$I_{0,\omega}(u_0) = \frac{1}{3}\|\nabla u_0\|_{L^2}^2, \tag{5.5}$$

$$0 = \int_{\mathbb{R}^3} \left(\frac{1}{6}|\nabla u_0|^2 + \frac{\omega}{2}|u_0|^2\right) dx - \frac{1}{p+1}\int_{\mathbb{R}^3} |u_0|^{p+1}\, dx. \tag{5.6}$$

Next we claim that

$$\lim_{e\to 0}\int_{\mathbb{R}^3} \left(\frac{1}{6}|\nabla u_e|^2 + \frac{\omega}{2}|u_e|^2\right) dx = \int_{\mathbb{R}^3} \left(\frac{1}{6}|\nabla u_0|^2 + \frac{\omega}{2}|u_0|^2\right) dx. \tag{5.7}$$

Indeed since u_e is a ground state of (1.9), it follows by Lemma 3.1 that

$$0 = P_{e,\omega}(u_e) = \int_{\mathbb{R}^3} \left(\frac{1}{6}|\nabla u_e|^2 + \frac{\omega}{2}|u_e|^2\right) dx + \frac{5}{12}e^2 A(u_e) - \frac{1}{p+1}\int_{\mathbb{R}^3} |u_e|^{p+1}\, dx.$$

Again, the uniform boundedness of $\|u_e\|_{H^1}$ and Lemma 3.3 (iii) ensure that $e^2 A(u_e) \to 0$ as $e \to 0$. Using the fact that $u_e \to u_0$ in $L^{p+1}_{loc}(\mathbb{R}^3)$ and Lemma 5.7, it transpires that $u_e \to u_0$ also in $L^{p+1}(\mathbb{R}^3)$ as $e \to 0$. Thus from (5.6), we get

$$\begin{aligned}
&\lim_{e\to 0}\int_{\mathbb{R}^3} \left(\frac{1}{6}|\nabla u_e|^2 + \frac{\omega}{2}|u_e|^2\right) dx \\
&= \lim_{e\to 0}\left\{-\frac{5}{12}e^2 A(u_e) + \frac{1}{p+1}\int_{\mathbb{R}^3} |u_e|^{p+1}\, dx\right\} \\
&= \frac{1}{p+1}\int_{\mathbb{R}^3} |u_0|^{p+1}\, dx = \int_{\mathbb{R}^3} \left(\frac{1}{6}|\nabla u_0|^2 + \frac{\omega}{2}|u_0|^2\right) dx,
\end{aligned}$$

which proves that (5.7) holds. As a consequence, one has $u_e \to u_0$ in $H^1(\mathbb{R}^3)$. Moreover by Lemma 5.5 and from (5.7), it follows that $u_0 \not\equiv 0$ and that u_0 is a nontrivial critical point of $I_{0,\omega}(u)$. Thus by Lemma 5.5 and (5.5), we obtain

$$m_0 \le I_{0,\omega}(u_0) = \frac{1}{3}\|\nabla u_0\|_{L^2}^2 = \lim_{e\to 0}\left\{\frac{1}{3}\|\nabla u_e\|_{L^2}^2\right\} = m_0,$$

from which we conclude that u_0 is a ground state of (1.12). □

Now we are ready to prove the asymptotic uniqueness of the ground states of (1.9) when $2 \le p < 5$.

Lemma 5.9. *Let $\omega > 0$ be given and suppose that $2 \le p < 5$. Then there exists $e_0 > 0$ such that for all $e \in (0, e_0)$, the ground state u_e of (1.9) is unique up to translation.*

Proof. We argue as in [1]. Suppose by contradiction that there exist a sequence $e_n \to 0$ and two ground states u_n, v_n solution to Equation (1.9) such that for all $y \in \mathbb{R}^3$, $u_n(\cdot) \neq v_n(\cdot - y)$. By Lemma 5.8, we may assume that there exists two ground states u_0, v_0 of (1.12) such that $u_n \to u_0$, $v_n \to v_0$ in $H^1(\mathbb{R}^3)$. Since ground state of (1.12) are unique, there exists $y_0 \in \mathbb{R}^3$ such that $v_0(x) = u_0(x - y_0)$ and hence $v_n(\cdot + y_0) \to u_0$ in $H^1(\mathbb{R}^3)$. Moreover up to translation, we may assume that u_0 is radially symmetric with respect to the origin. For $0 < e < \min\{1, e^*\}$ and $y \in \mathbb{R}^3$, we define

$$F_j(e, y) := \left(u_e(\cdot) - v_e(\cdot - y + y_0) \ , \ \frac{\partial u_0}{\partial x_j}(\cdot) \right)_{L^2}, \quad j = 1, 2, 3.$$

A direct computation provides

$$F_j(0, 0) = \left(u_0(\cdot) - u_0(\cdot) \ , \ \frac{\partial u_0}{\partial x_j} \right)_{L^2} = 0, \quad j = 1, 2, 3,$$

$$\frac{\partial F_j}{\partial y_j}(0, 0) = \left(\frac{\partial u_0}{\partial x_j}, \frac{\partial u_0}{\partial x_j} \right)_{L^2} \neq 0, \quad \frac{\partial F_j}{\partial y_k}(0, 0) = \left(\frac{\partial u_0}{\partial x_k}, \frac{\partial u_0}{\partial x_j} \right)_{L^2} = 0, \ j \neq k,$$

from which we deduce that

$$\frac{\partial(F_1, F_2, F_3)}{\partial(y_1, y_2, y_3)}(0, 0) \neq 0.$$

By the Implicit Function Theorem, there exists a sequence $\{y_n\} \subset \mathbb{R}^3$ with $y_n \to 0$ such that

$$F_j(e_n, y_n) = 0 \quad \text{for } j = 1, 2, 3. \tag{5.8}$$

Our contradiction will be obtained by a blow-up type argument. To this end, we introduce $\tilde{v}_n(x) = v_n(x - y_n + y_0)$ and $\phi_n = u_n - \tilde{v}_n$. Observe first that $\phi_n \not\equiv 0$. We define

$$\psi_n = \frac{\phi_n}{\|\phi_n\|}, \text{where } \|u\|^2 = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + \omega |u|^2 \right) dx.$$

From (5.8), one has $\left(\psi_n, \frac{\partial u_0}{\partial x_j} \right)_{L^2} = 0$ for $j = 1, 2, 3$ and since $\|\psi_n\| = 1$, we may assume that $\psi_n \rightharpoonup \psi_0$ in $H^1(\mathbb{R}^3)$ for some $\psi_0 \in H^1(\mathbb{R}^3)$. Next we look for an equation on ψ_n. Observe first that u_n and $\tilde{v}_n$ satisfy

$$\begin{cases} -\Delta u_n + \omega u_n + e^2 S(u_n) u_n - u_n^p = 0, \\ -\Delta \tilde{v}_n + \omega \tilde{v}_n + e^2 S(\tilde{v}_n) \tilde{v}_n - \tilde{v}_n^p = 0, \end{cases}$$

from which we deduce by taking the difference and using the Mean Value Theorem,

$$\begin{aligned}0 = &-\Delta\phi_n + \omega\phi_n + e_n^2\Big(S(u_n)\phi_n + \big(S(u_n) - S(\tilde{v}_n)\big)\tilde{v}_n\Big) \\ &- p\big(\kappa_n u_n + (1-\kappa_n)\tilde{v}_n\big)^{p-1}\phi_n,\end{aligned} \tag{5.9}$$

for some $\kappa_n \in (0,1)$. Note that we also have

$$\begin{aligned}S(u_n) - S(\tilde{v}_n) &= \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{|u_n|^2 - |\tilde{v}_n|^2}{|x-y|}\,dy = \frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{\phi_n(u_n + \tilde{v}_n)}{|x-y|}\,dy \\ &= \frac{1}{2}(-\Delta)^{-1}\big(\phi_n(u_n + \tilde{v}_n)\big).\end{aligned}$$

Thus multiplying (5.9) by $\frac{\psi_n}{\|\phi_n\|}$ and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned}0 = 1 &+ e_n^2\int_{\mathbb{R}^3} S(u_n)|\psi_n|^2\,dx + \frac{e_n^2}{2}\int_{\mathbb{R}^3} \psi_n\tilde{v}_n(-\Delta)^{-1}\big(\psi_n(u_n+\tilde{v}_n)\big)\,dx \\ &- p\int_{\mathbb{R}^3}\big(\kappa_n u_n + (1-\kappa_n)\tilde{v}_n\big)^{p-1}|\psi_n|^2\,dx.\end{aligned} \tag{5.10}$$

By Lemma 5.7 and the exponential decay of u_0, for every $R>0$, there exists $C>0$ independent of $n\in\mathbb{N}$ and $R>0$ such that

$$\int_{|x|\geq R}\big(\kappa_n u_n + (1-\kappa_n)\big)^{p-1}|\psi_n|^2\,dx \leq Ce^{-CR}\int_{|x|\geq R}|\psi_n|^2\,dx \leq Ce^{-CR}, \tag{5.11}$$

$$\int_{|x|\geq R} u_0^{p-1}|\psi_n|^2\,dx \leq Ce^{-CR}. \tag{5.12}$$

Since $y_n \to 0$ and $v_n(\cdot + y_0) \to u_0$ one can write $\tilde{v}_n(\cdot) \to u_0(\cdot)$ in $H^1(\mathbb{R}^3)$. Thus there exists $K \in L^{p+1}\big(B_R(0)\big)$ such that

$$u_n, \tilde{v}_n \to u_0 \text{ in } L^{p+1}\big(B_R(0)\big), \quad u_n, \tilde{v}_n \to u_0 \text{ a.e. in } B_R(0),$$

$$|u_n(x)| \leq K(x),\ |\tilde{v}_n(x)| \leq K(x) \text{ a.e. } x \in B_R(0).$$

Recalling that $\psi_n \rightharpoonup \psi_0$ in $H^1(\mathbb{R}^3)$, we deduce that

$$\psi_n \to \psi_0 \text{ in } L^{p+1}\big(B_R(0)\big), \quad \psi_n \to \psi_0 \text{ a.e. in } B_R(0),$$

$$|\psi_n(x)| \leq M(x) \text{ a.e. } x \in B_R(0) \quad \text{for some } M \in L^{p+1}\big(B_R(0)\big).$$

Then one gets

$$\big(\kappa_n u_n + (1-\kappa_n)\tilde{v}_n\big)^{p-1}|\psi_n|^2 \leq K^{p-1}M^2 \in L^1\big(B_R(0)\big),$$

$$\big(\kappa_n u_n + (1-\kappa_n)\tilde{v}_n\big)^{p-1}|\psi_n|^2 \to u_0^{p-1}|\psi_0|^2 \text{ a.e. in } B_R(0).$$

The Lebesgue Dominated Convergence Theorem provides

$$\lim_{n\to\infty}\int_{|x|\le R}\big(\kappa_n u_n + (1-\kappa_n)\tilde{v}_n\big)^{p-1}|\psi_n|^2\,dx = \int_{|x|\le R} u_0^{p-1}|\psi_0|^2\,dx. \tag{5.13}$$

From (5.11)-(5.13), it follows that

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^3}\big(\kappa_n u_n + (1-\kappa_n)\tilde{v}_n\big)^{p-1}|\psi_n|^2\,dx - \int_{\mathbb{R}^3} u_0^{p-1}|\psi_0|^2\,dx\right| \le Ce^{-CR}.$$

Since R is arbitrary, we can perform the limit $R\to\infty$ to obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\big(\kappa_n u_n + (1-\kappa_n)\tilde{v}_n\big)^{p-1}|\psi_n|^2\,dx = \int_{\mathbb{R}^3} u_0^{p-1}|\psi_0|^2\,dx.$$

By Lemma 3.3 (iii), one has

$$\begin{aligned} e_n^2\int_{\mathbb{R}^3} S(u_n)|\psi_n|^2\,dx &\le e_n^2\|S(u_n)\|_{L^6}\|\psi_n\|^2_{L^{\frac{12}{5}}} \\ &\le Ce_n^2\|u_n\|^2_{H^1}\|\psi_n\|^2_{H^1} \le Ce_n^2 \to 0 \quad \text{as } n\to\infty, \end{aligned}$$

and the Hardy-Littlewood-Sobolev inequality ([28, Theorem 4.3, P.106]) furnishes

$$\begin{aligned} & e_n^2\int_{\mathbb{R}^3}\psi_n\tilde{v}_n(-\Delta)^{-1}\big(\psi_n(u_n+\tilde{v}_n)\big)\,dx \\ & \le e_n^2\|\psi_n\tilde{v}_n\|_{L^{\frac{6}{5}}}\|\psi_n(u_n+\tilde{v}_n)\|_{L^{\frac{6}{5}}} \\ & \le Ce_n^2\|\psi_n\|^2_{H^1}\|\tilde{v}_n\|_{H^1}(\|u_n\|_{H^1}+\|\tilde{v}_n\|_{H^1}) \le Ce_n^2 \to 0 \quad \text{as } n\to\infty. \end{aligned}$$

Thus performing the limit $n\to\infty$ in (5.10), we obtain

$$0 = 1 - p\int_{\mathbb{R}^3} u_0^{p-1}|\psi_0|^2\,dx,$$

yielding that $\psi_0 \not\equiv 0$. On the other hand arguing as above, one can see that ψ_0 satisfies

$$-\Delta\psi_0 + \omega\psi_0 - u_0^{p-1}\psi_0 = 0, \quad \left(\psi_0, \frac{\partial u_0}{\partial x_j}\right)_{L^2} = 0,\ j=1,2,3.$$

Finally, the non-degeneracy of u_0 (see in (1.13)), provides $\psi_0 \equiv 0$, from which we obtain our contradiction. □

Remark 5.10. (i) *It is standard to show that there is a radial ground state of* (1.9)*. Denoting by* $\mathcal{G}_e^*$ *the set of radial ground states, one has* $\mathcal{G}_e^* \subset \mathcal{G}_e$*. However by Lemma 5.9, we know that, for small* e*,* $\mathcal{G}_e$ *consists of one element* u_e *up to translation, showing that* u_e *is radially symmetric with respect to the origin up to translation. Thus we obtain "a posteriori" the radiality of ground states for* $2 \le p < 5$ *and* $e < e_0$*.*

(ii) *Following the proof of Lemma 5.9, one can show that* $u_{e,\omega}$ *is asymptotically non-degenerate, that is,*

$$\mathrm{Ker}(\mathcal{L}_{e,\omega}) = \mathrm{span}\left\{ \frac{\partial u_{e,\omega}}{\partial x_1}, \frac{\partial u_{e,\omega}}{\partial x_2}, \frac{\partial u_{e,\omega}}{\partial x_3} \right\} \tag{5.14}$$

for sufficiently small $e > 0$*. Here* $\mathcal{L}_{e,\omega} : H^2(\mathbb{R}^3, \mathbb{R}) \to L^2(\mathbb{R}^3, \mathbb{R})$ *is the linearized operator of* (1.9) *with* $\mathrm{Dom}(\mathcal{L}_{e,\omega}) = H^1(\mathbb{R}^3, \mathbb{R})$*, which is defined by*

$$\begin{aligned} \mathcal{L}_{e,\omega}\varphi = & -\Delta\varphi + \omega\varphi + e^2(-\Delta)^{-1}\left(\frac{u_{e,\omega}^2}{2}\right)\varphi \\ & + e^2 u_{e,\omega}(-\Delta)^{-1}(u_{e,\omega}\varphi) - u_{e,\omega}^{p-1}\varphi, \ \ \varphi \in H^1(\mathbb{R}^3). \end{aligned}$$

For more details, we refer to [1].

Using Lemmas 5.3 and 5.9, we can give the following characterizations of $\mathcal{G}_e(\omega)$ and $\mathcal{M}_e(\mu)$.

Proposition 5.11. *Let* $\omega > 0$ *be given. Suppose that* $2 \le p < 5$ *and* $u_{e,\omega}$ *is a ground state of* (1.9)*.*

(i) *There exists* $e_0 = e_0(\omega) > 0$ *such that* $e_0(\omega) = e_0(1)\omega^{\frac{p-2}{p-1}}$ *and for all* $e < e_0(\omega)$*, it holds*

$$\mathcal{G}_e(\omega) = \left\{ e^{i\theta} u_{e,\omega}(\cdot - y) \ ; \ \theta \in [0, 2\pi), \ y \in \mathbb{R}^3 \right\}.$$

(ii) *If* $p = 2$*,* e_0 *is independent of* ω*. Moreover for any* $\mu > 0$*, it holds*

$$\mathcal{M}_e(\mu) = \left\{ e^{i\theta} u_{e,\omega}(\cdot - y) \ ; \ \theta \in [0, 2\pi), \ y \in \mathbb{R}^3 \right\}.$$

Proof. We observe by Lemma 3.3 (ii) that $u_{e,1}(x) = \omega^{-\frac{1}{p-1}} u_{e,\omega}\big(\omega^{-\frac{1}{2}}x\big)$ satisfies

$$-\Delta u_{e,1} + u_{e,1} + \left(e\omega^{-\frac{p-2}{p-1}}\right)^2 S(u_{e,1})u_{e,1} = |u_{e,1}|^{p-1}u_{e,1}.$$

This implies that the uniqueness of $u_{e,\omega}$ follows from that of $u_{e,1}$. Thus (i) is a direct consequence of Lemma 5.9. Moreover e_0 is independent of ω if $p = 2$. Next for given $\mu > 0$, let $\tilde{u}$ be a minimizer of $c_e(\mu)$ and $\tilde{\omega}(\mu) > 0$ be the corresponding Lagrange multiplier. Then as we have shown in the proof of Lemma 5.2, $u(x) = \frac{\omega}{\tilde{\omega}}\tilde{u}\left(\sqrt{\frac{\omega}{\tilde{\omega}}}x\right)$ is a ground state of (1.9). Thus $\tilde{u}$ is unique up to phase shift and translation, which completes the proof of (ii). □

Proof of Theorem 1.2. Combine Proposition 5.1, Lemma 5.2, Lemma 5.3, Lemma 5.9 and Proposition 5.11. □

Remark 5.12. *For* $2 < p < 5$*, we have shown the uniqueness of ground states of* (1.9)*. However, it does not imply the uniqueness of minimizers of* (1.10) *for the case* $2 < p < \frac{7}{3}$*. Indeed, we know that a minimizer* $\tilde{u}$ *is a solution of* (1.9) *with the Lagrange multiplier* $\tilde{\omega}(\mu) > 0$*. But contrary to the case* $p = 2$*, we are not able to show that* $\tilde{u}$ *is a ground state of* (1.9) *with* $\omega = \tilde{\omega}$*. Moreover, even if this is true, we cannot conclude to the uniqueness of* $\tilde{u}$*, because we don't know whether the map* $\mu \mapsto \tilde{\omega}(\mu)$ *is one-to-one. In other words, there remains a possibility that there exist* $\mu_1 < \mu_2$ *such that* $\tilde{\omega}(\mu_1) = \tilde{\omega}(\mu_2)$*. We also note that the injective property of the mapping* $\tilde{\omega}(\mu)$ *is equivalent to prove that Lemma 5.3 (i) is true for* $2 < p < 5$*, that is, any ground states of* (1.9) *share the same* L^2*-norm.*

6 Stability of standing waves

In this section, we prove Theorem 1.3. To this end, we prepare the following lemma, which is rather well-known. For the proof, we refer to [28, Theorem 7.21, P. 193].

Lemma 6.1. [Diamagnetic inequality]
Let $A \in L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)$ *and* $u : \mathbb{R}^N \to \mathbb{C}$*. Assume that*

$$u \in L^2(\mathbb{R}^N, \mathbb{C}) \text{ and } (\nabla + iA)u \in L^2(\mathbb{R}^N, \mathbb{C}).$$

Then $|u| \in H^1(\mathbb{R}^N, \mathbb{R})$ *and the following inequality holds:*

$$\big|\nabla|u|(x)\big| \le |(\nabla + iA)u(x)| \text{ a.e. } x \in \mathbb{R}^N.$$

Proof of Theorem 1.3. We fix $e < e^*$ and let $u_{e,\omega}$ be the unique (real-valued) ground state of (1.9) with $p = 2$. The proof follows the argument of [15, 34]. First we observe, since $\phi(t,\cdot) = \frac{e}{2}(-\Delta)^{-1}|\psi(t,\cdot)|^2$, that if

$$\sup_{t>0}\Big\{\inf_{y\in\mathbb{R}^3}\big\|\nabla|\psi(t,\cdot)|-\nabla u_{e,\omega}(\cdot+y)\big\|_{L^2}+\inf_{y\in\mathbb{R}^3}\inf_{\theta\in[0,2\pi)}\|\psi(t,\cdot)-e^{i\theta}u_{e,\omega}(\cdot+y)\|_{L^2}\Big\} < \varepsilon,$$

one also has

$$\sup_{t>0}\inf_{y\in\mathbb{R}^3}\Big\|\phi(t,\cdot) - \frac{e}{2}(-\Delta)^{-1}|u_{e,\omega}(\cdot + y)|^2\Big\|_{D^{1,2}} < C\varepsilon$$

for some $C > 0$ independent of ε. Thus it is enough to prove that for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for any initial data $(\psi_{(0)}, \mathbf{A}_{(0)}, \mathbf{A}_{(1)})$ satisfying (1.11) and

$$\|\psi_{(0)} - u_{e,\omega}\|_{H^1} + \|\nabla\mathbf{A}_{(0)}\|_{L^2} + \|\mathbf{A}_{(1)}\|_{L^2} < \delta,$$

the corresponding solution $(\psi, \mathbf{A})$ of (1.1)-(1.2) verifies

$$\sup_{t>0} \Big\{ \inf_{y\in\mathbb{R}^3} \big\| \nabla|\psi(t,\cdot)| - \nabla u_{e,\omega}(\cdot+y)\big\|_{L^2} + \inf_{y\in\mathbb{R}^3} \inf_{\theta\in[0,2\pi)} \|\psi(t,\cdot) - e^{i\theta} u_{e,\omega}(\cdot+y)\|_{L^2} + \|\nabla \mathbf{A}(t,\cdot)\|_{L^2} + \|\mathbf{A}_t(t,\cdot)\|_{L^2} \Big\} < \varepsilon.$$

For that purpose, we assume by contradiction that there exist $\varepsilon_0 > 0$,

$$\big(\psi_{(0)j}, \mathbf{A}_{(0)j}, \mathbf{A}_{(1)j}\big)_{j\in\mathbb{N}} \subset H^1(\mathbb{R}^3,\mathbb{C}) \times D^{1,2}(\mathbb{R}^3,\mathbb{R}) \times L^2(\mathbb{R}^3,\mathbb{R})$$

and $\{t_j\} \subset \mathbb{R}$ such that $\operatorname{div}\mathbf{A}_{(0)j} = \operatorname{div}\mathbf{A}_{(1)j} = 0$ and

$$\|\psi_{(0)j} - u_{e,\omega}\|_{H^1} + \|\nabla \mathbf{A}_{(0)j}\|_{L^2} + \|\mathbf{A}_{(1)j}\|_{L^2} \to 0 \text{ as } j\to\infty, \tag{6.1}$$

but the corresponding solution $(\psi_j, \mathbf{A}_j)$ of (1.1)-(1.2) satisfies

$$\inf_{y\in\mathbb{R}^3} \|\nabla|\psi_j(t_j,\cdot)| - \nabla u_{e,\omega}(\cdot+y)\|_{L^2} + \inf_{y\in\mathbb{R}^3} \inf_{\theta\in[0,2\pi)} \|\psi(t_j,\cdot) - e^{i\theta} u_{e,\omega}(\cdot+y)\|_{L^2} + \|\nabla \mathbf{A}_j(t_j,\cdot)\|_{L^2} + \|(\mathbf{A}_j)_t(t_j,\cdot)\|_{L^2} \geq \varepsilon_0. \tag{6.2}$$

For simplicity, we write $u_j = \psi_j(t_j,\cdot)$. Then by the charge conservation law (2.3), Lemma 5.3 (i) and from (6.1), one has

$$\|u_j\|_{L^2}^2 = \|\psi_{(0)j}\|_{L^2}^2 \to \|u_{e,\omega}\|_{L^2}^2 = \mu(\omega). \tag{6.3}$$

By the energy conservation law (2.7), we also have

$$\begin{aligned}
\mathcal{E}\big(u_j, \mathbf{A}_j(t_j,\cdot)\big) &= \mathcal{E}\big(\psi_{(0)j}, \mathbf{A}_j(0,\cdot)\big) \\
&= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\psi_{(0)j} - ie\mathbf{A}_{(0)j}\psi_{(0)j}|^2 + |\nabla \mathbf{A}_{(0)j}|^2 + |\mathbf{A}_{(1)j}|^2\,dx \\
&\quad + \frac{e^2}{4}\int_{\mathbb{R}^3} S(\psi_{(0)j})|\psi_{(0)j}|^2\,dx - \frac{1}{3}\int_{\mathbb{R}^3} |\psi_{(0)j}|^3\,dx.
\end{aligned}$$

From (6.1), Lemma 5.3 (ii) and $\|\mathbf{A}_{(0)j}\|_{L^6} \leq C\|\nabla \mathbf{A}_{(0)j}\|_{L^2} \to 0$, one gets

$$\mathcal{E}\big(u_j, \mathbf{A}_j(t_j,\cdot)\big) \to J_e(u_{e,\omega}) = c_e\big(\mu(\omega)\big). \tag{6.4}$$

Finally by Lemma 6.1, we obtain

$$\begin{aligned}
\mathcal{E}\big(u_j, \mathbf{A}_j(t_j,\cdot)\big) &\geq \frac{1}{2}\int_{\mathbb{R}^3} \big|\nabla|u_j|\big|^2\,dx + \frac{e^2}{4}\int_{\mathbb{R}^3} S(|u_j|)|u_j|^2\,dx - \frac{1}{3}\int_{\mathbb{R}^3} |u_j|^3\,dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^3} \big(|\nabla \mathbf{A}(t_j,x)|^2 + |\mathbf{A}_t(t_j,x)|^2\big)\,dx \\
&= J_e(|u_j|).
\end{aligned} \tag{6.5}$$

Since $J_e(|u_j|)$ is bounded, one has $J_e(|u_j|) \to d$ for some $d \in \mathbb{R}$ and $d \le c\big(\mu(\omega)\big)$. But by the definition of $c\big(\mu(\omega)\big)$ and from (6.3), it follows that $d = c\big(\mu(\omega)\big)$ and hence from (6.5),

$$\|\nabla \mathbf{A}_j(t_j,\cdot)\|_{L^2} + \|\mathbf{A}_t(t_j,\cdot)\|_{L^2} \to 0.$$

Again from (6.3) and (6.4), $\{|u_j|\} \subset H^1(\mathbb{R}^3)$ satisfies $J_e(|u_j|) \to c\big(\mu(\omega)\big)$ and $\big\||u_j|\big\|_{L^2}^2 \to \mu(\omega)$. By Lemma 4.5, Lemma 5.3 (ii) and Proposition 5.11, there exists $\{y_j\} \subset \mathbb{R}^3$ such that $|u_j|(\cdot) - u_{e,\omega}(\cdot + y_j) \to 0$ in $H^1(\mathbb{R}^3)$, in contradiction with (6.2). This ends the proof of Theorem 1.3. □

Remark 6.2. *In a similar way, one can prove that the minimizer set* $\mathcal{M}_e(\mu)$ *is stable for* $2 < p < \frac{7}{3}$ *and* $e < e^*$, *or* $1 < p < 2$ *and* $e < \tilde{e}$. *Here the notion of stability is the following one: For every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if an initial value* $(\psi_{(0)}, \mathbf{A}_{(0)}, \mathbf{A}_{(1)})$ *of* (1.1)-(1.3) *satisfies* (1.11) *and*

$$\inf_{\tilde{\psi} \in \mathcal{M}_e(\mu)} \|\psi_0 - \tilde{\psi}\|_{H^1} + \|\nabla \mathbf{A}_{(0)}\|_{L^2} + \|\mathbf{A}_{(1)}\|_{L^2} < \delta,$$

then the corresponding solution $(\psi, \mathbf{A}, \phi)$ *of* (1.1)-(1.3) *fulfills*

$$\sup_{t>0} \Bigg\{ \inf_{\tilde{\psi} \in \mathcal{M}_e(\mu)} \Big(\big\||\psi(t,\cdot)| - \tilde{\psi}\big\|_{H^1} + \Big\|\phi(t,\cdot) - \frac{e}{2}(-\Delta)^{-1}|\tilde{\psi}|^2\Big\|_{D^{1,2}} \Big) + \|\nabla \mathbf{A}(t,\cdot)\|_{L^2} + \|\mathbf{A}_t(t,\cdot)\|_{L^2} \Bigg\} < \varepsilon.$$

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