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A refined stability result for standing waves of the Schrödinger-Maxwell system

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Abstract In this paper, we are interested in standing waves of the nonlinear Schrödinger equation coupled with the Maxwell equation. Our aim is to formulate the orbital stability of standing waves in the full gauge invariant form. For this purpose, we study a new constraint minimization problem.

Keywords Schrödinger-Maxwell system, constraint minimization problem, orbital stability.

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1 Introduction

In this paper, we consider the following nonlinear Schrödinger equation coupled with Maxwell equation stated in $\mathbb{R}_+ \times \mathbb{R}^3$:

$$i\psi_t + \Delta\psi = e\phi\psi + e^2|\mathbf{A}|^2\psi + 2ie\nabla\psi \cdot \mathbf{A} + ie\psi \operatorname{div}\mathbf{A} - |\psi|^{p-1}\psi, \tag{1.1}$$

$$\mathbf{A}_{tt} - \Delta\mathbf{A} = e \operatorname{Im}(\bar{\psi}\nabla\psi) - e^2|\psi|^2\mathbf{A} - \nabla\phi_t - \nabla \operatorname{div}\mathbf{A}, \tag{1.2}$$

$$-\Delta\phi = \frac{e}{2}|\psi|^2 + \operatorname{div}\mathbf{A}_t, \tag{1.3}$$

where $\psi : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{C}$, $\mathbf{A} : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$, $\phi : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$, $e > 0$ and i denotes the unit complex number, that is, $i^2 = -1$. We recall that System (1.1)-(1.3) describes the interaction of the Schrödinger wave function ψ with the gauge potential $(\mathbf{A}, \phi)$ and the constant e represents the strength of the interaction. We refer to [18] for more physical backgrounds. Our aim

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is to complete the study of orbital stability of standing waves of (1.1)-(1.3) initiated in [15] by focusing on the particular case $p = 2$.

It is known that System (1.1)-(1.3) has a so-called *gauge ambiguity*. Namely if $(\psi, \mathbf{A}, \phi)$ is a solution of (1.1)-(1.3), then $(\exp(ie\chi)\psi, \mathbf{A}+\nabla\chi, \phi-\chi_t)$ is also a solution of (1.1)-(1.3) for any smooth function $\chi : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$. To rule out this ambiguity, we adopt the Coulomb gauge:

$$\mathrm{div}\mathbf{A} = 0, \tag{1.4}$$

which is propagated by the set of Equations (1.1)-(1.3). (See e.g. [14, **?**] for the proof.) In this setting, the last Equation (1.3) can be solved explicitly and the solution is given by

$$\phi = \frac{e}{2}(-\Delta)^{-1}|\psi|^2 = \frac{e}{8\pi|x|} * |\psi|^2.$$

Moreover from (1.4), one can observe that (1.1) can be written as

$$i\psi_t + L_A\psi - V(x)\psi + |\psi|^{p-1}\psi = 0, \tag{1.5}$$

where V is the non-local potential: $V(x) = \frac{e^2}{2}(-\Delta)^{-1}|\psi|^2$ and L_A is the *magnetic Schrödinger operator* which is defined by $\mathbf{A} = (A_1, A_2, A_3)$ and

$$L_A\psi := \sum_{j=1}^{3}\left(\frac{\partial}{\partial x_j} - ieA_j(x)\right)^2\psi = \Delta\psi - 2ie\nabla\psi\cdot\mathbf{A} - e^2|\mathbf{A}|^2\psi.$$

When $\mathbf{A} \equiv 0$, Equation (1.5) is also called the *Schrödinger-Poisson(-Slater)* equation:

$$i\psi_t + \Delta\psi - \left(\frac{e^2}{8\pi|x|} * |\psi|^2\right)\psi + |\psi|^{p-1}\psi = 0 \quad \text{in } \mathbb{R}^3. \tag{1.6}$$

The existence of ground states related with (1.6) as well as their orbital stability have been widely studied. (See [3], [7], [10], [20], [21], [28] and references therein.) The orbital stability of standing waves for the magnetic Schrödinger equation (1.5) with *given* magnetic potential has been considered in [1], [12], [17], [19]. Finally in [9], [15], the orbital stability of standing waves for the full system (1.1)-(1.3) has been studied. Our main result will enable us to generalize these previous results.

In the study of the stability of standing waves of (1.1)-(1.3), the following two conserved quantities play an fundamental role:

$$\frac{d}{dt}\int_{\mathbb{R}^3}|\psi|^2\,dx = 0, \qquad \text{(Charge conservation)} \tag{1.7}$$

$$\frac{d}{dt}\mathcal{E}(\psi, \mathbf{A}) = 0, \qquad \text{(Energy conservation)} \tag{1.8}$$

$$\begin{aligned}\mathcal{E}(\psi,\mathbf{A}) =&\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\psi - ie\mathbf{A}\psi|^2 + |\operatorname{rot}\mathbf{A}|^2 + |\mathbf{A}_t|^2\,dx \\ &+\frac{e^2}{8}\int_{\mathbb{R}^3}|\psi|^2(-\Delta)^{-1}|\psi|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^3}|\psi|^{p+1}\,dx.\end{aligned}$$

One can observe that the conserved energy $\mathcal{E}(\psi,\mathbf{A})$ still has the *general gauge invariance*:

$$\mathcal{E}\big(\exp(ie\chi)\psi, \mathbf{A}+\nabla\chi\big) = \mathcal{E}(\psi,\mathbf{A}) \quad \text{for any } \chi=\chi(x)\in C^2(\mathbb{R}^3,\mathbb{R}). \tag{1.9}$$

In this point of view, we have to take account of this strong invariance in order to obtain the complete orbital stability for (1.1)-(1.3).

By the standing wave of (1.1)-(1.3), we mean a global solution $(\psi,\mathbf{A},\phi)$ of the form:

$$\psi(t,x)=\exp(i\omega t)u(x),\ \mathbf{A}(t,x)=\mathbf{0} \text{ and } \phi(t,x)=\phi(x),$$

where where $\omega>0$ and (u,ϕ) solves

$$\begin{cases}-\Delta u + \omega u + e\phi u = |u|^{p-1}u \\ -\Delta\phi = \frac{e}{2}|u|^2.\end{cases} \tag{1.10}$$

Following the reduction method performed in [3], [27], we introduce the functional S as

$$S(u) := \frac{1}{2}(-\Delta)^{-1}|u|^2 \in D^{1,2}(\mathbb{R}^3,\mathbb{R})$$

where $D^{1,2}(\mathbb{R}^3,\mathbb{R})$ denotes the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3}|\nabla u|^2\,dx$. As a consequence, System (1.10) can be reduced to the single equation:

$$-\Delta u + \omega u + e^2 S(u)u = |u|^{p-1}u. \tag{1.11}$$

Solutions to (1.11) are no more than the critical points of the energy functional

$$I_{e,\omega}(u) = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + \omega|u|^2\,dx + \frac{e^2}{4}\int_{\mathbb{R}^3}S(u)|u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\,dx$$

The set of ground states $\mathcal{G}_e(\omega)$ can be defined as

$$\mathcal{G}_e(\omega) = \big\{u\in H^1(\mathbb{R}^3,\mathbb{C})\ ;\ \ I'_{e,\omega}(u)=0,\ I_{e,\omega}(u)=m_e(\omega)\big\},$$

where

$$m_e(\omega) = \inf\big\{I_{e,\omega}(u)\ ;\ I'_{e,\omega}(u)=0,\ u\in H^1(\mathbb{R}^3,\mathbb{C})\setminus\{0\}\big\}.$$

In [3] and [15], the existence of ground states of (1.11) has been established in the situation $2\le p<5$. Moreover, if $0<e<e_0$ is small enough, where e_0 depends also on ω and p, the ground state of (1.11) is unique up to translation and phase shift. We note that in the particular case $p=2$, e_0 does not depend

on ω and that, a posteriori, one can prove that the unique ground state of (1.11) is real-valued, positive and radially symmetric up to translations.

In view of introducing some notions of orbital stability, one first needs an initial value condition associated with System (1.1)-(1.3):

$$\psi(0,x) = \psi_{(0)}(x),\ \mathbf{A}(0,x) = \mathbf{A}_{(0)}(x),\ \mathbf{A}_t(0,x) = \mathbf{A}_{(1)}(x),$$
$$\mathrm{div}\mathbf{A}_{(0)} = 0,\ \mathrm{div}\mathbf{A}_{(1)} = 0. \tag{1.12}$$

In [15], the standing wave

$$(\psi_{e,\omega}, \mathbf{A}_{e,\omega}, \phi_{e,\omega}) := \Big(\exp(i\omega t)u_{e,\omega},\ \mathbf{0},\ \frac{e}{2}(-\Delta)^{-1}|u_{e,\omega}|^2\Big),$$

where $u_{e,\omega}$ is the unique ground state of (1.11), is proved to be stable in the following weak sense: For every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if an initial value $(\psi_{(0)}, \mathbf{A}_{(0)}, \mathbf{A}_{(1)})$ satisfies (1.12) and

$$\|\psi_{(0)} - u_{e,\omega}\|_{H^1} + \|\nabla \mathbf{A}_{(0)}\|_{L^2} + \|\mathbf{A}_{(1)}\|_{L^2} < \delta, \tag{1.13}$$

then the corresponding solution $(\psi, \mathbf{A}, \phi)$ of (1.1)-(1.3) satisfies

$$\sup_{t>0}\Big\{ \inf_{y\in\mathbb{R}^3} \big\|\nabla|\psi(t,\cdot)| - \nabla u_{e,\omega}(\cdot+y)\big\|_{L^2} + \|\nabla \mathbf{A}(t,\cdot)\|_{L^2} + \|\mathbf{A}_t(t,\cdot)\|_{L^2} \tag{1.14}$$
$$+ \inf_{y\in\mathbb{R}^3,\theta\in[0,2\pi)} \|\psi(t,\cdot) - \exp(i\theta)u_{e,\omega}(\cdot+y)\|_{L^2}$$
$$+ \inf_{y\in\mathbb{R}^3} \Big\|\phi(t,\cdot) - \frac{e}{2}(-\Delta)^{-1}|u_{e,\omega}(\cdot+y)|^2\Big\|_{D^{1,2}}\Big\} < \varepsilon.$$

The proof is based on the study of the following minimization problem:

$$\tilde{c}_e(\mu) := \inf_{u\in B(\mu)} \tilde{J}(u), \tag{1.15}$$

where $\mu > 0$, $B(\mu) = \{u \in H^1(\mathbb{R}^3, \mathbb{C}),\ \|u\|^2_{L^2(\mathbb{R}^3)} = \mu\}$ and

$$\tilde{J}(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{e^2}{4}\int_{\mathbb{R}^3} S(u)|u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\,dx.$$

As one may sees in (1.14), the main issue concerning the stability result of [15] lies in the fact that, in the previous definition of stability, one has to take the absolute value of ψ. This problem comes from the use of the diamagnetic inequality:

$$\big|\nabla|u|(x)\big| \le |(\nabla - ie\mathbf{A})u(x)| \quad \text{a.e. } x \in \mathbb{R}^3 \tag{1.16}$$

for $u \in L^2(\mathbb{R}^3, \mathbb{C})$ with $\Big(\frac{\partial}{\partial x_j} - ieA_j\Big)u \in L^2(\mathbb{R}^3, \mathbb{C})$ and $\mathbf{A} \in L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$. (See e.g. [22] for the proof.) The main object of this article is to get rid of this constraint in order to take into account the full gauge invariance.

For that purpose, we define the energy functional involving the static magnetic potential:

$$J(u,\mathbf{A}) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u - ie\mathbf{A}u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3}|\nabla \mathbf{A}|^2\,dx$$
$$+\frac{e^2}{4}\int_{\mathbb{R}^3} S(u)|u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}\,dx \quad \text{for} \quad (u,\mathbf{A})\in X,$$

$$X = \Big\{(u,\mathbf{A}) \in H^1(\mathbb{R}^3,\mathbb{C}) \times \big(D^{1,2}(\mathbb{R}^3,\mathbb{R}^3)\big)^3,\ \mathrm{div}\mathbf{A} = 0\Big\},$$

and consider the following new modified constraint minimization problem:

$$c_e(\mu) := \inf_{(u,\mathbf{A})\in X, \|u\|_{L^2}^2=\mu} J(u,\mathbf{A}). \tag{1.17}$$

One finds that the energy functional $J(u,\mathbf{A})$ fulfills the general gauge invariance:

$$J\big(\exp(ie\chi)u, \mathbf{A}+\nabla\chi) = J(u,\mathbf{A}) \quad \text{for any } \chi = \chi(x) \in C^2(\mathbb{R}^3,\mathbb{R}).$$

We also define the set of minimizers $\mathcal{M}_e(\mu)$ by

$$\mathcal{M}_e(\mu) := \big\{(u,\mathbf{A}) \in X\ ;\ \ \|u\|_{L^2}^2 = \mu,\ J(u,\mathbf{A}) = c_e(\mu)\big\}.$$

In this setting, ω in (1.11) appears as a Lagrange multiplier. It is known that one of the key point for the stability is to establish a link between $\mathcal{G}_e(\omega)$ and $\mathcal{M}_e(\mu)$, see [11], [13].

Our first result concerns the existence of solutions to the minimization problem (1.17) and can be stated as follows.

Theorem 1.1 *Let $\mu > 0$ be given and suppose that $1 < p < \frac{7}{3}$. Then there exists $e^* = e^*(\mu,p) > 0$ such that the following properties hold.*

(i) *If $2 < p < \frac{7}{3}$, the minimization problem* (1.17) *has a minimizer for $0 < e \le e^*$ and no minimizer for $e > e^*$.*
(ii) *If $p = 2$, it holds that $e^*(\mu,2) = e^*(2) \le \frac{2}{3}$ for any $\mu > 0$ and* (1.17) *has a minimizer for $0 < e < e^*$. Moreover* (1.17) *has no minimizer for $e > \frac{2}{3}$.*
(iii) *If $1 < p < 2$,* (1.17) *admits a minimizer for $0 < e < e^*$.*

Furthermore in any cases, the minimizer of (1.17) *has the form $(u_e,\mathbf{0})$.*

We now focus on the particular case $p = 2$. In this situation, one can prove that there exists $e_0 > 0$ such that (1.11) has a unique (real-valued) ground state for any $\omega > 0$ and $0 < e < e_0$. (See Proposition 4.1 below.) Then our refined stability result can be described as follows.

Theorem 1.2 *Let $\omega > 0$ be given. Suppose $p = 2$, $e < \min(e_0,e^*)$ and $u_{e,\omega}$ is the unique real-valued ground state of* (1.11)*. Then the standing wave*

$$(\psi_{e,\omega},\mathbf{A}_{e,\omega},\phi_{e,\omega}) := \Big(\exp(i\omega t)u_{e,\omega},\ \mathbf{0},\ \frac{e}{2}(-\Delta)^{-1}|u_{e,\omega}|^2\Big)$$

of (1.1)-(1.3) *is stable in the following sense: For every* $\varepsilon > 0$*, there exists* $\delta(\varepsilon) > 0$ *such that if an initial value* $(\psi_{(0)}, \mathbf{A}_{(0)}, \mathbf{A}_{(1)})$ *satisfies* (1.12) *and*

$$\|\nabla\psi_{(0)} - ie\mathbf{A}_{(0)}\psi_{(0)} - \nabla u_{e,\omega}\|_{L^2} + \|\psi_{(0)} - u_{e,\omega}\|_{L^2} + \|\operatorname{rot}\mathbf{A}_{(0)}\|_{L^2} + \|\mathbf{A}_{(1)}\|_{L^2} < \delta, \tag{1.18}$$

then the corresponding solution $(\psi, \mathbf{A}, \phi)$ *of* (1.1)-(1.3) *satisfies*

$$\begin{aligned}\sup_{t>0}\Big\{ &\inf_{y\in\mathbb{R}^3,\chi\in C^2} \big\|\nabla\psi(t,\cdot) - ie\mathbf{A}(t,\cdot)\psi(t,\cdot) - \exp\big(ie\chi(t,\cdot)\big)\nabla u_{e,\omega}(\cdot+y)\big\|_{L^2} \\ &+ \inf_{y\in\mathbb{R}^3,\chi\in C^2} \big\|\psi(t,\cdot) - \exp\big(ie\chi(t,\cdot)\big)u_{e,\omega}(\cdot+y)\big\|_{L^2} \\ &+ \inf_{\chi\in C^2} \Big(\big\|\operatorname{rot}\big(\mathbf{A}(t,\cdot) - \nabla\chi(t,\cdot)\big)\big\|_{L^2} + \|\big(\mathbf{A}(t,\cdot) - \nabla\chi(t,\cdot)\big)_t\|_{L^2}\Big) \\ &+ \inf_{y\in\mathbb{R}^3,\chi\in C^2} \big\|\phi(t,\cdot) - \phi_{e,\omega}(\cdot+y) + \chi_t(t,\cdot)\big\|_{D^{1,2}} \Big\} < \varepsilon.\end{aligned}$$

We note that the restriction $p = 2$ is due to the necessity of proving the one-to-one correspondence between $\mathcal{G}_e(\omega)$ and $\mathcal{M}_e(\mu)$, which can be obtained, for the moment, only for $p = 2$. In the case $1 < p < \frac{7}{3}$, $p \neq 2$ and $0 < e < e^*$, we are able to show the *stability of the minimizer set* $\mathcal{M}_e(\mu)$. For this result, we refer to Remark 4.2 below.

Compared to our previous result in [15], we have replaced the modulus of ψ by making use of the gauge invariant norm:

$$\inf_{y\in\mathbb{R}^3,\chi\in C^2(\mathbb{R}_+\times\mathbb{R}^3,\mathbb{R})} \big\|\nabla\psi(t,\cdot) - ie\mathbf{A}(t,\cdot)\psi(t,\cdot) - \exp\big(ie\chi(t,\cdot)\big)\nabla u_{e,\omega}(\cdot+y)\big\|_{L^2}.$$

This new term is consistent with the use of the energy $J(u, \mathbf{A})$ and the fact that our proof of Theorem 1.2 is based on the classical variational approach developed in [11], [13]. Moreover, as we will see in the proof of Theorem 1.2, $\|\psi_{(0)} - u_{e,\omega}\|_{H^1}$ can be controlled in terms of

$$\|\nabla\psi_{(0)} - ie\mathbf{A}_{(0)}\psi_{(0)} - \nabla u_{e,\omega}\|_{L^2} + \|\psi_{(0)} - u_{e,\omega}\|_{L^2} + \|\nabla\mathbf{A}_{(0)}\|_{L^2}.$$

Recalling that $\operatorname{rot}\mathbf{A}_{(0)} = \nabla\mathbf{A}_{(0)}$ when $\operatorname{div}\mathbf{A}_{(0)} = 0$, one observes that (1.18) implies (1.13). This yields that Theorem 1.2 is a complete extension of our previous stability result in [15], as well as the ones for (1.6) by taking $\mathbf{A} \equiv 0$ and $\chi \equiv \theta$.

We remark that our stability result, as the one exposed in [15], is still *conditional* in the following sense. In order to obtain the complete stability result, one requires a global well-posedness theory in the energy space. In this direction, we refer to [16] in which a local existence theory in higher order Sobolev spaces is proposed for system (1.1)-(1.3). To the best of our knowledge, it is still an open problem to obtain global solutions of (1.1)-(1.13) in the energy space $H^1 \times H^1 \times L^2$. Let us also introduce results concerning the solvability of the Cauchy problem related to (1.1)-(1.3). In [6], [26], the linear Schrödinger equation coupled with the Maxwell equations (namely, without

$|\psi|^{p-1}\psi$ in (1.1)) has been studied. Using the Strichartz estimate, the authors obtained the global well-posedness in the energy space. However, it is not clear that their argument can be applied to the nonlinear case. On the other hand, a huge attention has been paid in the magnetic Schrödinger equation, see [4], [12], [25]. Especially in [25], the local well-posedness for (1.5) in the energy space has been established in the case $V \equiv 0$. However, in this situation, the magnetic potential $\mathbf{A}$ is *given* and was assumed to be C^∞, which cannot be expected a priori for system (1.1)-(1.3).

Finally we explain the main difficulties and ideas for the study of $c_e(\mu)$. Firstly, we need to establish the sub-additivity condition for $c_e(\mu)$ in order to prove the existence of a minimizer. Since our functional $J(u, \mathbf{A})$ involves not only the non-local term $S(u)$ but also the magnetic potential $\mathbf{A}$, this cannot be obtained directly. However, using the diamagnetic inequality (1.16), we are able to prove the following fact:

$$c_e(\mu) = \tilde{c}_e(\mu). \tag{1.19}$$

Relation (1.19) enables us to conclude that we have only to study the sub-additivity of $\tilde{c}_e(\mu)$, which has been already established in [15].

The second difficulty is the main part of this paper. When we study the relative compactness of a minimizing sequence $(u_j, \mathbf{A}_j) \subset X$, we are led to the situation that $\|\nabla \mathbf{A}_j\|_{L^2}$ is bounded, which shows that, passing to a subsequence, $\nabla \mathbf{A}_j \rightharpoonup \nabla \mathbf{A}$ in $L^2(\mathbb{R}^3)$. In order to complete the proof of the compactness of the minimizing sequence $(u_j, \mathbf{A}_j)$, we need to establish that $\operatorname{div}\mathbf{A} = 0$. Although $\operatorname{div}\mathbf{A}_j = 0$ for all $j \in \mathbb{N}$, we cannot say that $\operatorname{div}\mathbf{A} = 0$ a priori, because the weak convergence provides us no information about the pointwise estimate.

At first sight, this problem can be avoided by restricting the function u_j to real-valued one. Indeed one can easily see that if the minimizing sequence u_j is real-valued, then $(u_j, \mathbf{0})$ is also a minimizing sequence for $c_e(\mu)$. However this approach does not work for our purpose, because solution ψ of (1.1) is complex-valued in general. Another approach is to assume some symmetry on the magnetic potential $\mathbf{A}$. This kind of arguments has been performed in [2], [5], [8], where stationary problems for the Maxwell equation was studied and $\mathbf{A}$ was supposed to be cylindrically symmetric. However this approach is not suitable for the study of the stability of standing waves, because the symmetry of $\mathbf{A}$ for all $t > 0$ requires a restriction of the initial values, causing the corresponding stability result to be weak.

The key idea is rather simple. Although the conserved energy $\mathcal{E}(\psi, \mathbf{A})$ in (1.8) is given in terms of $|\operatorname{rot}\mathbf{A}|$, we define the functional $J(u, \mathbf{A})$ which is associated with the stationary problem by using $|\nabla \mathbf{A}|$. If $(u_j, \mathbf{A}_j) \subset X$ is a minimizing sequence for $c_e(\mu)$, one can show that the weak limit $(u, \mathbf{A})$ satisfies $c_e(\mu) \geq J(u, \mathbf{A})$ by applying the concentration compactness principle due to [23], [24]. On the other hand, by the standard identity of vector calculus:

$$|\nabla \mathbf{A}|^2 = |\operatorname{rot}\mathbf{A}|^2 + |\operatorname{div}\mathbf{A}|^2,$$

we are able to show that $J(u,\mathbf{A}) > c_e(\mu)$ if $\operatorname{div}\mathbf{A} \neq 0$, yielding that a contradiction occurs. Once we could prove that $\operatorname{div}\mathbf{A} = 0$, one obtains the relative compactness of the minimizing sequence $(u_j, \mathbf{A}_j) \subset X$.

This paper is organized as follows. In Section 2, we collect some known results concerning the non-local term $S(u)$. We investigate the existence of minimizers for $c_e(\mu)$ and give the proof of Theorem 1.1 in Section 3. Especially the key result will be shown in Lemma 3.5, where the relative compactness of minimizing sequences for $c_e(\mu)$ is established. Finally in Section 4, we study the orbital stability of standing waves and complete the proof of Theorem 1.2.

2 Preliminaries

In this section, we present two lemmas which will be used later on. To this end, we denote

$$D(u) = \int_{\mathbb{R}^3} S(u)|u|^2\,dx = \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}\,dx\,dy.$$

Lemma 2.1 *For $u \in H^1(\mathbb{R}^3,\mathbb{C})$, $S(u)$ and $D(u)$ satisfy the following properties.*

(i) *$S(u)(x) \geq 0$ and $D(u) \geq 0$.*

(ii) *For $\lambda > 0$, $a \in \mathbb{R}$, $b \in \mathbb{R}$, denoting $u_\lambda(x) = \lambda^a u(\lambda^b x)$, one has*

$$S(u_\lambda)(x) = \lambda^{2a-2b} S(u)(\lambda^b x), \quad D(u_\lambda) = \lambda^{4a-5b} D(u).$$

(iii) *There exists $C > 0$ such that $D(u) \leq C\big\|\nabla|u|\big\|_{L^2}\|u\|_{L^2}^3$.*

(iv) *If $u_n \to u$ in $L^{\frac{12}{5}}(\mathbb{R}^3,\mathbb{C})$, then $D(u_n) \to D(u)$.*

Proof See e.g. [27] for the proof. □

Lemma 2.2 *Suppose $2 \leq p \leq \frac{7}{3}$. Then it holds*

(i) *There exists $C = C(p) > 0$ such that*

$$\|u\|_{L^{p+1}}^{p+1} \leq C D(u)^{\frac{7-3p}{2}} \big\|\nabla|u|\big\|_{L^2}^{3p-5}\|u\|_{L^2}^{4(p-2)} \text{ for all } u \in H^1(\mathbb{R}^3,\mathbb{C}).$$

(ii) *Let $C^* = C^*(p) > 0$ be the quantity defined by*

$$C^* = \sup_{u\in H^1(\mathbb{R}^3,\mathbb{C}),\, u\neq 0} \frac{\|u\|_{L^{p+1}}^{p+1}}{D(u)^{\frac{7-3p}{2}}\big\|\nabla|u|\big\|_{L^2}^{3p-5}\|u\|_{L^2}^{4(p-2)}}.$$

Then C^* is well-defined, namely $C^* < +\infty$. Moreover, for any $\tilde{C} < C^*$ and $\mu > 0$, there exists $\tilde{u} \in H^1(\mathbb{R}^3,\mathbb{C})$ such that $\|\tilde{u}\|_{L^2}^2 = \mu$ and

$$\|\tilde{u}\|_{L^{p+1}}^{p+1} > \tilde{C}\mu^{2(p-2)} D(\tilde{u})^{\frac{7-3p}{2}}\big\|\nabla|\tilde{u}|\big\|_{L^2}^{3p-5}.$$

(iii) *If $p = 2$, then it follows that $C^*(2) \leq \sqrt{2}$.*

Proof We refer to [15] for the proof. □

3 Existence of minimizers of the new constraint minimization problem

In this section, we establish the existence of minimizers for (1.17). First we begin with the following standard result.

Lemma 3.1 *Suppose that* $1 < p < \frac{7}{3}$, $e > 0$ *and* $\mu > 0$. *Then* $J(u, \mathbf{A})$ *is bounded from below on* $\{(u, \mathbf{A}) \in X; \|u\|_{L^2}^2 = \mu\}$, *and hence* $c_e(\mu)$ *is well-defined.*

Proof By Lemma 2.1 (i), one has

$$J(u, \mathbf{A}) \geq \frac{1}{2}\|\nabla u - ie\mathbf{A}u\|_{L^2}^2 + \frac{1}{2}\|\nabla \mathbf{A}\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}. \tag{3.1}$$

Moreover by the diamagnetic inequality:

$$\|\nabla u - ie\mathbf{A}u\|_{L^2} \geq \big\|\nabla|u|\big\|_{L^2}, \tag{3.2}$$

the Gagliardo-Nirenberg inequality, the Young inequality and from $\frac{3}{2}(p-1) < 2$, we find that

$$\begin{aligned} J(u, \mathbf{A}) &\geq \frac{1}{2}\big\|\nabla|u|\big\|_{L^2}^2 - C\big\|\nabla|u|\big\|_{L^2}^{\frac{3}{2}(p-1)}\|u\|_{L^2}^{\frac{5-p}{2}} \\ &\geq \frac{1}{4}\big\|\nabla|u|\big\|_{L^2}^2 - C\mu^{\frac{5-p}{7-3p}} \geq -C\mu^{\frac{5-p}{7-3p}}, \end{aligned}$$

from which we conclude. □

Next we establish the following fact, which will be crucially used to obtain the sub-additivity of $c_e(\mu)$, relative compactness of minimizing sequences and the existence of a minimizer.

Lemma 3.2 *Suppose that* $1 < p < \frac{7}{3}$, $e > 0$ *and* $\mu > 0$. *Let* $\tilde{c}_e(\mu)$ *and* $c_e(\mu)$ *be the minimum energies of* (1.15) *and* (1.17) *respectively. Then it holds that*

$$c_e(\mu) = \tilde{c}_e(\mu).$$

Moreover let $(u, \mathbf{A}) \in X$ *with* $\|u\|_{L^2}^2 = \mu$ *be a minimizer for* $c_e(\mu)$. *Then it follows that* $\mathbf{A} = \mathbf{0}$.

Proof By the definitions of $c_e(\mu)$ and $\tilde{c}_e(\mu)$, it is clear that $\tilde{c}_e(\mu) \geq c_e(\mu)$. On the other hand, by the diamagnetic inequality (3.2), one finds that

$$J(u, \mathbf{A}) \geq J(|u|, \mathbf{0}) \geq \tilde{c}_e(\mu) \quad \text{for all } u \in H^1(\mathbb{R}^3, \mathbb{C}) \text{ with } \|u\|_{L^2}^2 = \mu,$$

yielding that $c_e(\mu) \geq \tilde{c}_e(\mu)$, which ends the proof of the first part.

Next let $(u, \mathbf{A}) \in X$ with $\|u\|_{L^2}^2 = \mu$ be a minimizer of $c_e(\mu)$. Then by using the diamagnetic inequality again, we obtain

$$c_e(\mu) = J(u, \mathbf{A}) \geq J(|u|, \mathbf{0}) + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2\, dx \geq c_e(\mu) + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2\, dx,$$

from which one concludes that $\mathbf{A} = \mathbf{0}$. □

By Lemma 3.2, we are able to obtain the following results.

Lemma 3.3 *Suppose that $1 < p < \frac{7}{3}$, $e > 0$ and $\mu > 0$.*

(i) *$c_e(\mu)$ is non-positive and non-increasing with respect to μ.*

(ii) *When $2 \le p < \frac{7}{3}$, let $e^* = e^*(\mu, p) > 0$ be the constant defined by*

$$e^* = \frac{\sqrt{2}(7-3p)^{\frac{1}{2}}(3p-5)^{\frac{3p-5}{2(7-3p)}}}{(p+1)^{\frac{1}{7-3p}}} \mu^{\frac{2(p-2)}{7-3p}} (C^*)^{\frac{1}{7-3p}}, \tag{3.3}$$

where $C^* = C^*(p)$ is the constant in Lemma 2.2 (ii). Then it follows that

$$\begin{cases} c_e(\mu) < 0 & \text{for } 0 < e < e^*, \\ c_e(\mu) = 0 & \text{for } e \ge e^*. \end{cases} \tag{3.4}$$

Moreover if $p = 2$, e^* is independent of μ and $e^*(2) \le \frac{2}{3}$.

(iii) *If $1 < p < 2$, then $c_e(\mu) < 0$ for all $e > 0$.*

Proof By Lemma 3.2, it suffices to consider $\tilde{c}_e(\mu)$. Then the results follow by [15, Lemma 4.1, Lemma 4.2]. □

Lemma 3.4 *Let $\mu > 0$ be given and assume that $c_e(\mu) < 0$.*

(i) *If $2 \le p < \frac{7}{3}$, then it holds*

$$c_e(\lambda\mu) < \lambda c_e(\mu) \quad \text{for any } \lambda > 1. \tag{3.5}$$

Especially, $c_e(\mu)$ satisfies the sub-additivity condition:

$$c_e(\mu) < c_e(\mu') + c_e(\mu - \mu') \quad \text{for all } \mu' \in (0, \mu). \tag{3.6}$$

Moreover in the case $2 < p < \frac{7}{3}$ and $c_e(\mu) = 0$, we assume that there exists a minimizer $(u, \mathbf{A}) \in X$ such that $c_e(\mu) = J(u, \mathbf{A})$. Then (3.5) also holds true.

(ii) *If $1 < p < 2$, then there exists $e^* > 0$ such that (3.6) holds for $0 < e < e^*$.*

Proof (i) Let $(u, \mathbf{A}) \in X$ be fixed and consider $u^\lambda(x) := \lambda^2 u(\lambda x)$, $\mathbf{A}^\lambda(x) := \lambda \mathbf{A}(\lambda x)$. Then by applying Lemma 2.1 (ii) with $a = 2$ and $b = 1$, one has $\|u^\lambda\|_{L^2}^2 = \lambda \|u\|_{L^2}^2$ and

$$\begin{aligned} &J(u^\lambda, \mathbf{A}^\lambda) \\ &= \frac{\lambda^3}{2} \int_{\mathbb{R}^3} |\nabla u - ie\mathbf{A}u|^2 \, dx + \frac{\lambda}{2} \|\nabla \mathbf{A}\|_{L^2}^2 + \frac{\lambda^3 e^2}{4} D(u) - \frac{\lambda^{2p-1}}{p+1} \|u\|_{L^{p+1}}^{p+1} \\ &= \lambda^3 J(u, \mathbf{A}) + \frac{\lambda - \lambda^3}{2} \|\nabla \mathbf{A}\|_{L^2}^2 + \frac{\lambda^3 - \lambda^{2p-1}}{p+1} \|u\|_{L^{p+1}}^{p+1}. \end{aligned}$$

Since $2 \le p < \frac{7}{3}$ and $\lambda > 1$, it follows that $\lambda < \lambda^3$ and $\lambda^3 \le \lambda^{2p-1}$, from which we deduce that $J(u^\lambda, \mathbf{A}^\lambda) < \lambda^3 J(u, \mathbf{A})$. Choosing $u \in H^1(\mathbb{R}^3, \mathbb{C})$ so that $\|u\|_{L^2}^2 = \mu$, one gets from $c_e(\mu) < 0$ that

$$c_e(\lambda\mu) < \lambda^3 c_e(\mu) < \lambda c_e(\mu).$$

Moreover the second assertion follows from (3.5). (See [23, Lemma II.1, P. 120].) In the case $2 < p < \frac{7}{3}$ and $c_e(\mu) = 0$, we choose $(u, \mathbf{A})$ as a minimizer of $c_e(\mu)$. Then one can see that (3.5) holds.

(ii) By Lemma 3.2, it is sufficient to study $\tilde{c}_e(\mu)$. Then the claim follows by the result in [15, Lemma 4.4]. □

The next lemma deals with the compactness of any minimizing sequence for $c_e(\mu)$, which plays an essential role in the study of the orbital stability of standing waves.

Lemma 3.5 *Suppose* $1 < p < \frac{7}{3}$ *and* $\mu > 0$. *Assume that* $c_e(\mu) < 0$ *and* $c_e(\mu)$ *satisfies* (3.6). *Let* $\{(u_j, \mathbf{A}_j)\} \subset X$ *be a sequence such that* $\|u_j\|_{L^2}^2 \to \mu$ *and* $J(u_j, \mathbf{A}_j) \to c_e(\mu)$.

Then there exist a subsequence of $\{(u_j, \mathbf{A}_j)\}$ *which is still denoted by the same, a sequence* $\{y_j\} \subset \mathbb{R}^3$ *and* $u \in H^1(\mathbb{R}^3, \mathbb{C})$ *such that*

$$\|\nabla u_j(\cdot - y_j) - ie\mathbf{A}_j(\cdot - y_j)u_j(\cdot - y_j) - \nabla u\|_{L^2(\mathbb{R}^3)} \to 0,$$

$$\|u_j(\cdot - y_j) - u\|_{L^2(\mathbb{R}^3)} \to 0, \ \|\nabla \mathbf{A}_j(\cdot - y_j)\|_{L^2(\mathbb{R}^3)} \to 0 \quad and \quad J(u, \mathbf{0}) = c_e(\mu).$$

As a consequence, Problem (1.17) *admits a minimizer.*

Proof First we observe that from (3.1) and (3.2) that

$$\begin{aligned} c_e(\mu) + o(1) &\geq \frac{1}{4}\|\nabla u_j - ie\mathbf{A}_j u_j\|_{L^2}^2 + \frac{1}{2}\|\nabla \mathbf{A}_j\|_{L^2}^2 - C\mu^{\frac{5-p}{7-3p}} \\ &\geq \frac{1}{4}\|\nabla |u_j|\|_{L^2}^2 + \frac{1}{2}\|\nabla \mathbf{A}_j\|_{L^2}^2 - C\mu^{\frac{5-p}{7-3p}}, \end{aligned}$$

yielding that the sequences

$$\|\nabla u_j - ie\mathbf{A}_j u_j\|_{L^2}, \ \|\nabla \mathbf{A}_j\|_{L^2}, \ \|u_j\|_{L^q} \ (2 \leq q \leq 6) \text{ are bounded.} \tag{3.7}$$

Moreover by replacing u_j by $\frac{\sqrt{\mu}}{\|u_j\|_{L^2}} u_j$, we may assume that $\{(u_j, \mathbf{A}_j)\} \subset X$ is a minimizing sequence of $c_e(\mu)$.

Now we apply the concentration compactness principle [23, Lemma I.1, P.115] to the sequence

$$\rho_j(x) = |u_j(x)|^2 + |\nabla u_j - ie\mathbf{A}_j u_j|^2 + |\nabla \mathbf{A}_j|^2.$$

Then from (3.7), there exists $C > 0$ such that

$$\int_{\mathbb{R}^3} \rho_j(x)\, dx \leq C \text{ for all } j \in \mathbb{N}.$$

Without loss of generality, one may assume that

$$\int_{\mathbb{R}^3} \rho_j(x)\, dx \to \lambda \text{ as } j \to \infty \text{ for some } \lambda > 0.$$

It is well-known that the behavior of the sequence $(\rho_j)_{j\in\mathbb{N}}$ is governed by the three possibilities: Compactness, Vanishing and Dichotomy. Our goal is to show that Compactness occurs.

If Vanishing occurs, there exists a subsequence of $\{\rho_j\}$, still denoted by $\{\rho_j\}$, such that

$$\lim_{j\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_R(y)}\rho_j(x)\,dx=0\quad\text{for all } R>0.$$

Here $B_R(y)$ describes a ball of radius R with the center at $y\in\mathbb{R}^3$. Then by [24, Lemma I.1, P. 231], it follows that $u_j\to 0$ in $L^q(\mathbb{R}^3)$ for any $q\in(2,6)$. On the other hand since $\{u_j\}$ is a minimizing sequence for $c_e(\mu)$, one has

$$\begin{aligned}
&c_e(\mu)+o(1)=J(u_j,\mathbf{A}_j)\\
&=\frac{1}{2}\|\nabla u_j-ie\mathbf{A}_ju_j\|_{L^2}^2+\frac{e^2}{4}D(u_j)+\frac{1}{2}\|\nabla\mathbf{A}_j\|_{L^2}^2-\frac{1}{p+1}\|u_j\|_{L^{p+1}}^{p+1}\\
&\ge-\frac{1}{p+1}\|u_j\|_{L^{p+1}}^{p+1}.
\end{aligned}$$

Passing a limit $j\to\infty$, we get $0>c_e(\mu)\ge 0$. This is a contradiction, which rules out Vanishing.

Next we assume that Dichotomy occurs. Then by a standard argument (see [24, Section I.2] or [11, Proposition 1.7.6, P. 23]), there exist $\mu'\in(0,\mu)$ and $\{(u_{j,1},\mathbf{A}_{j,1})\}$, $\{(u_{j,2},\mathbf{A}_{j,2})\}\subset X$ such that

$$\|u_{j,1}\|_{L^2}^2\to\mu',\quad \|u_{j,2}\|_{L^2}^2\to\mu-\mu',$$

$$\operatorname{supp}(u_{j,1})\cap\operatorname{supp}(u_{j,2})=\emptyset,\ \delta_j:=\operatorname{dist}\big(\operatorname{supp}(u_{j,1}),\operatorname{supp}(u_{j,2})\big)\to\infty,\tag{3.8}$$

$$\|u_j-u_{j,1}-u_{j,2}\|_{L^q}\to 0\quad\text{for all } 2\le q<6,\tag{3.9}$$

$$\liminf_{j\to\infty}\int_{\mathbb{R}^3}|\nabla\mathbf{A}_j|^2-|\nabla\mathbf{A}_{j,1}|^2-|\nabla\mathbf{A}_{j,2}|^2\,dx\ge 0,\tag{3.10}$$

$$\liminf_{j\to\infty}\int_{\mathbb{R}^3}|\nabla u_j-ie\mathbf{A}_ju_j|^2-|\nabla u_{j,1}-ie\mathbf{A}_{j,1}u_{j,1}|^2-|\nabla u_{j,2}-ie\mathbf{A}_{j,2}u_{j,2}|^2\,dx\ge 0.\tag{3.11}$$

Moreover replacing $u_{j,1}$, $u_{j,2}$ by $\frac{\sqrt{\mu'}}{\|u_{j,1}\|_{L^2}}u_{j,1}$, $\frac{\sqrt{\mu-\mu'}}{\|u_{j,2}\|_{L^2}}u_{j,2}$ respectively, we may assume that $\|u_{j,1}\|_{L^2}^2=\mu'$, $\|u_{j,2}\|_{L^2}^2=\mu-\mu'$. Now from (3.8), one has

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_{j,1}(x)|^2|u_{j,2}(y)|^2}{|x-y|}\,dx\,dy\\
&\quad=\int_{\operatorname{supp}(u_{j,2})}\int_{\operatorname{supp}(u_{j,1})}\frac{|u_{j,1}(x)|^2|u_{j,2}(y)|^2}{|x-y|}\,dx\,dy\\
&\quad\le\frac{1}{\delta_j}\|u_{j,1}\|_{L^2}^2\|u_{j,2}\|_{L^2}^2\to 0\quad\text{as } j\to\infty.
\end{aligned}$$

Using (3.9) and arguing as in the proof of Lemma 2.2 in [29], a direct computation furnishes

$$\begin{aligned}
&D(u_j) - D(u_{j,1}) - D(u_{j,2}) \\
&= \int_{\mathbb{R}^3} S(u_j)|u_j|^2 - S(u_{j,1})|u_{j,1}|^2 - S(u_{j,2})|u_{j,2}|^2 \, dx \\
&= \int_{\mathbb{R}^3} \Big\{ \big(S(u_j)|u_j| + S(u_{j,1})|u_{j,1}| + S(u_{j,2})|u_{j,2}|\big)\big(|u_j| - |u_{j,1}| - |u_{j,2}|\big) \\
&\qquad + |u_j|\big(|u_{j,1}| + |u_{j,2}|\big)\big(S(u_j) - S(u_{j,1}) - S(u_{j,2})\big) \\
&\qquad + |u_{j,1}||u_{j,2}|\big(S(u_{j,1}) + S(u_{j,2})\big) + |u_j|\big(|u_{j,1}|S(u_{j,2}) + |u_{j,2}|S(u_{j,1})\big)\Big\} \, dx \\
&\to 0 \quad \text{as } j \to \infty.
\end{aligned}$$

Thus from (3.9)-(3.11) and by the diamagnetic inequality, we obtain

$$\begin{aligned}
c_e(\mu) &= \liminf_{j\to\infty} J(u_j, \mathbf{A}_j) \\
&\geq \liminf_{j\to\infty} J(u_{j,1}, \mathbf{A}_{j,1}) + \liminf_{j\to\infty} J(u_{j,2}, \mathbf{A}_{j,2}) \\
&\geq \liminf_{j\to\infty} J(|u_{j,1}|, \mathbf{0}) + \liminf_{j\to\infty} J(|u_{j,2}|, \mathbf{0}) \\
&\geq c_e(\mu') + c_e(\mu - \mu'),
\end{aligned}$$

which contradicts (3.6). Thus Dichotomy does not occur.

The only remaining possibility is Compactness, that is, there exists $\{y_j\} \subset \mathbb{R}^3$ such that for all $\varepsilon > 0$, there exists $R_\varepsilon > 0$ satisfying

$$\int_{B_{R_\varepsilon}(y_j)} \rho_j(x) \, dx \geq \lambda - \varepsilon. \tag{3.12}$$

From (3.7), one has

$$\begin{aligned}
\|\nabla u_j\|_{L^2} &\leq \|\nabla u_j - ie\mathbf{A}_j u_j\|_{L^2} + \|ie\mathbf{A}_j u_j\|_{L^2} \\
&\leq \|\nabla u_j - ie\mathbf{A}_j u_j\|_{L^2} + e\|\mathbf{A}_j\|_{L^6}\|u_j\|_{L^3},
\end{aligned}$$

showing that $\|u_j\|_{H^1}$ is bounded. Thus passing to a subsequence, we may assume that

$$u_j(\cdot - y_j) \rightharpoonup u \text{ in } H^1(\mathbb{R}^3, \mathbb{C}) \quad \text{and} \quad \mathbf{A}_j(\cdot - y_j) \rightharpoonup \mathbf{A} \text{ in } D^{1,2}(\mathbb{R}^3, \mathbb{R}^3).$$

Then from (3.12), it holds that $u_j(\cdot - y_j) \to u$ in $L^q(\mathbb{R}^3, \mathbb{C})$ for any $2 \leq q < 6$.

Next we claim that $\mathbf{A} = \mathbf{0}$ and $(u, \mathbf{0})$ is a minimizer for $c_e(\mu)$. Indeed by the weak lower semi-continuity of $\|\nabla \cdot\|_{L^2}$ and by Lemma 2.1 (iv), one finds

$$\begin{aligned}
\|u\|_{L^2}^2 &= \lim_{j\to\infty} \|u_j(\cdot - y_j)\|_{L^2}^2 = \mu, \\
c_e(\mu) &= \liminf_{j\to\infty} J\big(u_j(\cdot - y_j), \mathbf{A}_j(\cdot - y_j)\big) \geq J(u, \mathbf{A}).
\end{aligned} \tag{3.13}$$

At this stage, we don't know whether $\operatorname{div}\mathbf{A} = 0$ and hence cannot conclude that $J(u, \mathbf{A}) \geq c_e(\mu)$. To conclude, one observes that

$$\int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 \, dx = \int_{\mathbb{R}^3} |\operatorname{rot}\mathbf{A}|^2 \, dx + \int_{\mathbb{R}^3} (\operatorname{div}\mathbf{A})^2 \, dx.$$

Thus if $\operatorname{div}\mathbf{A} \neq 0$, it follows that $\int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 \, dx > \int_{\mathbb{R}^3} |\operatorname{rot}\mathbf{A}|^2 \, dx$ and hence

$$\begin{aligned} c_e(\mu) &\geq J(u, \mathbf{A}) \\ &> \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u - ie\mathbf{A}u|^2 + |\operatorname{rot}\mathbf{A}|^2 \right) dx + \frac{e^2}{4} D(u) - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \, dx \\ &\geq \inf_{(v,\mathbf{B}) \in X, \|v\|_{L^2}^2 = \mu, \operatorname{div}\mathbf{B} = 0} J(v, \mathbf{B}) = c_e(\mu), \end{aligned}$$

yielding that $\operatorname{div}\mathbf{A} = 0$. Then from (3.13), we obtain $J(u, \mathbf{A}) = c_e(\mu)$. Thus by Lemma 3.2, it holds that $\mathbf{A} = \mathbf{0}$ and $(u, \mathbf{0})$ is a minimizer for $c_e(\mu)$, as claimed.

Finally using (3.13) again, one has from $\operatorname{div}\mathbf{A} = \mathbf{0}$ that

$$\begin{aligned} &\lim_{j \to \infty} \left(\|\nabla u_j(\cdot - y_j) - ie\mathbf{A}_j(\cdot - y_j) u_j(\cdot - y_j)\|_{L^2}^2 + \|\nabla \mathbf{A}_j(\cdot - y_j)\|_{L^2}^2 \right) \\ &\quad = \|\nabla u\|_{L^2}^2. \end{aligned} \tag{3.14}$$

This implies that

$$\begin{aligned} &\|\nabla u_j(\cdot - y_j) - ie\mathbf{A}_j(\cdot - y_j) u_j(\cdot - y_j) - \nabla u\|_{L^2}^2 \\ &= \|\nabla u_j(\cdot - y_j) - ie\mathbf{A}_j(\cdot - y_j) u_j(\cdot - y_j)\|_{L^2}^2 \\ &\quad - 2\Big(\nabla u_j(\cdot - y_j) - ie\mathbf{A}_j(\cdot - y_j) u_j(\cdot - y_j), \nabla u \Big)_{L^2} + \|\nabla u\|_{L^2}^2 \\ &\leq \|\nabla u_j(\cdot - y_j) - ie\mathbf{A}_j(\cdot - y_j) u_j(\cdot - y_j)\|_{L^2}^2 + \|\nabla \mathbf{A}_j(\cdot - y_j)\|_{L^2}^2 \\ &\quad - 2\Big(\nabla u_j(\cdot - y_j) - ie\mathbf{A}_j(\cdot - y_j) u_j(\cdot - y_j), \nabla u \Big)_{L^2} + \|\nabla u\|_{L^2}^2 \to 0 \text{ as } j \to \infty, \end{aligned}$$

and hence

$$\nabla u_j(\cdot - y_j) - ie\mathbf{A}_j(\cdot - y_j) u_j(\cdot - y_j) \to \nabla u \text{ in } L^2.$$

Then from (3.14), we infer that $\nabla \mathbf{A}_j(\cdot - y_j) \to \mathbf{0}$ in L^2, which ends the proof. □

By Lemmas 3.2-3.5, one can see that for $1 < p < \frac{7}{3}$ and $0 < e < e^*$, there exists a minimizer $(u_e, \mathbf{0})$ of (1.17). Next we prove the existence of a minimizer in the borderline case $e = e^*$ when $2 < p < \frac{7}{3}$.

Lemma 3.6 *Suppose $2 < p < \frac{7}{3}$ and let $e^* = e^*(\mu, p)$ be the constant defined in Lemma 3.3. Then for $e = e^*$, the minimization problem* (1.17) *admit a minimizer.*

Proof We fix $\mu > 0$ and put $e_n = e^* - \frac{1}{n}$. Then from (3.4), it holds that $c_{e_n}(\mu) < 0$ and $c_{e_n}(\mu) \to c_e(\mu) = 0$. Moreover by lemmas 3.3-3.5, there exists $\{u_n\} \subset H^1(\mathbb{R}^3)$ with $\|u_n\|_{L^2}^2 = \mu$ such that $J(u_n, \mathbf{0}) = c_{e_n}(\mu)$. Our strategy is to prove that, after a suitable translation, u_n converges to a minimizer for $c_{e^*}(\mu)$.

First we show that $\big\||u_n|\big\|_{H^1}$ is bounded. Indeed from (3.1) and (3.2), one has

$$\begin{aligned}\frac{1}{2}\big\|\nabla|u_n|\big\|_{L^2}^2 + \frac{e_n^2}{4}D(u_n) &\le J(u_n, \mathbf{0}) + \frac{1}{p+1}\|u_n\|_{L^{p+1}}^{p+1} \\ &= c_{e_n}(\mu) + \frac{1}{p+1}\|u_n\|_{L^{p+1}}^{p+1} \\ &< \frac{1}{p+1}\|u_n\|_{L^{p+1}}^{p+1}. \qquad (3.15)\end{aligned}$$

By Lemma 2.1 (iii) and Lemma 2.2 (i), we get

$$\begin{aligned}\frac{1}{2}\big\|\nabla|u_n|\big\|_{L^2}^2 &< CD(u_n)^{\frac{7-3p}{2}}\big\|\nabla|u_n|\big\|_{L^2}^{3p-5}\|u_n\|_{L^2}^{4(p-2)} \\ &\le C\big\|\nabla|u_n|\big\|_{L^2}^{\frac{3}{2}(p-1)}\|u_n\|_{L^2}^{\frac{5-p}{2}} \\ &= C\mu^{\frac{5-p}{2}}\big\|\nabla|u_n|\big\|_{L^2}^{\frac{3}{2}(p-1)}.\end{aligned}$$

Since $\frac{3}{2}(p-1) < 2$, it follows that $\big\|\nabla|u_n|\big\|_{L^2}$ is bounded and so is $\big\||u_n|\big\|_{H^1}$.

Next we claim that $\|u_n\|_{L^{p+1}} \not\to 0$ as $n \to \infty$. For this purpose, we suppose by contradiction that $\|u_n\|_{L^{p+1}} \to 0$. Then from (3.15) and $e_n \to e^* > 0$, one has $\big\|\nabla|u_n|\big\|_{L^2} \to 0$ and $D(u_n) \to 0$. Since $2 < p < \frac{7}{3} < 3$, one can apply the Hölder inequality to obtain

$$\int_{\mathbb{R}^3} |u_n|^{p+1}\,dx = \int_{\mathbb{R}^3} |u_n|^{3(3-p)}|u_n|^{4(p-2)}\,dx \le \|u_n\|_{L^3}^{3(3-p)}\|u_n\|_{L^4}^{4(p-2)}.$$

By using Lemma 2.2 (i) with $p = 2$ and applying the interpolation inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^3} |u_n|^{p+1}\,dx &\le CD(u_n)^{\frac{3-p}{2}}\big\|\nabla|u_n|\big\|_{L^2}^{3-p}\|u_n\|_{L^6}^{3(p-2)}\|u_n\|_{L^2}^{p-2} \\ &\le C\mu^{p-2}D(u_n)^{\frac{3-p}{2}}\big\|\nabla|u_n|\big\|_{L^2}^{2p-3}.\end{aligned}$$

Then by the Young inequality, it holds that

$$\begin{aligned}J(u_n, \mathbf{0}) &= \frac{1}{2}\|\nabla u_n\|_{L^2}^2 + \frac{e_n^2}{4}D(u_n) - \frac{1}{p+1}\|u_n\|_{L^{p+1}}^{p+1} \\ &\ge \frac{1}{2}\big\|\nabla|u_n|\big\|_{L^2}^2 + \frac{e_n^2}{4}D(u_n) - CD(u_n)^{\frac{3-p}{2}}\big\|\nabla|u_n|\big\|_{L^2}^{2p-3} \\ &\ge \frac{1}{2}\big\|\nabla|u_n|\big\|_{L^2}^2 - \frac{C}{\varepsilon}\big\|\nabla|u_n|\big\|_{L^2}^{\frac{2(2p-3)}{p-1}} + \left(\frac{e_n^2}{4} - \varepsilon\right)D(u_n)\end{aligned}$$

for any $\varepsilon > 0$. Taking $\varepsilon = \frac{4}{e_n^2}$ and noticing that $2 < \frac{2(2p-3)}{p-1}$ for $p > 2$, we infer that $J(u_n, \mathbf{0}) \geq 0$ for sufficiently large $n \in \mathbb{N}$, contradicting to the fact that $J(u_n, \mathbf{0}) = c_{e_n}(\mu) < 0$.

Now since $\int_{\mathbb{R}^3} |u_n|^{p+1} \, dx \not\to 0$, $\{u_n\}$ does not vanish, that is, there exist $\delta > 0$ and $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{B_1(y_n)} |u_n|^2 \, dx \geq \delta.$$

Putting $v_n(\cdot) = u_n(\cdot + y_n)$, then $\big\| |v_n| \big\|_{H^1}$ is bounded and $\int_{B_1(0)} |v_n|^2 \, dx \geq \delta$. This implies that there exists $v_0 \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that $|v_n| \rightharpoonup v_0$ in $H^1(\mathbb{R}^3)$. Moreover by the Fatou lemma, we also have

$$\|v_0\|_{L^2}^2 \leq \liminf_{n \to \infty} \|v_n\|_{L^2}^2 = \mu.$$

Next we suppose that $\|v_0\|_{L^2}^2 < \mu$. Since $e^* = e^*(\mu)$ is increasing with respect to μ by (3.3), it follows that $e^*(\|v_0\|_{L^2}^2) < e^*(\mu)$. Then by the definition of e^* in (3.4) and the monotonicity of $c_e(\mu)$ with respect to e, we have

$$0 = c_{e^*(\|v_0\|_{L^2}^2)}(\|v_0\|_{L^2}^2) \leq c_{e^*(\mu)}(\|v_0\|_{L^2}^2) \leq 0,$$

from which we conclude that

$$c_{e^*(\mu)}(\|v_0\|_{L^2}^2) = 0. \tag{3.16}$$

On the other hand, since $\big\| |v_n| \big\|_{H^1}$ is bounded, one has by Lemma 2.1 (iii) that

$$J_{e_n}(v_n, \mathbf{0}) - J_{e^*(\mu)}(v_n, \mathbf{0}) = \frac{e_n^2 - e^*(\mu)^2}{4} D(v_n) \to 0.$$

Here we write $J(u, \mathbf{A}) = J_e(u, \mathbf{A})$ to emphasize the dependence on e. Then by the splitting lemma due to [29], we get

$$\begin{aligned} 0 = c_{e^*(\mu)}(\mu) &= \lim_{n \to \infty} c_{e_n}(\mu) = \lim_{n \to \infty} J_{e_n}(u_n, \mathbf{0}) \\ &= \lim_{n \to \infty} J_{e_n}(v_n, \mathbf{0}) \\ &= \lim_{n \to \infty} \Big(J_{e_n}(v_n, \mathbf{0}) - J_{e^*(\mu)}(v_n, \mathbf{0}) \Big) + \lim_{n \to \infty} J_{e^*(\mu)}(v_n, \mathbf{0}) \\ &= \lim_{n \to \infty} J_{e^*(\mu)}(v_n - v_0, \mathbf{0}) + J_{e^*(\mu)}(v_0, \mathbf{0}). \end{aligned} \tag{3.17}$$

Moreover from $v_n \rightharpoonup v_0$ in $L^2(\mathbb{R}^3)$, it follows that

$$\mu = \lim_{n \to \infty} \|v_n\|_{L^2}^2 = \lim_{n \to \infty} \|v_n - v_0\|_{L^2}^2 + \|v_0\|_{L^2}^2$$

and hence

$$\lim_{n \to \infty} J_{e^*(\mu)}(v_n - v_0, \mathbf{0}) \geq \lim_{n \to \infty} c_{e^*(\mu)}\big(\|v_n - v_0\|_{L^2}^2\big) = c_{e^*(\mu)}\big(\mu - \|v_0\|_{L^2}^2\big). \tag{3.18}$$

Since $\mu - \|v_0\|_{L^2}^2 < \mu$, one has $e^*\big(\mu - \|v_0\|_{L^2}^2\big) < e^*(\mu)$, yielding that

$$0 = c_{e^*\left(\mu - \|v_0\|_{L^2}^2\right)}\big(\mu - \|v_0\|_{L^2}^2\big) \le c_{e^*(\mu)}\big(\mu - \|v_0\|_{L^2}^2\big) \le 0,$$

and hence

$$c_{e^*(\mu)}\big(\mu - \|v_0\|_{L^2}^2\big) = 0.$$

Then from (3.17) and (3.18), we infer that

$$\lim_{n\to\infty} J_{e^*(\mu)}(v_n - v_0, \mathbf{0}) \ge 0 \quad \text{and} \quad J_{e^*(\mu)}(v_0, \mathbf{0}) \le 0.$$

If $J_{e^*(\mu)}(v_0, \mathbf{0}) < 0$, we have $c_{e^*(\mu)}\big(\|v_0\|_{L^2}^2\big) < 0$, contradicting to (3.16). Thus it holds that

$$J_{e^*(\mu)}(v_0, \mathbf{0}) = 0 = c_{e^*(\mu)}\big(\|v_0\|_{L^2}^2\big) \tag{3.19}$$

and $(v_0, \mathbf{0})$ is a minimizer for $c_{e^*(\mu)}\big(\|v_0\|_{L^2}^2\big)$. Then by Lemma 3.4 (i) and from $\|v_0\|_{L^2}^2 < \mu$, we find that (3.5) holds for $\lambda = \frac{\mu}{\|v_0\|_{L^2}^2}$, that is,

$$c_{e^*(\mu)}(\mu) < \frac{\mu}{\|v_0\|_{L^2}^2} c_{e^*(\mu)}\big(\|v_0\|_{L^2}^2\big) = 0.$$

This is a contradiction to the definition of $e^*(\mu)$ in (3.4). Thus one concludes that $\|v_0\|_{L^2}^2 = \mu$.

Finally from (3.19), we obtain

$$J_{e^*(\mu)}(v_0, \mathbf{0}) = c_{e^*(\mu)}\big(\|v_0\|_{L^2}^2\big) = c_{e^*(\mu)}(\mu),$$

from which one finds that $(v_0, \mathbf{0})$ is a minimizer for $c_{e^*(\mu)}(\mu)$, as claimed. □

Finally we establish the non-existence of minimizers if $2 < p < \frac{7}{3}$ and $e > e^*$.

Lemma 3.7 *Suppose $2 < p < \frac{7}{3}$ and let $e^* = e^*(\mu, p)$ be the constant defined in Lemma 3.3. Then for $e > e^*$, the minimization problem* (1.17) *does not admit any minimizers. When $p = 2$, there is no minimizer of* (1.17) *if $e > \frac{2}{3}$.*

Proof We suppose by contradiction that there exist $\mu > 0$ and $\hat{e} > e^*(\mu, p)$ such that $c_{\hat{e}}(\mu)$ has a minimizer if $p > 2$. By the definition of e^* in (3.3), we find that $e^*(\mu, p)$ is continuous and increasing in μ. Thus one can choose $\hat{\mu} > \mu$ so that $e^*(\hat{\mu}, p) \in \big(e^*(\mu, p), \hat{e}\big)$.

On the other hand by Lemma 3.3 and the characterization of $e^*(\mu, p)$ in (3.4), it follows that $c_{\hat{e}}(\mu) = 0$. Then by Lemma 3.4, we get $c_{\hat{e}}(\hat{\mu}) < \frac{\hat{\mu}}{\mu} c_{\hat{e}}(\mu) = 0$, from which we deduce that $\hat{e} < e^*(\hat{\mu}, p)$. This is a contradiction and hence the proof is complete when $2 < p < \frac{7}{3}$.

Finally if $p = 2$, using Lemma 2.2 and the Young inequality, we find that

$$J(u, \mathbf{A}) \ge \frac{1}{2}\big\|\nabla|u|\big\|_{L^2}^2 + \frac{e^2}{4} D(u) - \frac{\sqrt{2}}{3} D(u)^{\frac{1}{2}} \big\|\nabla|u|\big\|_{L^2} \ge \left(\frac{e^2}{4} - \frac{1}{9}\right) D(u).$$

This implies that if $e > \frac{2}{3}$, $J(u, \mathbf{A}) > 0$ for any $(u, \mathbf{A}) \in X$. Thus by Lemma 3.3, the non-existence of minimizers holds. □

Proof (Proof of Theorem 1.1) It is a straightforward consequence of Lemmas 3.2-3.7. □

4 A refined stability of standing waves

In this section, we establish the orbital stability of standing waves of (1.1)-(1.3) by using the relative compactness of minimizing sequences obtained in Lemma 3.5.

First we recall known results for ground states of the following elliptic problem:

$$-\Delta u + \omega u + e^2 S(u)u = |u|^{p-1}u \ \text{ in } \ \mathbb{R}^3. \tag{4.1}$$

Proposition 4.1 *Let $\omega > 0$ be given. Suppose that $2 \le p < 5$ and $e < e^*$ if $p = 2$, where e^* is the constant defined in Lemma 3.3.*

(i) *The problem (4.1) has a ground state $u_{e,\omega}$, which can be assumed to be real-valued. Moreover if $p = 2$, there exists a unique $\mu(\omega) > 0$ such that u is a ground state of (4.1) if and only if $(u, \mathbf{0})$ is a minimizer of $c_e\big(\mu(\omega)\big)$.*

(ii) *There exists $e_0 = e_0(\omega, p) > 0$ such that for $e < e_0$, the ground state of (4.1) is unique up to phase shift and translation, that is, it holds*

$$\mathcal{G}_e(\omega) = \big\{ \exp(i\theta) u_{e,\omega}(\cdot - y) \ ; \ \theta \in [0, 2\pi), \ y \in \mathbb{R}^3 \big\}.$$

Moreover if $p = 2$, e_0 is independent of ω.

(iii) *If $p = 2$, for any $\mu > 0$ and $0 < e < \min\{e^*, e_0\}$, the minimizer $(u_{e,\mu}, \mathbf{0})$ of (1.17) is unique up to phase shift and translation, that is, it holds*

$$\mathcal{M}_e(\mu) := \Big\{ \big(\exp(i\theta) u_{e,\mu}(\cdot - y), \mathbf{0} \big) \ ; \ \theta \in [0, 2\pi), \ y \in \mathbb{R}^3 \Big\}.$$

The proof can be done by using Lemma 3.2 and from Theorem 1.2 in [15]. Now we are able to complete the proof of Theorem 1.2.

Proof (Proof of Theorem 1.2) We fix $0 < e < \min(e_0, e^*)$ and let $u_{e,\omega}$ be the unique (real-valued) ground state of (4.1). We adapt the argument developed in [12], [13]. For this purpose, let us assume by contradiction that there exist $\varepsilon_0 > 0$,

$$(\psi_{(0)j}, \mathbf{A}_{(0)j}, \mathbf{A}_{(1)j})_{j \in \mathbb{N}} \subset H^1(\mathbb{R}^3, \mathbb{C}) \times D^{1,2}(\mathbb{R}^3, \mathbb{R}^3) \times L^2(\mathbb{R}^3, \mathbb{R}^3)$$

and $\{t_j\} \subset \mathbb{R}$ such that $\operatorname{div} \mathbf{A}_{(0)j} = \operatorname{div} \mathbf{A}_{(1)j} = 0$ and

$$\begin{aligned} &\|\nabla \psi_{(0)j} - ie\mathbf{A}_{(0)j}\psi_{(0)j} - \nabla u_{e,\omega}\|_{L^2} + \|\psi_{(0)j} - u_{e,\omega}\|_{L^2} \\ &\quad + \|\operatorname{rot} \mathbf{A}_{(0)j}\|_{L^2} + \|\mathbf{A}_{(1)j}\|_{L^2} \to 0 \quad \text{as } j \to \infty \end{aligned} \tag{4.2}$$

but the corresponding solution $(\psi_j, \mathbf{A}_j, \phi_j)$ of (1.1)-(1.3) satisfies

$$\begin{aligned} &\inf_{y \in \mathbb{R}^3, \chi \in C^2} \Big\| \nabla \psi_j(t_j, \cdot) - ie\mathbf{A}_j(t_j, \cdot)\psi_j(t_j, \cdot) - \exp\big(ie\chi(t_j, \cdot)\big) \nabla u_{e,\omega}(\cdot + y) \Big\|_{L^2} \\ &+ \inf_{y \in \mathbb{R}^3, \chi \in C^2} \Big\| \psi_j(t_j, \cdot) - \exp\big(ie\chi(t_j, \cdot)\big) u_{e,\omega}(\cdot + y) \Big\|_{L^2} \\ &+ \inf_{\chi \in C^2} \Big\{ \|\operatorname{rot} \mathbf{A}_j(t_j, \cdot) - \nabla \chi(t_j, \cdot)\|_{L^2} + \|(\mathbf{A}_j)_t(t_j, \cdot) - \nabla \chi_t(t_j, \cdot)\|_{L^2} \Big\} \\ &+ \inf_{y \in \mathbb{R}^3, \chi \in C^2} \|\phi_j(t_j, \cdot) - \phi_{e,\omega}(\cdot + y) + \chi_t(t_j, \cdot)\|_{D^{1,2}} \ge \varepsilon_0. \end{aligned}$$

We fix $\chi \in C^2(\mathbb{R}_+ \times \mathbb{R}^3, \mathbb{R})$ with $\chi(t,x) = \chi(x), \Delta\chi = 0$ arbitrarily and put

$$\tilde{\psi}_j = \psi_j \exp(-ie\chi),\ \ \tilde{\mathbf{A}}_j = \mathbf{A}_j - \nabla\chi,\ \ \tilde{\phi}_j = \phi_j + \chi_t = \phi_j,$$

$$u_j(x) = \tilde{\psi}_j(t_j, x),\ \ \mathbf{B}_j(x) = \tilde{\mathbf{A}}_j(t_j, x),\ \ \mathbf{C}_j(x) = (\tilde{\mathbf{A}}_j)_t(t_j, x),\ \ \varphi_j(x) = \tilde{\phi}_j(t_, x).$$

Then one has

$$\begin{aligned} &\inf_{y\in\mathbb{R}^3} \Big\{ \|\nabla u_j - ie\mathbf{B}_j u_j - \nabla u_{e,\omega}(\cdot + y)\|_{L^2} + \|u_j - u_{e,\omega}(\cdot + y)\|_{L^2} \Big\} \\ &\quad + \|\operatorname{rot}\mathbf{B}_j\|_{L^2} + \|\mathbf{C}_j\|_{L^2} + \inf_{y\in\mathbb{R}^3} \|\varphi_j - \phi_{e,\omega}(\cdot + y)\|_{D^{1,2}} \geq \varepsilon_0. \end{aligned} \tag{4.3}$$

Now by the charge conservation law (1.7), Proposition 4.1 (i) and from (4.2), one finds that

$$\|u_j\|_{L^2}^2 = \|\tilde{\psi}_j(t_j, \cdot)\|_{L^2}^2 = \|\psi_j(t_j, \cdot)\|_{L^2}^2 = \|\psi_{(0)j}\|_{L^2}^2 \to \|u_{e,\omega}\|_{L^2}^2 = \mu(\omega). \tag{4.4}$$

Moreover by the energy conservation law (1.8) and the gauge invariance of the energy stated in (1.9), we also have

$$\begin{aligned} \mathcal{E}(u_j, \mathbf{B}_j, \varphi_j) &= \mathcal{E}(\tilde{\psi}_j(t_j, \cdot), \tilde{\mathbf{A}}_j(t_j, \cdot), \tilde{\phi}_j(t_j, \cdot)) \\ &= \mathcal{E}(\psi_j(t_j, \cdot), \mathbf{A}_j(t_j, \cdot), \phi_j(t_j, \cdot)) \\ &= \mathcal{E}(\psi_{(0)j}, \mathbf{A}_{(0)j}, \phi_{(0)j}) \\ &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\psi_{(0)j} - ie\mathbf{A}_{(0)j}\psi_{(0)j}|^2 + |\operatorname{rot}\mathbf{A}_{(0)j}|^2 + |\mathbf{A}_{(1)j}|^2\, dx \\ &\quad + \frac{e^2}{4}\int_{\mathbb{R}^3} S(\psi_{(0)j})|\psi_{(0)j}|^2\, dx - \frac{1}{3}\int_{\mathbb{R}^3} |\psi_{(0)j}|^3\, dx. \end{aligned} \tag{4.5}$$

From (4.2), we know that $\|\operatorname{rot}\mathbf{A}_{(0)j}\|_{L^2} \to 0$. But since $\operatorname{div}\mathbf{A}_{(0)j} = 0$, it follows that $\|\operatorname{rot}\mathbf{A}_{(0)j}\|_{L^2} = \|\nabla\mathbf{A}_{(0)j}\|_{L^2}$ and hence $\|\mathbf{A}_{(0)j}\|_{L^6} \to 0$ by the Sobolev inequality. Next we claim that $\psi_{(0)j} \to u_{e,\omega}$ in $H^1(\mathbb{R}^3)$. Indeed one has from (4.2) that $\|\psi_{(0)j} - u_{e,\omega}\|_{L^2} \to 0$ and

$$\begin{aligned} &\|\nabla\psi_{(0)j} - \nabla u_{e,\omega}\|_{L^2} \\ &\leq \|\nabla\psi_{(0)j} - ie\mathbf{A}_{(0)j}\psi_{(0)j} - \nabla u_{e,\omega}\|_{L^2} + \|ie\mathbf{A}_{(0)j}\psi_{(0)j}\|_{L^2} \\ &\leq \|\nabla\psi_{(0)j} - ie\mathbf{A}_{(0)j}\psi_{(0)j} - \nabla u_{e,\omega}\|_{L^2} + e\|\mathbf{A}_{(0)j}\|_{L^6}\|\psi_{(0)j}\|_{L^3}. \end{aligned}$$

Moreover by the diamagnetic inequality and the Sobolev inequality, we find

$$\begin{aligned} \|\psi_{(0)j}\|_{L^6} &\leq C\big\|\nabla|\psi_{(0)j}|\big\|_{L^2} \leq C\|\nabla\psi_{(0)j} - ie\mathbf{A}_{(0)j}\psi_{(0)j}\|_{L^2} \\ &\leq C\|\nabla\psi_{(0)j} - ie\mathbf{A}_{(0)j}\psi_{(0)j} - \nabla u_{e,\omega}\|_{L^2} + C\|\nabla u_{e\omega}\|_{L^2} \leq C. \end{aligned}$$

Thus by the interpolation, it holds that

$$\|\nabla\psi_{(0)j} - \nabla u_{e,\omega}\|_{L^2} \leq \|\nabla\psi_{(0)j} - ie\mathbf{A}_{(0)j}\psi_{(0)j} - \nabla u_{e,\omega}\|_{L^2} + C\|\mathbf{A}_{(0)j}\|_{L^6} \to 0,$$

yielding that $\psi_{(0)j} \to u_{e,\omega}$ in $H^1(\mathbb{R}^3)$. Then from (4.5) and by Proposition 4.1 (i), one gets

$$\mathcal{E}(u_j, \mathbf{B}_j, \varphi_j) = \mathcal{E}(\psi_{(0)j}, \mathbf{A}_{(0)j}, \phi_{(0)j}) \to J(u_{e,\omega}, \mathbf{0}) = c_e\big(\mu(\omega)\big). \tag{4.6}$$

On the other hand, recalling that $\mathrm{div}\mathbf{A}_{(0)j} = \mathrm{div}\mathbf{A}_{(1)j} = 0$, it follows that $\mathrm{div}\mathbf{A}_j(t,\cdot) = 0$ for all $t > 0$, from which we conclude that $\mathrm{div}\mathbf{B}_j = \mathrm{div}\mathbf{A}_j(t_j,\cdot) + \Delta\chi = 0$. Thus we obtain

$$\mathcal{E}(u_j, \mathbf{B}_j, \varphi_j) = J(u_j, \mathbf{B}_j) + \frac{1}{2}\|\mathbf{C}_j\|_{L^2}^2 \geq J(u_j, \mathbf{B}_j) \tag{4.7}$$

showing that $J(u_j, \mathbf{B}_j)$ is bounded. Thus passing to a subsequence, we may assume that $J(u_j, \mathbf{B}_j) \to d$ for some $d \in \mathbb{R}$ and $d \leq c_e\big(\mu(\omega)\big)$ by (4.6). But by the definition of $c_e\big(\mu(\omega)\big)$ and from (4.4), it holds that $d = c_e\big(\mu(\omega)\big)$. Thus from (4.6) and (4.7), we infer that $\|\mathbf{C}_j\|_{L^2} \to 0$ and $(u_j, \mathbf{B}_j) \subset X$ is a minimizing sequence for $c_e\big(\mu(\omega)\big)$. Then by Lemma 3.5, one concludes that there exists $\{y_j\} \subset \mathbb{R}^3$ such that

$$\nabla u_j(\cdot - y_j) - ie\mathbf{B}_j(\cdot - y_j)u_j(\cdot - y_j) \to \nabla u \text{ in } L^2(\mathbb{R}^3),\ u_j(\cdot - y_j) \to u \text{ in } L^2(\mathbb{R}^3),$$
$$\text{and } \ \mathbf{B}_j(\cdot - y_j) \to \mathbf{0} \text{ in } D^{1,2}(\mathbb{R}^3) \ \text{ with } \ J(u, \mathbf{0}) = c_e\big(\mu(\omega)\big).$$

Since $\mathrm{div}\mathbf{B}_j = 0$, this yields that $\|\,\mathrm{rot}\mathbf{B}_j\|_{L^2} \to 0$. Moreover by Proposition 4.1 (iii), it follows that $u(\cdot) = e^{i\theta}u_{e,\omega}(\cdot + y_0)$ for some $y_0 \in \mathbb{R}^3$ and $\theta \in [0, 2\pi)$. Thus by applying the gauge transformation again, one gets

$$\|\nabla u_j - ie\mathbf{B}_j u_j - \nabla u_{e,\omega}(\cdot + z_j)\|_{L^2} + \|u_j - u_{e,\omega}(\cdot + z_j)\|_{L^2} \to 0, \text{ where } z_j = y_j + y_0.$$

Finally we notice that $\varphi_j = \frac{e}{2}(-\Delta)^{-1}|u_j|^2$ and $\phi_{e,\omega} = \frac{e}{2}(-\Delta)^{-1}|u_{e,\omega}|^2$. Thus by the Hardy-Littlewood-Sobolev inequality, one has

$$\begin{aligned}
&\|\varphi_j - \phi_{e,\omega}(\cdot + z_j)\|_{D^{1,2}} \\
&= \left\| \nabla \left(\frac{e}{2}(-\Delta)^{-1}\big(|u_j|^2 - |\phi_{e,\omega}(\cdot + z_j)|^2\big) \right) \right\|_{L^2} \\
&\leq C\left\| (-\Delta)^{-\frac{1}{2}}\big(|u_j|^2 - |\phi_{e,\omega}(\cdot + z_j)|^2\big) \right\|_{L^2} \\
&\leq C\big\| |u_j|^2 - |\phi_{e,\omega}(\cdot + z_j)|^2 \big\|_{L^{\frac{6}{5}}} \\
&\leq C\|u_j - \phi_{e,\omega}(\cdot + z_j)\|_{L^{\frac{12}{5}}} \|u_j + \phi_{e,\omega}(\cdot + z_j)\|_{L^{\frac{12}{5}}} \to 0,
\end{aligned}$$

from which one deduces that

$$\begin{aligned}
&\|\nabla u_j - ie\mathbf{B}_j u_j - \nabla u_{e,\omega}(\cdot + z_j)\|_{L^2} + \|u_j - u_{e,\omega}(\cdot + z_j)\|_{L^2} \\
&\quad + \|\,\mathrm{rot}\mathbf{B}_j\|_{L^2} + \|\mathbf{C}_j\|_{L^2} + \|\varphi_j - \phi_{e,\omega}(\cdot + z_j)\|_{D^{1,2}} \to 0.
\end{aligned}$$

This contradicts to (4.3) and hence the proof is complete. □

Remark 4.2 In a similar way, one can prove that the minimizer set $\mathcal{M}_e(\mu)$ is stable for $1 < p < \frac{7}{3}$ and $e < e^*$. Here the stability of the minimizer set is given as follows: For every $\varepsilon > 0$, there exists $\delta > 0$ such that if an initial value $(\psi_{(0)}, \mathbf{A}_{(0)}, \mathbf{A}_{(1)})$ of (1.1)-(1.3) satisfies (1.12) and

$$\begin{aligned}
&\inf_{(\tilde{\psi}, \mathbf{0}) \in \mathcal{M}_e(\mu)} \Big\{ \big\|\nabla\psi_{(0)} - ie\mathbf{A}_{(0)}\psi_{(0)} - \nabla\tilde{\psi}\big\|_{L^2} + \|\psi_{(0)} - \tilde{\psi}\|_{L^2} \Big\} \\
&\quad + \|\nabla\mathbf{A}_{(0)}\|_{L^2} + \|\mathbf{A}_{(1)}\|_{L^2} < \delta,
\end{aligned}$$

then the corresponding solution $(\psi, \mathbf{A}, \phi)$ of (1.1)-(1.3) fulfills

$$\sup_{t>0}\left\{ \inf_{(\tilde{\psi},\mathbf{0})\in\mathcal{M}_e(\mu)} \left(\left\| \nabla\psi(t,\cdot) - ie\mathbf{A}(t,\cdot)\psi(t,\cdot) - \nabla\tilde{\psi} \right\|_{L^2} + \|\psi(t,\cdot) - \tilde{\psi}\|_{L^2} \right) \right.$$
$$+ \|\nabla\mathbf{A}(t,\cdot)\|_{L^2} + \|\mathbf{A}_t(t,\cdot)\|_{L^2}$$
$$\left. + \inf_{(\tilde{\psi},\mathbf{0})\in\mathcal{M}_e(\mu)} \left\| \phi(t,\cdot) - \frac{e}{2}(-\Delta)^{-1}|\tilde{\psi}|^2 \right\|_{D^{1,2}} \right\} < \varepsilon.$$

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