On the existence of solitary waves for Boussinesq type equations and a new conservative model.

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July 1, 2014

Abstract

In this paper, we present a local Cauchy theory for a new enhanced Boussinesq-type system with constant bathymetry written in a conservative form. We also prove the existence of solitary wave for a large class of asymptotic models, including Beji-Nadaoka, Madsen-Sorensen and Nwogu equations. Furthermore, we give a procedure to calculate numerically these particular solutions and we present some effective computations.

1 Introduction

The description of the motion of the free surface and the evolution of the velocity field of an ideal, incompressible and irrotational fluid under the gravity is well-described by the Euler equations, especially when surface tension and dissipative effects are neglected. However, in many physical situations, the Euler equations appear too complex compare to the complexity of the flow one wants to describe. Consequently, many asymptotic models have been introduced, fitting restricted physical regimes. In this direction, significant effort has been made during the last two decades to develop systems of depth averaged equations which correctly reproduce the dispersion characteristic of wave propagation in the near-shore region. In the context of coastal applications, most of them are based on the nonlinear shallow water equations (NSW). The NSW equations provides a good description of the nonlinear phenomenas occuring in the near-shore region such as wave breaking for

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example. Nevertheless, by neglecting the dispersive effects, they appear not completely satisfactory. Roughly speaking, they are derived from the Euler equations by neglecting the terms of order σ^2 where

$$\sigma^2 = \frac{d_0^2}{L^2},$$

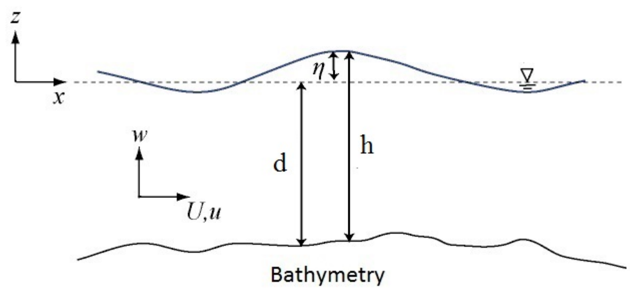
represents the shallowness coefficient (d_0 is the typical depth and L is the typical horizontal scale). By pushing the asymptotic approximation one degree further, one can obtain the Green-Naghdi equations which contains almost all of the dispersive effects (see [17]). However, simpler models of Boussinesq type can also be considered, providing more smallness assumptions on the amplitude of the waves and on the topography variations (see for example [5]). Note that dealing with complex bathymetries in this situation is one of the main challenge of coastal engineering.

Starting from the Boussinesq equations of Peregrine [20], several Boussinesq systems have been developed such that, for example, the enhanced equations of Madsen-Sorensen [18], the equations of Beji-Nadaoka [2], the equations of Nwogu [19], the nonlinear Serre-Green-Naghdi equations [16] and shallow water type models [12].

The literature concerning numerical treatment of the previous systems is abondant and contains very promising approach involving finite differences, finite volumes or finite elements approached. Due to the presence of higher order partial derivatives in the different models cited above, the finite difference scheme is widely used despite its poor local mesh adaptivity potential (see [2, 11, 13, 19]). Note that a few tentative of using multi-dimensional unstructured finite volume discretization of Boussinesq equations have been successfully performed (see [1, 9]). Otherwise, continuous finite element and residual based schemes included upwind stabilization have been used in [10].

Before going further, let us introduce some notations. For simplicity, in this article, we only deal with 2-D and 1-D problems. Denote by (x, z) respectively the horizontal and the vertical spatial dimension. We consider an incompressible and irrotational flow of an ideal fluid under gravity, acting in the direction of the negative z-values. Denote by $d(x)$ the depth at still water and $\eta(t, x)$ the surface elevation from its rest position. The total depth is then $h(t, x) = d(x) + \eta(t, x)$ (see Figure 1).

Owning these notations, the usual 2-D Euler system governing the evo-



lution of the free surface reads

$$u_t + uu_x + wu_z + \frac{p_x}{\rho} = 0, \tag{1.1}$$

$$v_t + uv_x + ww_z + \frac{p_z}{\rho} + g = 0, \tag{1.2}$$

$$u_x + w_z = 0, \tag{1.3}$$

$$u_z - w_x = 0. \tag{1.4}$$

The boundary conditions at the free surface $z = \eta$ reads

$$w = \eta_t + u\eta_x, \tag{1.5}$$

$$p = 0, \tag{1.6}$$

and at the bottom $z = -d$, one has

$$w = -ud_x, \tag{1.7}$$

where u and w denote respectively the horizontal and vertical velocities, ρ the constant density of the fluid, p the pressure and g the acceleration due to gravity. Starting from System (1.1)-(1.7), one can distinguish two family-type of asymptotic models. The first one uses the velocity u and the surface elevation η as unknowns. This is the case for the so-called Boussinesq system (see [6])

$$\begin{aligned} &\eta_t + u_x + (u\eta)_x + au_{xxx} - b\eta_{xxt} = 0, \\ &u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} = 0, \end{aligned} \tag{1.8}$$

for Beji-Nadaoka equations (see [2])

$$\begin{aligned} &\eta_t + [(d+\eta)\bar{u}]_x = 0, \\ &\bar{u}_t + \bar{u}\bar{u}_x + g\eta_x + (1+3\alpha_B)(\tfrac{d^2}{6}\bar{u}_{txx} - \tfrac{d}{2}[d\bar{u}]_{txx}) \\ &+3g\alpha_B(\tfrac{d^2}{6}\eta_{xxx} - \tfrac{d}{2}[d\eta_x]_{xx}) = 0, \end{aligned} \tag{1.9}$$

where $\bar{u}$ denotes the depth averaged horizontal velocity, and for the Nwogu formulation (see [19])

$$\begin{aligned} &\eta_t + [(d+\eta)U]_x + \left[\beta_1 d^2 [dU]_{xx} + \beta_2 d^3 U_{xx}\right]_x = 0, \\ &U_t + UU_x + g\eta_x + \alpha_1 d [dU_t]_{xx} + \alpha_2 d^2 U_{txx} = 0, \end{aligned} \tag{1.10}$$

with $\theta \in [-1, 0]$, where U denotes the horizontal velocity at depth d, and

$$\alpha_1 = \theta,\ \alpha_2 = \frac{\theta^2}{2},$$

$$\beta_1 = \theta + \frac{1}{2},\ \beta_2 = \frac{\theta^2}{2} - \frac{1}{6}.$$

Another different point of view consists in writing the asymptotic models with a different variable $q(t,x) = h(t,x)u(t,x)$, where q is homogeneous to a flux. This formulation was used, for example, by Madsen and Sorensen to derive the following set of equations

$$\begin{aligned} &\eta_t + \bar{q}_x = 0 \\ &\bar{q}_t + gh\eta_x + \left(\tfrac{\bar{q}^2}{h}\right)_x + (1+\alpha_M)\left(\tfrac{d^3}{6}\left(\tfrac{\bar{q}}{d}\right)_{txx} - \tfrac{d^2}{2}\bar{q}_{txx}\right) \\ &+ g\alpha_M\left(\tfrac{d^3}{6}\eta_{xxx} - \tfrac{d^2}{2}[d\eta_x]_{xx}\right) = 0. \end{aligned} \tag{1.11}$$

In this direction, in [3], a new system, referred as the Nwogu-Abott system, was introduced and reads

$$\begin{cases} \eta_t + Q_x + \left[\beta_1 d^2 Q_{xx} + \beta_2 d^3 \left(\dfrac{Q}{d}\right)_{xx}\right]_x = 0 & (1.12) \\ Q_t + \left(\dfrac{Q^2}{h}\right)_x + gh\eta_x + \alpha_1 d^2 Q_{txx} + \alpha_2 d^3 \left(\dfrac{Q}{d}\right)_{txx} = 0, & (1.13) \end{cases}$$

with

$$\beta_1 = \theta + \frac{1}{2},\ \beta_2 = \frac{\theta^2}{2} - \frac{1}{6};$$

$$\alpha_1 = \theta,\ \alpha_2 = \frac{\theta^2}{2}.$$

System (1.12)-(1.13) is similar to the Nwogu equations, in a sense that both systems share the same linear dispersion properties, but appears very promising for practical application in coastal engineering due to its conservative form (see [3] for the first introduction of this new system and for a comparaison between the different models cited above). In this paper, after recalling the derivation of (1.12)-(1.13) and in order to justify its accuracy, we present a local Cauchy theory showing that it is well-posed under some technical conditions on the initial data. Note that Cauchy problem for the water waves problems has been widely studied for decades. We refer to the work of D. Lannes [14] (see also [15] and the references therein). The question of local well-posedness for asymptotic models is also of interest [6] and widely studied for the systems using the (η, u)-formulation. To our knowledge, no existence results are known for the (η, q)-formulation. The theorem concerning (1.12)-(1.13) reads

Theorem 1. *Assume that* $(\eta_0, Q_0) \in H^2(\mathbb{R}) \times H^4(\mathbb{R})$ *satisfy*

$$d + \varepsilon\eta_0(x) > 0 \quad \text{on } \mathbb{R}, \tag{1.14}$$

$$g(d + \varepsilon\eta_0) - \varepsilon\frac{Q_0^2}{(d + \varepsilon\eta_0)^2} > 0 \quad \text{on } \mathbb{R}, \tag{1.15}$$

and that the bathymetry d is constant. Then there exists a time $T > 0$ with $T = O(\frac{1}{\varepsilon})$ and a unique local solution to the Cauchy problem (1.12)-(1.13) *sucht that*

$$\eta \in C([0,T]; L^2(\mathbb{R})) \cap L^\infty(0,T; H^2(\mathbb{R})),$$
$$Q \in C([0,T]; L^2(\mathbb{R})) \cap L^\infty(0,T; H^4(\mathbb{R})),$$

with

$$\eta(0,x) = \eta_0(x), \quad Q(0,x) = Q_0(x).$$

Eventhough all the systems presented above have similar linear dispersion characteristics and formulations, they are not nonlinearly equivalent and their behaviors differ when one deals with practical applications (see [3]). One possibly way to calibrate all these systems is to study the existence of solitary waves for each equations. A solitary wave is a global positive and regular solution (of class $\mathcal{C}^2$ for instance), which propagates at a constant speed c with a constant shape. In addition, a solitary wave and its first and second derivatives tends to zero at infinity. As a consequence, they appear very useful to check the accuracy of numerical schemes as well as for practical applications in coastal engineering and to estimate the validity of models. In this paper, we prove the existence and the unicity of solitary waves for the equations cited above (namely, Beji-Nadaoka, Madsen and Sorensen and Nwogu equations) for a wide range of speed c. We also give a relation between the amplitude A of one of the component of the solitary wave and the speed of propagation c, that enables to make some effective computations. Note that we are not able yet to handle the new

Nwogu-Abott system and we postponed the existence of a solitary wave to a future work. The results obtained here are summed up in the following theorem :

Theorem 2. *We recall that* $c_0 = \sqrt{gd}$ *and consider the three following set of equations.*

1) Beji-Nadaoka equations.

We define $\gamma = \alpha_B + 1/3$ *and assume that one of the following alternative is satisfied :*
i) $\gamma = 0$, $c > c_0$,
ii) $\alpha_B < 0$, $\gamma > 0$, $c > c_0$,
iii) $\alpha_B > 0$, $\gamma > 0$, $c \in (c_0, c_0 \frac{2-\frac{\alpha_B}{\gamma}}{\sqrt{\frac{\alpha_B}{\gamma}}})$.
Then Equations (1.9) *admit a unique solitary wave of the form* $(\eta_c(x-ct), \bar{u}_c(x-ct))$*. Furthermore, the relation between parameter* c *and the amplitude* A *of* η_c *is given by*

$$c^4\gamma\left[\frac{A^3}{6(d+A)^3} - \frac{A^2}{2(d+A)^2}\right] + c^2c_0^2\left[\gamma\log(\frac{A+d}{d}) - \gamma\frac{A}{d+A} + \frac{\alpha_B}{2}\frac{A^2}{d(d+A)}\right] - c_0^4\frac{\alpha_B A^2}{2d^2} = 0. \quad (1.16)$$

Conversely, if $\gamma < 0$, $\alpha_B < 0$ *and* $c \geq 0$*, then Equations* (1.9) *have no positive solutions of the previous form.*

2) Madsen and Sorensen equations.

For all $c > c_0$*, Equations* (1.11) *admit a unique solitary wave of the form* $(\eta_c(x-ct), \bar{q}_c(x-ct))$.
In addition, the relation between parameter c *and the amplitude* A *of* η_c *is given by*

$$c^2 = c_0^2\frac{\frac{A^2}{2} + \frac{A^3}{6d}}{dA - d^2\log(\frac{d+A}{d})}. \quad (1.17)$$

3) Nwogu equations.

Assume that one of the following alternative is satisfied :
i) $\alpha \in (-\frac{1}{2}, -\frac{1}{9})$ *and* $c > c_0$,
ii) $\alpha \in (-\frac{1}{9}, 0)$ *and* $c > c_0\sqrt{\frac{\frac{\beta^2}{\alpha^2}}{2-\frac{\beta}{\alpha}}}$.
Then Equations (1.10) *admit a unique solitary wave of the form* $(\eta_c(x-$

$ct), U_c(x-ct))$. The amplitude A of U_c and parameter c satisfy

$$-\frac{1}{18\alpha}A^3+\left(\frac{c}{4\alpha}+\frac{\beta c_0^2-\alpha c^2}{12c\alpha}\right)A^2+\left(\frac{c_0^2-c^2}{3\alpha}-\frac{\beta c_0^2-\alpha c^2}{2\alpha^2}-\frac{(\beta c_0^2-\alpha c^2)^2}{6\alpha^2c^2}\right)A$$
$$+\left(\frac{(\beta c_0^2-\alpha c^2)^4}{6c^3\alpha^3}+\frac{(\beta c_0^2-\alpha c^2)^3}{2c\alpha^3}-\frac{(c_0^2-c^2)(\beta c_0^2-\alpha c^2)^2}{3c\alpha^2}\right)\log(1+\frac{c\alpha}{\beta c_0^2-\alpha c^2}A)=0. \quad (1.18)$$

Conversely, if $c \leq c_0$, then Equations (1.10) *have no positive solutions of the previous form.*

Finally, in all cases 1), 2) and 3), the solitary waves decay exponentially at infinity in the sense of Theorem 3.

Remark 1. Considering the Nwogu system in the case $\alpha \in (-\frac{1}{9}, 0)$, and $c_0 < c \leq c_0\sqrt{\frac{\frac{\beta^2}{\alpha^2}}{2-\frac{\beta}{\alpha}}}$, we are not able to conclude concerning the existence and uniqueness of a solitary wave. See the end of Section 3.3 for more informations.

The paper is organized as follows. In Section 2, we recall the derivation of the new extended Boussinesq equations, namely Nwogu-Abbott equations. Then, we present the linear dispersion characteristics and we propose a local Cauchy theory for this system. In Section 3, we prove the existence and the uniqueness of a solitary wave solution to Beji-Nadaoka, Nwogu and Madsen&Sorensen equations. Furthermore, we give a relationship between the solitary wave amplitude and the propagation speed c for every models cited above.

2 A new extended Boussinesq equations system

2.1 Derivation

In this section, starting from Euler equations, we recall the derivation of Nwogu-Abbott equations adapting the method used by Walkley to derive the so-called Nwogu system (see [21]). We refer to [3] for the introduction and a complete description of this system. The method consists in writing a non-dimensionalized version of the Euler equation and then to perform an asymptotic analysis on this system.

We first recall the Euler equations describing the evolution of an inviscid, incompressible fluid with free surface

$$\begin{aligned}
&u_t + uu_x + wu_z + \frac{p_x}{\rho} = 0,\\
&w_t + uw_x + ww_z + \frac{p_z}{\rho} + g = 0,\\
&u_x + w_z = 0,\\
&u_z - w_x = 0.
\end{aligned}$$

The associated boundary conditions reads:
• at the free surface $z = \eta$,

$$\begin{aligned}
&w = \eta_t + u\eta_x,\\
&p = 0,
\end{aligned}$$

• at the bed $z = -d$,

$$w = -ud_x.$$

In view of performing asymptotic analysis, it is useful to introduce non-dimensionalized variables in such a way that all the dependent quantities involved are of order one. Consequently, the small amplitude and long wave-length assumptions appear explicitly connected with the small parameters of the equations. For that purpose, we denote by d_0 the average bathymetry and introduce a typical wave amplitude a and a typical wavelength λ. We then consider the following new set of variables

$$\tilde{x} = \frac{x}{L},\ \tilde{z} = \frac{z}{d_0},\ \tilde{t} = \frac{\sqrt{gd_0}}{L}t,\ \tilde{\eta} = \frac{\eta}{a},\ \tilde{d} = \frac{d}{d_0},$$

$$\tilde{u} = \frac{d_0}{a\sqrt{gd_0}}u,\ \tilde{w} = \frac{L}{a}\frac{1}{\sqrt{gd_0}}v,\ \tilde{p} = \frac{p}{gd_0\rho},$$

and define the nonlinearity and dispersion parameters ε and σ respectively:

$$\varepsilon = \frac{a}{d_0},\ \sigma = \frac{d_0}{L}.$$

A direct computation furnishes the non-dimensionalized version of the Euler system as follows

$$\varepsilon\tilde{u}_{\tilde{t}} + \varepsilon^2\tilde{u}\tilde{u}_{\tilde{x}} + \varepsilon^2\tilde{w}\tilde{u}_{\tilde{z}} + \tilde{p}_{\tilde{x}} = 0 \qquad (2.1)$$

$$\varepsilon\sigma^2\tilde{w}_{\tilde{t}} + \varepsilon^2\sigma^2\tilde{u}\tilde{w}_{\tilde{x}} + \varepsilon^2\sigma^2\tilde{w}\tilde{w}_{\tilde{z}} + \tilde{p}_{\tilde{z}} + 1 = 0 \tag{2.2}$$

$$\tilde{u}_{\tilde{x}} + \tilde{w}_{\tilde{z}} = 0 \tag{2.3}$$

$$\tilde{u}_{\tilde{z}} - \sigma^2\tilde{w}_{\tilde{x}} = 0. \tag{2.4}$$

The boundary conditions are transformed into :

• at the free surface $\tilde{z} = \varepsilon\tilde{\eta}$

$$\tilde{w} = \tilde{\eta}_{\tilde{t}} + \varepsilon\tilde{u}\tilde{\eta}_{\tilde{x}}, \tag{2.5}$$

$$\tilde{p} = 0. \tag{2.6}$$

• at the bed $\tilde{z} = -\tilde{d}$

$$\tilde{w} = -\tilde{d}_{\tilde{x}}\tilde{u}. \tag{2.7}$$

We now perform the usual method of Nwogu and then integrate the equation (2.3) with respect to $\tilde{z}$ between $-\tilde{d}$ and $\tilde{z}$. Using Liebnitz' Rule and the boundary condition (2.7), we obtain

$$\tilde{w} = -\frac{\partial}{\partial\tilde{x}}\left(\int_{-\tilde{d}}^{\tilde{z}} \tilde{u}d\tilde{z}\right). \tag{2.8}$$

Substituting (2.8) in Equation (2.4), we derive

$$\frac{\partial\tilde{u}}{\partial\tilde{z}} = -\sigma^2\frac{\partial^2}{\partial\tilde{x}^2}\left(\int_{-\tilde{d}}^{\tilde{z}} \tilde{u}d\tilde{z}\right). \tag{2.9}$$

Next, we consider a Taylor expansion of the function $\tilde{z} \mapsto \tilde{u}(\tilde{t},\tilde{x},\tilde{z})$ around $\tilde{z} = \tilde{z}_\alpha$, where $\tilde{z}_\alpha$ stands for a chosen arbitrary depth, (we denote $\tilde{U} = \tilde{u}(\tilde{t},\tilde{x},\tilde{z}_\alpha)$),

$$\tilde{u} = \tilde{U} + (\tilde{z}-\tilde{z}_\alpha)\frac{\partial\tilde{u}}{\partial\tilde{z}}|_{\tilde{z}=\tilde{z}_\alpha} + \frac{(\tilde{z}-\tilde{z}_\alpha)^2}{2}\frac{\partial^2\tilde{u}}{\partial\tilde{z}^2}|_{\tilde{z}=\tilde{z}_\alpha} + \cdots \tag{2.10}$$

The integration of (2.10) from $-\tilde{d}$ to $\tilde{z}$ furnishes

$$\int_{-\tilde{d}}^{\tilde{z}} \tilde{u}d\tilde{z} = (\tilde{z}+\tilde{d})\tilde{U} + \left(\frac{(\tilde{z}-\tilde{z}_\alpha)^2}{2} - \frac{(\tilde{d}+\tilde{z}_\alpha)^2}{2}\right)\frac{\partial\tilde{u}}{\partial\tilde{z}}|_{\tilde{z}=\tilde{z}_\alpha} + \left(\frac{(\tilde{z}-\tilde{z}_\alpha)^3}{6} + \frac{(\tilde{d}+\tilde{z}_\alpha)^3}{6}\right)\frac{\partial^2\tilde{u}}{\partial\tilde{z}^2}|_{\tilde{z}=\tilde{z}_\alpha} + \cdots$$

which provides using Equation (2.9),

$$\tilde{u}_{\tilde{z}} = -\sigma^2 \frac{\partial^2}{\partial \tilde{x}^2} \left((\tilde{z} + \tilde{d})\tilde{U} + \left[\frac{(\tilde{z} - \tilde{z}_\alpha)^2}{2} - \frac{(\tilde{d} + \tilde{z}_\alpha)^2}{2} \right] \frac{\partial \tilde{u}}{\partial \tilde{z}}|_{\tilde{z}=\tilde{z}_\alpha} + \cdots \right). \tag{2.11}$$

Applying succesively $\partial_{\tilde{z}}$ and $\partial^2_{\tilde{z}}$ on Equation (2.11), we obtain by (2.9)

$$\tilde{u}_{\tilde{z}\tilde{z}} = -\sigma^2 \frac{\partial^2 \tilde{U}}{\partial \tilde{x}^2} + \mathcal{O}(\sigma^4),\ \tilde{u}_{\tilde{z}\tilde{z}\tilde{z}} = \mathcal{O}(\sigma^4). \tag{2.12}$$

Plugging (2.12) into (2.10) furnishes

$$\tilde{u} = \tilde{U} - \sigma^2 \left((\tilde{z} - \tilde{z}_\alpha) \frac{\partial^2}{\partial \tilde{x}^2}[(\tilde{d} + \tilde{z}_\alpha)\tilde{U}] + \frac{(\tilde{z} - \tilde{z}_\alpha)^2}{2} \frac{\partial^2 \tilde{U}}{\partial \tilde{x}^2} \right) + \mathcal{O}(\sigma^4). \tag{2.13}$$

Then we substitute relation (2.13) in Equation (2.8) which provides an expression for $\tilde{w}$ as follows

$$\tilde{w} = -\frac{\partial}{\partial \tilde{x}} \left((\tilde{d} + \tilde{z})\tilde{U} + \sigma^2 \Big(\frac{(\tilde{d} + \tilde{z}_\alpha)^2}{2} \frac{\partial^2}{\partial \tilde{x}^2}[(\tilde{d} + \tilde{z}_\alpha)\tilde{U}] \right.$$
$$\left. - \frac{(\tilde{d} + \tilde{z}_\alpha)^3}{6} \frac{\partial^2 \tilde{U}}{\partial \tilde{x}^2} \Big) \right) + \mathcal{O}(\sigma^4). \tag{2.14}$$

Equation (2.2) then can be rewritten as

$$-\varepsilon\sigma^2 \frac{\partial^2}{\partial \tilde{t} \partial \tilde{x}} \left((\tilde{z} + \tilde{d})\tilde{U} \right) + \tilde{p}_{\tilde{z}} + 1 + \mathcal{O}(\varepsilon^2\sigma^2, \varepsilon\sigma^4) = 0. \tag{2.15}$$

We now integrate (2.15) with respect to $\tilde{z}$ between $\varepsilon\tilde{\eta}$ and $\tilde{z}$ to obtain, using the boundary condition at the free surface (2.6)

$$\tilde{p} = \varepsilon\tilde{\eta} - \tilde{z} + \varepsilon\sigma^2 \frac{\partial^2}{\partial \tilde{t} \partial \tilde{x}} \left((\tilde{d}\tilde{z} + \frac{\tilde{z}^2}{2})\tilde{U} \right) + \mathcal{O}(\varepsilon\sigma^4, \varepsilon^2\sigma^2). \tag{2.16}$$

Substituting Equations (2.13), (2.14) and (2.16) in the horizontal momentum Equation (2.1) evaluated at $\tilde{z} = \tilde{z}_\alpha$ provides finally the first equation of our new system

$$\tilde{U}_{\tilde{t}} + \varepsilon\tilde{U}\tilde{U}_{\tilde{x}} + \tilde{\eta}_{\tilde{x}} + \sigma^2 \left(\frac{\tilde{z}^2_\alpha}{2} \tilde{U}_{\tilde{t}\tilde{x}\tilde{x}} + \tilde{z}_\alpha [\tilde{d}\tilde{U}_{\tilde{t}}]_{\tilde{x}\tilde{x}} \right) = \mathcal{O}(\varepsilon\sigma^2, \sigma^4). \tag{2.17}$$

Let us now derive the second equation of the system. First, we integrate Equation (2.3) to obtain using Liebnitz' Rule and the kinematic boundary conditions (2.6)-(2.7)

$$\tilde{\eta}_{\tilde{t}} + \frac{\partial}{\partial \tilde{x}} \left(\int_{-\tilde{d}}^{\varepsilon\tilde{\eta}} \tilde{u} d\tilde{z} \right) = 0,$$

which provides by (2.13),

$$\tilde{\eta}_{\tilde{t}} + \Big[(\varepsilon\tilde{\eta} + \tilde{d})\tilde{U} \Big]_{\tilde{x}} + \sigma^2 \frac{\partial}{\partial \tilde{x}} \left(\Big(\tilde{d}\tilde{z}_\alpha + \frac{(\tilde{d})^2}{2} \Big) [\tilde{d}\tilde{U}]_{\tilde{x}\tilde{x}} + \Big(\frac{\tilde{d}\tilde{z}_\alpha^2}{2} - \frac{(\tilde{d})^3}{6} \Big) \tilde{U}_{\tilde{x}\tilde{x}} \right) = \mathcal{O}(\varepsilon\sigma^2, \sigma^4). \qquad (2.18)$$

Equations (2.17) and (2.18) consist in the so-called Nwogu system. To go further, let us denote

$$\tilde{Q} := \tilde{h}\tilde{U} = \tilde{d}\tilde{U} + \mathcal{O}(\varepsilon), \qquad (2.19)$$

since $\tilde{h} = \tilde{d} + \varepsilon\tilde{\eta}$. Differentiating Equation (2.19) twice with respect to $\tilde{x}$ and one time with respect to $\tilde{t}$, we obtain successively

$$(\tilde{d}\tilde{U})_{\tilde{x}\tilde{x}} = \tilde{Q}_{\tilde{x}\tilde{x}} + \mathcal{O}(\varepsilon).$$

$$\tilde{U}_{\tilde{t}} = \left(\frac{\tilde{Q}}{\tilde{d} + \varepsilon\tilde{\eta}} \right)_{\tilde{t}} = \frac{\tilde{Q}_{\tilde{t}}}{\tilde{d}} + \mathcal{O}(\varepsilon).$$

from which we deduce that

$$\tilde{U}_{\tilde{t}\tilde{x}\tilde{x}} = \left(\frac{\tilde{Q}_{\tilde{t}}}{\tilde{d}} \right)_{\tilde{x}\tilde{x}} + \mathcal{O}(\varepsilon), \qquad (2.20)$$

$$\tilde{U}_{\tilde{x}\tilde{x}\tilde{x}} = \left(\frac{\tilde{Q}}{\tilde{d}} \right)_{\tilde{x}\tilde{x}\tilde{x}} + \mathcal{O}(\varepsilon). \qquad (2.21)$$

Plugging (2.20) and (2.21) in Equations (2.17) and (2.18), multiplying Equation (2.18) by $\varepsilon\tilde{u}$ and Equation (2.17) by $\tilde{h}$, we get

$$\tilde{Q}_{\tilde{t}} + \varepsilon \left(\frac{\tilde{Q}^2}{\tilde{h}} \right)_{\tilde{x}} + \tilde{h}\tilde{\eta}_{\tilde{x}} + \sigma^2 \tilde{d} \left(\tilde{z}_\alpha \tilde{Q}_{\tilde{t}\tilde{x}\tilde{x}} + \frac{\tilde{z}_\alpha^2}{2} \Big(\frac{\tilde{Q}_{\tilde{t}}}{\tilde{d}} \Big)_{\tilde{x}\tilde{x}} \right) = \mathcal{O}(\varepsilon\sigma^2, \sigma^4),$$

$$\tilde{\eta}_{\tilde{t}} + \tilde{Q}_{\tilde{x}} + \sigma^2 \frac{\partial}{\partial \tilde{x}} \left(\Big(\tilde{d}\tilde{z}_\alpha + \frac{(\tilde{d})^2}{2} \Big) \tilde{Q}_{\tilde{x}\tilde{x}} + \Big(\frac{\tilde{d}\tilde{z}_\alpha^2}{2} - \frac{(\tilde{d})^3}{6} \Big) \Big(\frac{\tilde{Q}}{\tilde{d}} \Big)_{\tilde{x}\tilde{x}} \right) = \mathcal{O}(\varepsilon\sigma^2, \sigma^4).$$

Coming back to the physical variables and setting the arbitrary depth $z_\alpha = \theta d,\ \theta \in [-1, 0]$, we obtain the new system of Nwogu-Abbott

$$\begin{cases} \eta_t + Q_x + \left[\beta_1 d^2 Q_{xx} + \beta_2 d^3 \left(\dfrac{Q}{d}\right)_{xx}\right]_x = 0, & (2.22) \\ Q_t + \left(\dfrac{Q^2}{h}\right)_x + gh\eta_x + \alpha_1 d^2 Q_{txx} + \alpha_2 d^3 \left(\dfrac{Q}{d}\right)_{txx} = 0, & (2.23) \end{cases}$$

with

$$\beta_1 = \theta + \frac{1}{2},\ \beta_2 = \frac{\theta^2}{2} - \frac{1}{6},$$

$$\alpha_1 = \theta,\ \alpha_2 = \frac{\theta^2}{2}.$$

Remark 2. A similar procedure in three dimensions leads to the two dimensional form of Nwogu-Abbott equations

$$\eta_t + \nabla \cdot \mathbf{Q} + \nabla \cdot \left[\beta_1 d^2 \nabla (\nabla \cdot \mathbf{Q}) + \beta_2 d^3 \nabla \left(\nabla \cdot \left(\frac{\mathbf{Q}}{d}\right)\right)\right] = 0, \qquad (2.24)$$

$$\mathbf{Q}_t + \nabla(\mathbf{Q} \cdot \mathbf{u}) + gh\nabla\eta + \alpha_1 d^2 \nabla (\nabla \cdot \mathbf{Q}_t) + \alpha_2 d^3 \nabla \left(\nabla \cdot \left(\frac{\mathbf{Q}_t}{d}\right)\right) = 0, \qquad (2.25)$$

where $\mathbf{Q}$ and $\mathbf{u}$ are the two-dimensional flux and velocity fields.

A check of the above calculations is to compute the dispersion relation of the linearized version around the rest state of Equations (2.22)-(2.23), which is, assuming that the bathymetry d is constant,

$$\begin{cases} \eta_t + Q_x + \beta d^2 Q_{xxx} = 0, \\ Q_t + gd\eta_x + \alpha d^2 Q_{txx} = 0, \end{cases}$$

with $\beta = \frac{\theta^2}{2} + \theta + \frac{1}{3}$ and $\alpha = \frac{\theta^2}{2} + \theta$.
Following the procedure described by Walkley in [21], the dispersion relation is obtained by looking for a steady periodic wave solution of amplitude a, period $2\pi/\omega$ and wavelength $2\pi/k$. To this end, we search for a solution (η, Q) under the form

$$\eta(t, x) = a \sin(kx - \omega t),$$

$$Q(t, x) = b \sin(kx - \omega t).$$

By direct computations, one can see that the coefficients a, b, k and ω have to satisfy the following algebraic system

$$\left\{\begin{array}{c} -\omega a + kb - \beta d^2 k^3 b = 0, \\ -\omega b + akc_0^2 + b\alpha\omega k^2 d^2 = 0, \end{array}\right.$$

which leads to the dispersion relation

$$\frac{\omega^2}{k^2} = c_0^2 \frac{1-\beta k^2 d^2}{1-\alpha k^2 d^2}.$$

This is the same relation as for Nwogu system (see [21]). Moreover, expanding the right-hand-side of the last equation for small value of k gives an expression for the phase speed c

$$c^2(k) = \frac{\omega^2}{k^2} = c_0^2\big(1 + (\alpha-\beta)d^2k^2 + \alpha(\alpha-\beta)d^4k^4 + O(k^6)\big).$$

For a suitable choice of α and β, namely

$$\alpha = -\frac{2}{5}, \;\; \beta = -\frac{1}{15}$$

(which corresponds to $\theta = -1 + \sqrt{1/15}$), one can see that the previous relation match the one of the Euler system up to order 4.

2.2 Local Cauchy theory for Nwogu-Abbott equations.

In Section 2.1, we have discussed the dispersive properties of the linear part of System (2.22)-(2.23), showing that the dispersion relation matches up to order 4 the one of the full classical Euler system. A second important step consists in solving the associated Cauchy Problem in Sobolev spaces. For that purpose, we introduce the following problem written in a dimensionless form with a constant flat bathymetry d for $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$

$$\begin{cases} \partial_t \eta + Q_x + \beta\sigma^2 d^2 Q_{xxx} = 0, & (2.26) \\ Q_t + \varepsilon \left(\dfrac{Q^2}{d+\varepsilon\eta}\right)_x + g(d+\varepsilon\eta)\eta_x + \alpha\sigma^2 d^2 Q_{txx} = 0, & (2.27) \end{cases}$$

with the initial condition

$$\begin{cases} \eta(0,x) = \eta_0(x), \;\; x \in \mathbb{R}, & (2.28) \\ Q(0,x) = Q_0(x), \;\; x \in \mathbb{R}, & (2.29) \end{cases}$$

and where $(\alpha,\beta)\in\mathbb{R}_-^2$ are fixed negative real. The aim of this section is to prove Theorem 1 claiming that System (2.26)-(2.29) is locally well-posed on a time interval $[0,T)$ with $T=O(\frac{1}{\varepsilon})$. We recall that the small parameter ε and σ^2 are of the same order. For convenience, we first introduce a constant C such that $\sigma^2=C\varepsilon$ and rewrite System (2.26)-(2.27) into, by denoting $\tilde{\alpha}=-Cd^2\alpha$ and $\tilde{\beta}=-Cd^2\beta$ and dropping the tilde

$$\begin{cases}\partial_t\eta+Q_x-\beta\varepsilon Q_{xxx}=0, & (2.30)\\ \partial_t(1-\alpha\varepsilon\partial_x^2)Q+\varepsilon\left(\dfrac{Q^2}{d+\varepsilon\eta}\right)_x+g(d+\varepsilon\eta)\eta_x=0. & (2.31)\end{cases}$$

Note now that coefficients α and β are positive. In order to apply the usual energy estimates one have to deal with the presence of the third order derivative in space in Equation (2.30) and the conservative terms of (2.31). Moreover, one has to prove that the quantity $d+\eta$ stay bounded away from 0. In this paper, standard notations will be used. For $1\le p\le\infty$, the $L^p(\mathbb{R})$-norm is denoted by $|\cdot|_p$ and the $H^s(\mathbb{R})$-norm by $||\cdot||_s$ for $s\in\mathbb{R}_+$. Moreover, we introduce the following Banach space $\mathcal{T}=H^2(\mathbb{R})\times H^4(\mathbb{R})$ endowed with the norm $||(\eta,Q)||_{\mathcal{T}}=\left(||\eta||_2^2+||Q||_4^2\right)^{\frac{1}{2}}$ and for $s>0$ and $T>0$, we denote

$$||\cdot||_{T,s}=\sup_{t\in[0,T]}||\cdot||_s$$

the norm of the space $L^\infty(0,T;H^s(\mathbb{R}))$.

Proof of Theorem 1. First note that we only give a sketch of the proof, pointing out the difficulties and omitting the classical details. As usual, the proof is based on a fixed point theorem in the function space

$$X_T=\left\{\begin{array}{c}U=(\eta,Q)\ :\ \eta\in C([0,T);L^2(\mathbb{R}))\cap L^\infty([0,T);H^2(\mathbb{R})),\\ Q\in C([0,T);L^2(\mathbb{R}))\cap L^\infty([0,T);H^4(\mathbb{R})),\ \sup\limits_{t\in[0,T]}||U(t)||_{\mathcal{T}}<+\infty\end{array}\right\}.$$

For $M\in\mathbb{R}_+^*$, $a>0$, $b>0$ and $r>0$, we denote

$$X_T(M,a,b,r)=\left\{\begin{array}{c}(\eta,Q)\in X_T\ :\ ||\eta||_{T,2}\le M,\ ||Q||_{T,4}\le M\\ ||\eta_t||_{T,1}\le r,\ \ ||Q_t||_{T,1}\le r,\ \ d+\varepsilon\eta\ge a\\ g(d+\varepsilon\eta)-\varepsilon\frac{Q^2}{(d+\varepsilon\eta)^2}\ge b,\\ \eta(0,x)=\eta_0(x),\ \ Q(0,x)=Q_0(x)\end{array}\right\},$$

and $\Omega=[0,T]\times\mathbb{R}$. Take $(\eta_0,Q_0)\in\mathcal{T}$ satisfying Conditions (1.14) and (1.15). Let $V=(\rho,P)\in X_T(M,a,b,r)$ and considered a linearized version of (2.30)-(2.31) as follows

$$\begin{cases} \partial_t \eta + Q_x - \beta\varepsilon Q_{xxx} = 0, & (2.32)\\ \partial_t(1-\alpha\varepsilon\partial_x^2)Q + \varepsilon \dfrac{2P}{d+\varepsilon\rho}Q_x + \Big(g(d+\varepsilon\rho) - \varepsilon\dfrac{P^2}{(d+\varepsilon\rho)^2}\Big)\eta_x = 0. & (2.33)\end{cases}$$

The Cauchy problem (2.32)-(2.33) with initial conditions $\eta(t,x) = \eta_0(x)$, $Q(0,x) = Q_0(x)$ defines a mapping

$$\begin{aligned} \mathcal{S} \ : \ X_T &\longrightarrow X_T \\ (\rho, P) &\longmapsto (\eta, Q). \end{aligned}$$

We want to show that there exists a time $T > 0$ and constants M, a, b and r such that $\mathcal{S}$ maps the closed ball $X_T(M,a,b,r)$ into itself and $\mathcal{S}$ is a contraction mapping under the constraint that it acts on $X_T(M,a,b,r)$ in the norm $||\cdot||_{T,0}$. We begin with the high order estimates on η and Q. Applying ∂_x on (2.32), we derive

$$\partial_t \eta_x + Q_{xx} - \beta\varepsilon Q_{xxxx} = 0. \tag{2.34}$$

The same procedure on (2.33) gives

$$\begin{aligned} &\partial_t(1-\alpha\varepsilon\partial_x^2)Q_x + \varepsilon\frac{2P}{d+\varepsilon\rho}Q_{xx} + \varepsilon\Big(\frac{2P}{d+\varepsilon\rho}\Big)_x Q_x \\ &+ \Big(g(d+\varepsilon\rho) - \varepsilon\frac{P^2}{(d+\varepsilon\rho)^2}\Big)\eta_{xx} + \Big(g(d+\varepsilon\rho) - \varepsilon\frac{P^2}{(d+\varepsilon\rho)^2}\Big)_x\eta_x = 0. \end{aligned} \tag{2.35}$$

Applying again ∂_x on Equations (2.34) and (2.35) we get

$$\partial_t \eta_{xx} + Q_{xxx} - \beta\varepsilon Q_{xxxxx} = 0, \tag{2.36}$$

$$\begin{aligned} &\partial_t(1-\alpha\varepsilon\partial_x^2)Q_{xx} + \varepsilon\frac{2P}{d+\varepsilon\rho}Q_{xxx} + 2\varepsilon\Big(\frac{2P}{d+\varepsilon\rho}\Big)_x Q_{xx} + \varepsilon\Big(\frac{2P}{d+\varepsilon\rho}\Big)_{xx}Q_x \\ &+ \Big(g(d+\varepsilon\rho) - \varepsilon\frac{P^2}{(d+\varepsilon\rho)^2}\Big)\eta_{xxx} + 2\Big(g(d+\varepsilon\rho) - \varepsilon\frac{P^2}{(d+\varepsilon\rho)^2}\Big)_x\eta_{xx} \\ &+ \Big(g(d+\varepsilon\rho) - \varepsilon\frac{P^2}{(d+\varepsilon\rho)^2}\Big)_{xx}\eta_x = 0. \end{aligned} \tag{2.37}$$

In order to cancel the fith-order derivative terms in (2.36), we multiply Equation (2.36) by $\Big(g(d+\varepsilon\rho) - \varepsilon\frac{P^2}{(d+\varepsilon\rho)^2}\Big)\eta_{xx}$ and Equation (2.37) by $Q_{xx} -$

$\beta\varepsilon Q_{xxxx}$ and integrate the resulting equations over $\mathbb{R}$. For simplicity, we denote

$$s(\rho,P)=g(d+\varepsilon\rho)-\varepsilon\frac{P^2}{(d+\varepsilon\rho)^2}.$$

We then obtain after integration by parts and cancellations

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\Big[\Big(g(d+\varepsilon\rho)-\varepsilon\frac{P^2}{(d+\varepsilon\rho)^2}\Big)\partial_t(\eta_{xx}^2)\\
&+\partial_t\Big(Q_{xx}^2+\varepsilon(\beta+\alpha)Q_{xxx}^2+\varepsilon^2\alpha\beta Q_{xxxx}^2\Big)\Big]dx\\
&=-\int_{\mathbb{R}}s(\rho,P)Q_{xxx}\eta_{xx}dx-\varepsilon\int_{\mathbb{R}}\Big(\frac{2P}{d+\varepsilon\rho}\Big)Q_{xxx}\Big(Q_{xx}-\varepsilon\beta Q_{xxxx}\Big)dx\\
&-2\varepsilon\int_{\mathbb{R}}\Big(\frac{2P}{d+\varepsilon\rho}\Big)_xQ_{xx}\Big(\varepsilon\beta Q_{xxxx}-Q_{xx}\Big)dx\\
&-\varepsilon\int_{\mathbb{R}}\Big(\frac{2P}{d+\varepsilon\rho}\Big)_{xx}Q_x\Big(\varepsilon\beta Q_{xxxx}-Q_{xx}\Big)dx\\
&-\int_{\mathbb{R}}s(\rho,P)\eta_{xxx}Q_{xx}dx+2\int_{\mathbb{R}}\Big(s(\rho,P)\Big)_x\eta_{xx}\Big(Q_{xx}-\varepsilon\beta Q_{xxxx}\Big)dx\\
&-\int_{\mathbb{R}}\Big(s(\rho,P)\Big)_{xx}\eta_x\Big(Q_{xx}-\varepsilon\beta Q_{xxxx}\Big)dx.
\end{aligned}\tag{2.38}$$

Note that in (2.38), the terms

$$\int_{\mathbb{R}}s(\rho,P)Q_{xxx}\eta_{xx}dx,\quad \int_{\mathbb{R}}s(\rho,P)\eta_{xxx}Q_{xx}dx$$

are not of order ε. Nevertheless, a direct integration by parts furnishes

$$\begin{aligned}
&\int_{\mathbb{R}}s(\rho,P)Q_{xxx}\eta_{xx}dx+\int_{\mathbb{R}}s(\rho,P)\eta_{xxx}Q_{xx}dx\\
&=-\int_{\mathbb{R}}\Big(s(\rho,P)\Big)_xQ_{xx}\eta_{xx}dx\\
&=-\varepsilon\int_{\mathbb{R}}\Big(g\rho-\frac{P^2}{(d+\varepsilon\rho)^2}\Big)_xQ_{xx}\eta_{xx}dx,
\end{aligned}\tag{2.39}$$

since

$$\Big(s(\rho,P)\Big)_x=\varepsilon\Big(g\rho-\frac{P^2}{(d+\varepsilon\rho)^2}\Big)_x.$$

Using (2.39), the Cauchy-Schwartz estimate, the continuous embedding $H^1(\mathbb{R})\hookrightarrow$

$L^\infty(\mathbb{R})$, the fact that $d+\varepsilon\rho \geq a$ and $(\rho, P) \in X_T(M,a,b,r)$, we get

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} \Big[\Big(g(d+\varepsilon\rho) - \varepsilon\frac{P^2}{(d+\varepsilon\rho)^2}\Big)\partial_t(\eta_{xx}^2) \\
&+ \partial_t\Big(Q_{xx}^2 + \varepsilon(\beta+\alpha)Q_{xxx}^2 + \varepsilon^2\alpha\beta Q_{xxxx}^2\Big)\Big]dx \\
&\leq \varepsilon C_0||\eta_{xx}||_0||Q_{xx}||_0 + \varepsilon C_1\Big(||Q_x||_0^2 + ||Q_{xx}||_0^2 + \varepsilon||Q_{xxx}||_0^2 + \varepsilon^2||Q_{xxxx}||_0^2\Big) \\
&+ \varepsilon C_2||\eta||_2\Big(||Q_{xx}||_0 + \varepsilon||Q_{xxxx}||_0\Big), \qquad (2.40)
\end{aligned}$$

where the constants C_0, C_1 and C_2 depends only on M, a, r, β. We now treat the non-standard term of Estimate (2.40), namely

$$\begin{aligned}
&\int_{\mathbb{R}} \Big(g(d+\varepsilon\rho) - \varepsilon\frac{P^2}{(d+\varepsilon\rho)^2}\Big)\partial_t(\eta_{xx}^2)dx \\
&= \partial_t\int_{\mathbb{R}} \Big(g(d+\varepsilon\rho) - \varepsilon\frac{P^2}{(d+\varepsilon\rho)^2}\Big)(\eta_{xx}^2)dx \\
&- \int_{\mathbb{R}} \partial_t\Big(\varepsilon g\rho - \varepsilon\frac{P^2}{(d+\varepsilon\rho)^2}\Big)\eta_{xx}^2 dx. \qquad (2.41)
\end{aligned}$$

Since $(\rho, Q) \in X_T(M,a,b,r)$, we deduce that

$$\Big|\int_{\mathbb{R}} \Big(g(d+\varepsilon\rho) - \varepsilon\frac{P^2}{(d+\varepsilon\rho)^2}\Big)_t(\eta_{xx}^2)dx\Big| \leq \varepsilon C_3||\eta_{xx}||_0^2, \qquad (2.42)$$

where the constant C_3 depends only on M, a and r. Integrating inequality (2.40) from 0 to t we obtain, using (2.41) and (2.42)

$$\begin{aligned}
&\int_{\mathbb{R}} \Big[s(\rho,P)\eta_{xx}^2 + Q_{xx}^2 + \varepsilon(\beta+\alpha)Q_{xxx}^2 + \varepsilon^2\alpha\beta Q_{xxxx}^2\Big]dx \\
&\leq \int_{\mathbb{R}} \Big[s(\rho_0,Q_0)\eta_{0xx}^2 + Q_{0xx}^2 + \varepsilon(\beta+\alpha)Q_{0xxx}^2 + \varepsilon^2\alpha\beta Q_{0xxxx}^2\Big]dx \\
&+ \varepsilon\Big(C_3\int_0^t ||\eta_{xx}(s)||_0^2 ds + C_0\int_0^t ||\eta_{xx}(s)||_0||Q_{xx}(s)||_0 ds \\
&+ C_1\int_0^t \big(||Q_{xx}(s)||_0^2 + \varepsilon||Q_{xxx}(s)||_0^2 + \varepsilon^2||Q_{xxxx}(s)||_0^2\big)ds \\
&+ C_2\int_0^t ||\eta(s)||_2\big(||Q_{xx}(s)||_0 + \varepsilon||Q_{xxxx}(s)||_0\big)ds\Big) \qquad (2.43)
\end{aligned}$$

Note that similar estimates can be obtained on Systems (2.32)-(2.33) and (2.34)-(2.35). Using the fact that

$$g(d+\varepsilon\rho) - \varepsilon\frac{P^2}{(d+\varepsilon\rho)^2} \geq b,$$

and choising M such that

$$\begin{aligned}M^2 \geq &\frac{2}{b}\int_{\mathbb{R}}\Big[s(\eta_0,Q_0)\eta_{0xx}^2+Q_{0xx}^2+\varepsilon(\beta+\alpha)Q_{0xxx}^2+\varepsilon^2\alpha\beta Q_{0xxxx}^2\Big]dx\\&+\frac{2}{b}\int_{\mathbb{R}}\Big[s(\eta_0,Q_0)\eta_{0x}^2+Q_{0x}^2+\varepsilon(\beta+\alpha)Q_{0xx}^2+\varepsilon^2\alpha\beta Q_{0xxx}^2\Big]dx\\&+\frac{2}{b}\int_{\mathbb{R}}\Big[s(\eta_0,Q_0)\eta_{0}^2+Q_{0}^2+\varepsilon(\beta+\alpha)Q_{0x}^2+\varepsilon^2\alpha\beta Q_{0xx}^2\Big]dx\end{aligned}$$

we deduce from (2.43) and Gronwall's inequality that one can find a time $T=O(\frac{1}{\varepsilon})$ depending only on M, a, b and r such that

$$||\eta||_{T,2}\leq M,\;\; ||Q||_{T,4}\leq M\;\text{ on } [0,T]. \tag{2.44}$$

It remains to prove that the solution (η,Q) belongs to the ball $X_T(M,a,b,r)$. By (2.32), (2.33) and (2.44), we get directly

$$||\eta_t||_{T,1}\leq ||Q_x||_{T,1}+\beta\varepsilon||Q_{xxx}||_{T,1}\leq C_4,$$

$$\begin{aligned}||Q_t||_{T,1}&=||(1-\alpha\varepsilon\partial_x^2)^{-1}\Big(\varepsilon\frac{2P}{d+\varepsilon\rho}Q_x+\Big(g(d+\varepsilon\rho)-\varepsilon\frac{P^2}{(d+\varepsilon\rho)^2}\Big)\eta_x\Big)||_{T,1}\\&\leq ||\varepsilon\frac{2P}{d+\varepsilon\rho}Q_x+\Big(g(d+\varepsilon\rho)-\varepsilon\frac{P^2}{(d+\varepsilon\rho)^2}\Big)\eta_x||_{T,0}\\&\leq C_5,\end{aligned}$$

where C_4 and C_5 depends only on M. Then choising r such that $r\geq \max(C_4,C_5)$, we get

$$||\eta_t||_{T,1}\leq r,\;\; ||Q_t||_{T,1}\leq r.$$

Here we recall that (η_0,Q_0) satisfy Conditions (1.14) and (1.15). Then choising $a>0$ such that

$$d+\varepsilon\eta_0(x)\geq 2a\;\text{ on }\mathbb{R},$$

and writting

$$\eta(t,x)=\eta_0(x)+\int_0^t\partial_s\eta(s,x)ds,$$

leads to

$$\begin{aligned}d+\varepsilon\eta(t,x)&=d+\varepsilon\eta_0(x)+\varepsilon\int_0^t\partial_s\eta(s,x)ds\\&\geq 2a-\varepsilon C(r)T\\&\geq a\;\text{ on }\Omega,\end{aligned}$$

where the constant $C(r)$ depends only on r and for $T = O(\frac{1}{\varepsilon})$ small enough. The same argument provides, choising $b > 0$ such that

$$g(d+\varepsilon\eta_0) - \varepsilon\frac{Q_0^2}{(d+\varepsilon\eta_0)^2} \geq 2b \text{ on } \mathbb{R},$$

$$\begin{aligned} g(d+\varepsilon\eta) - \varepsilon\frac{Q^2}{(d+\varepsilon\eta)^2} &= g(d+\varepsilon\eta_0) - \varepsilon\frac{Q_0^2}{(d+\varepsilon\eta_0)^2} \\ &\quad + \varepsilon\int_0^t \partial_s\Big(g\eta - \frac{Q^2}{(d+\varepsilon\eta)^2}\Big)ds \\ &\geq 2b - \varepsilon C(M,a,r)T \\ &\geq b \text{ on } \Omega, \end{aligned}$$

where the constant $C(M,a,r)$ depends only on M, a and r and for $T = O(\frac{1}{\varepsilon})$ small enough. This proves that $(\eta, Q) \in X_T(M,a,b,r)$. Hence $\mathcal{S}$ maps the closed ball $X_T(M,a,b,r)$ into itself. We then prove that $\mathcal{S}$ is a contracting mapping in the ball $X_T(M,a,b,r)$ endowed with the metric Y_T where

$$Y_T = \Big\{(\eta,Q) \in \Big(C([0,T];L^2(\mathbb{R}))\Big)^2 \;/\; ||(\eta,Q)||_{Y_T}^2 = ||\eta||_{T,0}^2 + ||Q||_{T,0}^2 < +\infty\Big\}.$$

Let $(\rho_1, P_1) \in X_T(M,a,b,r)$, $(\rho_2,P_2) \in X_T(M,a,b,r)$ and put $(\eta_1,Q_1) = \mathcal{S}(\rho_1,P_1)$, $(\eta_2,Q_2) = \mathcal{S}(\rho_2,P_2)$ the corresponding solutions to Equations (2.32)-(2.33) with the same initial conditions. The equations satisfied by $\lambda = \eta_1 - \eta_2$ and $R = Q_1 - Q_2$ read

$$\partial_t\lambda + R_x - \beta\varepsilon R_{xxx} = 0, \tag{2.45}$$

$$\begin{aligned} \partial_t(1-\alpha\varepsilon\partial_x^2)R + \varepsilon\Big(\frac{2P_1}{d+\varepsilon\rho_1} - \frac{2P_2}{d+\varepsilon\rho_2}\Big)(Q_2)_x + \frac{2P_1}{d+\varepsilon\rho_1}R_x \\ + \varepsilon\Big(g(\rho_1-\rho_2) - \frac{P_1^2}{(d+\varepsilon\rho_1)^2} + \frac{P_2^2}{(d+\varepsilon\rho_2)^2}\Big)(\eta_2)_x \\ + \Big(g(d+\varepsilon\rho_1) - \varepsilon\frac{P_1^2}{(d+\varepsilon\rho_1)^2}\Big)\lambda_x = 0. \end{aligned} \tag{2.46}$$

Since all the nonlinear terms in Equation (2.46) are of class C^1 with respect to their arguments, we deduce directly that there exists a constant $C(M,a,b,r)$ such that

$$||(\eta_1,Q_1) - (\eta_2,Q_2)||_{Y_T} \leq C(M,a,b,r)\varepsilon T||(\rho_1,P_1) - (\rho_2,P_2)||_{Y_T},$$

proving the desired result for $T = O(\frac{1}{\varepsilon})$ small enough. Moreover, the ball $X_T(M, a, b, r)$, endowed with the metric of Y_T is closed. By the contraction mapping principle, there exists a unique solution (η, Q) to (2.26)-(2.29) defined on $[0, T]$ satisfying

$$\eta \in C([0,T]; L^2(\mathbb{R})) \cap L^\infty(0,T; H^2(\mathbb{R})),$$

$$Q \in C([0,T]; L^2(\mathbb{R})) \cap L^\infty(0,T; H^4(\mathbb{R})).$$

which ends the proof of Theorem 1. □

3 Solitary waves solutions for Boussinesq systems

In this section, we prove the existence and unicity of solitary waves for Equations (1.9) (Beji-Nadaoka), Equations (1.11) (Madsen-Sorensen) and for Equations (1.10) (Nwogu) assuming that the bathymetry d is constant. We recall that a solitary wave is a global solution of the generic form $u(t,x) = u_c(x - ct)$ where c is the celerity of the wave and u_c is a profile independent of the time t satisfying the two conditions

$$\lim_{x\to\pm\infty} u_c(x) = 0,\ u_c(x_0) > 0 \ \text{ for some } x_0 \in \mathbb{R}. \tag{3.1}$$

Plugging this relation into the previous equation leads to stationary equations. We present the details for each equations in the next sections. Note that, in view of numerical computations, we also give the relation between the celerity and the amplitude of the solitary wave. We recall now the classical theorem by H. Berestycki and P.-L.Lions (see [4]), which gives a necessary and sufficient condition for the existence and unicity of a positive solution to the stationary equation $-u'' = f(u)$.

Theorem 3. *Let f be a locally Lipschitz continuous real function with $f(0) = 0$. Let $F(z) = \int_0^z f(s)ds$. Consider the problem*

$$-u" = f(u),\ u \in \mathcal{C}^2(\mathbb{R}), \tag{3.2}$$

$$\lim_{x\to\pm\infty} u(x) = 0,\ u(x_0) > 0 \ \textit{for some } x_0 \in \mathbb{R}.$$

There is a unique solution $u \in H^1(\mathbb{R}) \cap C^2(\mathbb{R})$ up to translations to problem (3.2) *if and only if*

$$\xi_0 = \inf\{\xi > 0,\ F(\xi) = 0\} \ \textit{exists, and satisfies}\ \xi_0 > 0,\ f(\xi_0) > 0. \tag{3.3}$$

In addition, if one assumes that

$$\lim_{s\to 0}\frac{g(s)}{s} \le -m < 0,$$

then u, u', u'' decay exponentially at inanity, that is there exists C, $\delta > 0$, such that for $n = 0, 1, 2$,

$$|\partial_x^n u(x)| \le C\exp(-\delta|x|),\ x \in \mathbb{R}.$$

Remark 3. Under the assumptions of Theorem 3, the solution u of (3.2) can be obtained as the solution of the Cauchy's problem

$$\begin{cases} -u" = f(u), \\ u(0) = \xi_0,\ u'(0) = 0. \end{cases}$$

Furthermore, this solution satisfies
(i) $u(x) = u(-x)$, $x \in \mathbb{R}$,
(ii) $u'(x) < 0,\ x \in \mathbb{R}$,
(iii) $0 < u(x) \le \xi_0,\ x \in \mathbb{R}$.

3.1 Solitary waves solutions for Beji-Nadaoka equations

We recall the Beji-Nadaoka equations (1.9) with constant bathymetry. We assume that $\alpha_B \in \mathbb{R}$ and define $\gamma = \alpha_B + 1/3$

$$\begin{cases} \eta_t + [h\bar{u}]_x = 0 & (3.4)\\ \bar{u}_t + g\eta_x + \bar{u}\bar{u}_x - \gamma d^2\bar{u}_{txx} - \alpha_B g d^2\eta_{xxx} = 0. & (3.5)\end{cases}$$

We look for solitary waves under the form

$$\begin{cases} \eta(t,x) = \eta_c(x-ct) \\ \bar{u}(t,x) = \bar{u}_c(x-ct), \end{cases}$$

where $\eta_c, \bar{u}_c \in H^1(\mathbb{R})$. A direct computation shows that the system satisfied by $\eta_c, \bar{u}_c$ reads, after one integration,

$$\begin{cases} \eta_c = \dfrac{d\bar{u}_c}{c-\bar{u}_c}, & (3.6)\\ -c\bar{u}_c + g\eta + \dfrac{\bar{u}_c^2}{2} + c\gamma d^2\bar{u}_c'' - \alpha_B g d^2\eta_c'' = 0. & (3.7)\end{cases}$$

Remark 4. From Equation (3.6), we obtain

$$\bar{u}_c = c\frac{\eta_c}{d+\eta_c}, \tag{3.8}$$

from which we deduce that

$$\bar{u}_c < c,$$

since $d \geq 0$.

In Equation (3.7), it is obvious that parameter γ plays a crucial role and we discuss now the existence of a solution of System (3.6)-(3.7) considering successively the case $\gamma = 0$ and $\gamma \neq 0$.

3.1.1 Case 1 : $\gamma = 0$.

The Beji-Nadaoka system (3.6)-(3.7) reduces to

$$\begin{cases} \bar{u}_c = \frac{c\eta_c}{d+\eta_c}, \\ -c\bar{u}_c + g\eta_c + \frac{\bar{u}_c^2}{2} + \frac{gd^2}{3}\eta_c^{''} = 0. \end{cases}$$

from which we deduce by substitution the following autonomous ODE on η_c

$$-\frac{gd^2}{3}\eta_c^{''} = -c^2\frac{\eta_c}{d+\eta_c} + g\eta_c + \frac{c^2\eta_c^2}{2(d+\eta_c)^2},$$

which can be rewritten, after simplifications, as

$$-\eta'' = \frac{3}{gd^2}\Big(\frac{c^2d^2}{2(d+\eta)^2} - \frac{c^2}{2} + c_0^2\frac{\eta}{d}\Big). \tag{3.9}$$

Introducing

$$g(s) = \frac{3}{gd^2}\left(\frac{c^2d^2}{2(d+s)^2} - \frac{c^2}{2} + c_0^2\frac{s}{d}\right),$$

it is obvious that Equation (3.9) can be put under the form (3.2). A direct integration provides

$$G(s) := \int_0^s g(t)\,dt = \frac{3}{gd^2}\left(\frac{c^2d}{2} - \frac{c^2d^2}{2(d+s)} - \frac{c^2}{2}s + c_0^2\frac{s^2}{2d}\right).$$

In order to apply Theorem 3, we give now the behavior of the functions g and G for $s \geq 0$. Remark first that $g(0) = G(0) = 0$. Moreover it is easy to see that

$$g(s) = 0 \Longleftrightarrow \Big(s = 0 \text{ or } s = s_1 \text{ or } s = s_2\Big),$$

where

$$s_1 := d\frac{(\frac{c^2}{c_0^2}-4)-\sqrt{8\frac{c^2}{c_0^2}+\frac{c^4}{c_0^4}}}{4},\ s_2 := d\frac{(\frac{c^2}{c_0^2}-4)+\sqrt{8\frac{c^2}{c_0^2}+\frac{c^4}{c_0^4}}}{4}.$$

Note that we always have $s_1 < 0$ and then we concentrate on s_2. For $c \le c_0$, we have $s_2 \le 0$ and so g is positive on $\mathbb{R}_+^*$ and then Theorem 3 ensures that Equation (3.9) has no solutions. On the contrary, if $c > c_0$, then $s_2 > 0$ and g is negative on $(0, s_2)$ and positive on $(s_2, +\infty)$. From $G(0) = 0$, we deduce that $G(s_2) < 0$ and, since $\lim_{s\to+\infty} G(s) = +\infty$, it is clear that G has a unique zero s_3 in $(s_2, +\infty)$. Moreover, since $s_3 > s_2$, one has $g(s_3) > 0$. Applying again Theorem 3, we deduce that Equation (3.9) admits a unique solution $\eta_c \in H^1(\mathbb{R})$. Denoting

$$\bar{u}_c = \frac{c\eta_c}{d+\eta_c},$$

it is clear that $(\bar{u}_c, \eta_c)$ is the unique solution to (3.6)-(3.7). Moreover, denoting by A the amplitude of the solution η_c and recalling that necessarily $A = s_3$, that is $G(A) = 0$, one has

$$c^2 = c_0^2\frac{d+A}{d}.$$

Remark that we find this relation replacing γ by 0 in Equation (1.16). Finally, one has

$$\frac{g(s)}{s} = \frac{3}{gd^2}\left(\frac{-2c^2d-c^2s}{2(d+s)^2}+\frac{c_0^2}{d}\right) \longrightarrow \frac{3}{gd^3}(c_0^2-c^2) < 0,\ \text{when } s \to 0.$$

Applying Theorem 3, we deduce that η_c decay exponentially at infinity and that $\eta_c(x) > 0,\ \forall x \in \mathbb{R}$. We also have

$$\begin{aligned}
|\bar{u}_c(x)| &= \left|\frac{c\eta_c(x)}{d+\eta_c(x)}\right| \le \left|\frac{c}{d}\eta_c(x)\right| \le K_1\exp(-\delta|x|),\\
|\bar{u}_c'(x)| &= \left|\frac{cd\eta_c'(x)}{(d+\eta_c(x))^2}\right| \le K_2\exp(-\delta|x|),\\
|\bar{u}_c''(x)| &= \left|\frac{cd\eta_c'(x)^2}{(d+\eta_c(x))^3}+\frac{cd|\eta_c''(x)|}{(d+\eta_c(x))^2}\right| \le K_3\exp(-\delta|x|),
\end{aligned}$$

where $K_1,\ K_2,\ K_3$ are positive constant.

3.1.2 Case 2 : $\gamma \neq 0$.

In this case, Equation (3.6) provides that

$$\eta_c^{''} = cd(\frac{\bar{u}_c^{''}}{(c-\bar{u}_c)^2} + \frac{2\bar{u}_c'^2}{(c-\bar{u}_c)^3}),$$

which gives by Equation (3.7)

$$(c\gamma d^2(c-\bar{u}_c)^2 - \alpha_B d^2 cc_0^2)\bar{u}_c^{''} - \frac{2cc_0^2\alpha_B d^2}{(c-\bar{u}_c)}\bar{u}_c'^2 - c\bar{u}_c(c-\bar{u}_c)^2$$
$$+c_0^2\bar{u}_c(c-\bar{u}_c) + \frac{\bar{u}_c^2(c-\bar{u}_c)^2}{2} = 0. \qquad (3.10)$$

Since Equation (3.10) is a quasilinear elliptic equation, we perform a change of variable to transform (3.10) into a semilinear one (see [7, 8]). More precisely, let us define $\bar{u}_c = f(v_c)$ where $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$. One has

$$\bar{u}_c' = v_c' f'(v_c),$$

and

$$\bar{u}_c^{''} = v_c'' f'(v_c) + v_c'^2 f''(v_c).$$

Performing this change of functions into (3.10) leads to

$$\begin{aligned}
&(c\gamma d^2(c-\bar{u}_c)^2 - \alpha_B d^2 cc_0^2)\bar{u}_c^{''} - \frac{2cc_0^2\alpha_B d^2}{(c-\bar{u}_c)}\bar{u}_c'^2 \\
&= \left(c\gamma d^2(c-\bar{u}_c)^2 - \alpha_B d^2 cc_0^2\right) f'(v_c)v_c'' \\
&+ \left((c\gamma d^2(c-f(v_c))^2 - \alpha_B d^2 cc_0^2)f''(v_c) - \frac{2cc_0^2\alpha_B d^2}{(c-f(v_c))} f'(v_c)^2\right) v_c'^2.
\end{aligned}$$

We choose $f(s)$ satisfying the following ODE

$$((c-f(s))^2 - \frac{\alpha_B}{\gamma}c_0^2)f''(s) - \frac{2c_0^2\alpha_B}{\gamma(c-f(s))}f'(s)^2 = 0. \qquad (3.11)$$

Assuming that $f^{'}(s) \neq 0$, then (3.11) can be rewritten into

$$\frac{f''(s)}{f'(s)} = \frac{2c_0^2\alpha_B}{\gamma(c-f(s))((c-f(s))^2 - \frac{\alpha_B}{\gamma}c_0^2)}f'(s),$$

or equivalently

$$\frac{f''(s)}{f'(s)} = -\frac{2f'(s)}{(c-f(s))} + \frac{2(c-f(s))f'(s)}{((c-f(s))^2 - \frac{\alpha_B}{\gamma}c_0^2)}.$$

After integrations, we obtain successively, taking $f(0)=0$,

$$f'(s)=\frac{(c-f(s))^2}{(c-f(s))^2-c_0^2\frac{\alpha_B}{\gamma}}, \tag{3.12}$$

$$s=f(s)-\frac{c_0^2\alpha_B}{\gamma(c-f(s))}+\frac{c_0^2\alpha_B}{c\gamma}. \tag{3.13}$$

We expect f to be a diffeomorphism on an interval of $\mathbb{R}_+$ containing 0. Let us compute

$$(f^{-1})'(t)=\frac{1}{f'(f^{-1}(t))}=1-\frac{c_0^2\alpha_B}{\gamma(c-t)^2}.$$

We deduce that :

- if $\alpha_B\gamma>0$ and $c^2>c_0^2\frac{\alpha_B}{\gamma}$, then $(f^{-1})'(t)>0$ on $[0,c-c_0\sqrt{\frac{\alpha_B}{\gamma}})$, which implies that f is a diffeomorphism from $[0,c(1-\frac{c_0}{c}\sqrt{\frac{\alpha_B}{\gamma}})^2)$ to $[0,c-c_0\sqrt{\frac{\alpha_B}{\gamma}})$,
- If $\alpha_B\gamma<0$, then $(f^{-1})'(t)>0$ on $[0,c)$ which implies that f is a diffeomorphism from $[0,+\infty)$ to $[0,c)$.

In the sequel, we assume that $c^2>c_0^2\frac{\alpha_B}{\gamma}$. Performing the change of variable $\bar{u}_c=f(v_c)$ in Equation (3.10) where f satisfies (3.11), one obtains the following ODE of the form (3.2) on v_c

$$-v_c''=\frac{1}{c\gamma d^2}\Big(-cf(v_c)+\frac{cc_0^2}{c-f(v_c)}-c_0^2+\frac{f(v_c)^2}{2}\Big). \tag{3.14}$$

We introduce

$$g(s)=\frac{1}{c\gamma d^2}\left(-cf(s)+\frac{cc_0^2}{c-f(s)}-c_0^2+\frac{f(s)^2}{2}\right),$$

and compute, by using the change of variable $t=f(x)$ and relation (3.12),

$$\begin{aligned}G(s)&=\int_0^s g(t)\,dt\\&=\frac{1}{c\gamma d^2}\Big(\frac{f(s)^3}{6}-\frac{c}{2}f(s)^2-c_0^2f(s)-c_0^2\frac{\alpha_B}{2\gamma}f(s)-cc_0^2\log(\frac{c-f(s)}{c})\\&+\frac{c_0^2\alpha_B}{2\gamma}\frac{c^2}{c-f(s)}+\frac{c_0^2\alpha_B}{\gamma}\frac{c_0^2}{c-f(s)}-\frac{c_0^4\alpha_B}{2\gamma}\frac{c}{(c-f(s))^2}-\frac{cc_0^2\alpha_B}{2\gamma}-\frac{c_0^4\alpha_B}{2\gamma c}\Big).\end{aligned}$$

We want to show that the function G has a unique zero on $(0,c)$, for suitable values of c. For convenience, we define $t=f(s)$ and introduce

$$h(t)=g(s)=\frac{1}{c\gamma d^2}\left(-ct+\frac{cc_0^2}{c-t}-c_0^2+\frac{t^2}{2}\right),$$

and

$$H(t) = G(s).$$

Then

$$h(t) = 0 \Leftrightarrow t = 0 \text{ or } -\frac{t^2}{2} + \frac{3c}{2}t + (c_0^2 - c^2) = 0.$$

This implies that h has 3 zeros in $\mathbb{R} \setminus \{c\}$, $\forall \alpha_B \in \mathbb{R} \setminus \{-\frac{1}{3}\}$,

$$t_0 = 0,\ t_1 = c\frac{3 - \sqrt{1 + 8\frac{c_0^2}{c^2}}}{2},\ t_2 = c\frac{3 + \sqrt{1 + 8\frac{c_0^2}{c^2}}}{2}.$$

We first remark that for any $c \in \mathbb{R}$, one has $t_2 > c$, then t_2 doesn't play any role in our analysis. Since our method depends on the properties of f, we have to separate the case $\alpha_B\gamma > 0$ from the case $\alpha_B\gamma < 0$.

- case 1 : $\alpha_B\gamma > 0$.

We recall that in this case f is a diffeomorphism from $[0, c(1 - \frac{c_0}{c}\sqrt{\frac{\alpha_B}{\gamma}})^2)$ to $[0, c - c_0\sqrt{\frac{\alpha_B}{\gamma}})$. A direct computations shows that

$$t_1 \in (0, c - c_0\sqrt{\frac{\alpha_B}{\gamma}}) \Longleftrightarrow c \in (c_0, c_0\frac{2 - \frac{\alpha_B}{\gamma}}{\sqrt{\frac{\alpha_B}{\gamma}}})$$

As a consequence, if $c \notin (c_0, c_0\frac{2-\frac{\alpha_B}{\gamma}}{\sqrt{\frac{\alpha_B}{\gamma}}})$, then for all $s \in [0, c(1 - \frac{c_0}{c}\sqrt{\frac{\alpha_B}{\gamma}})^2)$, $g(s) \geq 0$ which implies that G is increasing. By Theorem 3, we deduce that Equation (3.10) does not admit solutions in this case. Conversely, if $c \in (c_0, c_0\frac{2-\frac{\alpha_B}{\gamma}}{\sqrt{\frac{\alpha_B}{\gamma}}})$, then $t_1 \in (0, c - c_0\sqrt{\frac{\alpha_B}{\gamma}})$, and $f^{-1}(t_1) \in [0, c(1 - \frac{c_0}{c}\sqrt{\frac{\alpha_B}{\gamma}})^2)$. Thus if $\gamma > 0$ (resp. $\gamma < 0$) G decreases (resp. increases) on $(0, f^{-1}(t_1)]$ and increases (resp. decreases) on $[f^{-1}(t_1), c(1 - \frac{c_0}{c}\sqrt{\frac{\alpha_B}{\gamma}})^2)$. Starting from $G(0) = 0$, we obtain $G(f^{-1}(t_1)) < 0$ (resp. $G(f^{-1}(t_1)) > 0$). Since

$$\lim_{s \to c(1 - \frac{c_0}{c}\sqrt{\frac{\alpha_B}{\gamma}})^2} G(s) = \lim_{t \to c - c_0\sqrt{\frac{\alpha_B}{\gamma}}} H(t) = +\infty \text{ (resp. } -\infty),$$

we deduce by the mean-value Theorem that G admits a unique zero $s_m \in (f^{-1}(t_1), c(1 - \frac{c_0}{c}\sqrt{\frac{\alpha_B}{\gamma}})^2)$, proving that $g(s_m) > 0$ (resp. $g(s_m) < 0$). By Theorem 3, we obtain that Equation (3.10) admits a unique positive solution satisfying (3.1) only if $\gamma > 0$ and $\alpha_B > 0$ and we conclude that Equation (3.6)-(3.7) admits a unique solution as in the case $\gamma = 0$.

- case 2 : $\alpha_B \gamma < 0$.

In this case, since $\gamma = \alpha_B + \frac{1}{3}$, one has necessarily $\gamma > 0$ and $\alpha_B < 0$. If $c \leq c_0$, then $t_1 \leq 0$ which implies that g is negative in $(0, +\infty)$ and G is increasing on $(0, +\infty)$ showing that Equation (3.14) has no solution in this case. If $c > c_0$, t_1 is the only zero in $(0, c)$ of h. G decreases on $[0, f^{-1}(t_1)]$ and increases on $[f^{-1}(t_1), +\infty)$. Since $G(0) = 0$, one has $G(f^{-1}(t_1)) < 0$. Moreover, using the fact that $\lim_{s \to +\infty} G(s) = \lim_{t \to c} H(t) = +\infty$, and applying the mean-value Theorem, one proves that G has a unique zero $s_m \in (f^{-1}(t_1), +\infty)$ with $g(s_m) > 0$, and again the conclusion is the same as in the case $\gamma = 0$.

We are now ready to give the relation between the amplitude A of η_c and the propagation velocity c. Indeed, by Theorem 3, using (3.8), we deduce that the quantity

$$f^{-1}(\frac{cA}{A+d})$$

is a zero of the function G that is

$$c^4 \left[\frac{A^3}{6(d+A)^3} - \frac{A^2}{2(d+A)^2} \right] + c^2 c_0^2 \left[\log(\frac{A+d}{d}) - \frac{A}{d+A} + \frac{\alpha_B}{2\gamma} \frac{A^2}{d(d+A)} \right] - c_0^4 \frac{\alpha_B A^2}{2\gamma d^2} = 0. \quad (3.15)$$

which leads to (1.16).

Furthermore, we recall that $c^2 > c_0^2 \frac{\alpha_B}{\gamma}$, $0 < v_c \leq s_m$, and that $s_m < c(1 - \frac{c_0}{c}\sqrt{\frac{\alpha_B}{\gamma}})^2$ if $\alpha_B \gamma > 0$. Since $t \to 0$ when $s \to 0$, we deduce

$$\frac{g(s)}{s} = \frac{h(t)}{f^{-1}(t)} = \frac{1}{c\gamma d^2} \left(\frac{c_0^2 - c^2 - t^2/2}{-ct + c^2 - c_0^2 \frac{\alpha_B}{\gamma}} \right),$$

$$\underset{s \to 0}{\longrightarrow} \frac{1}{c\gamma d^2} \left(\frac{c_0^2 - c^2}{c^2 - c_0^2 \frac{\alpha_B}{\gamma}} \right) < 0.$$

We deduce, applying Theorem 3, v_c decays exponentially at infinity. Coming back to $u_c = f(v_c)$, we observe, since f is increasing, that

$$0 < u_c(x) \leq f(s_m) < \min(c - c_0 \sqrt{\frac{\alpha_B}{\gamma}}, c),$$

$$|f'(v_c)| = |\frac{(c-f(v_c))^2}{(c-f(v_c))^2 - c_0^2\frac{\alpha_B}{\gamma}}| \leq M_1,$$

where $M_1 \in \mathbb{R}_+^*$. In the same way, we can show that there exists $M_2 \in \mathbb{R}_+^*$ such that

$$|f''(s)| = |\frac{2c_0^2\alpha_B}{\gamma((c-f(s))^2 - \frac{\alpha_B}{\gamma}c_0^2)(c-f(s))}f'(s)^2| \leq M_2.$$

Applying the Finite-Increments Theorem on f, we deduce

$$|u_c(x)| = |f(v_c(x)) - f(0)| \leq M_1|v_c(x)| \leq C_1\exp(-\delta|x|),$$

with $C_1 \in \mathbb{R}_*^+$.
In addition,

$$|u_c'(x)| = |v_c'(x)f'(v_c(x))| \leq M_1|v_c'(x)| \leq C_2\exp(-\delta|x|),$$

$$\begin{aligned} |u_c''(x)| &\leq |v_c''(x)f'(v_c(x))| + |v_c'(x)|^2 f''(v_c(x))| \\ &\leq M_1|v_c''(x)| + M_2|v_c'(x)|^2 \\ &\leq C_3\exp(-\delta|x|), \end{aligned}$$

where $C_2,\ C_3 \in \mathbb{R}_*^+$. Recalling that

$$\eta_c = \frac{du_c}{c-u_c},\ \ u_c(x) \leq f(s_m) < c.$$

one has

$$\eta_c \leq \frac{d}{c-f(s_m)}u_c(x) \leq K_1\exp(-\delta|x|).$$

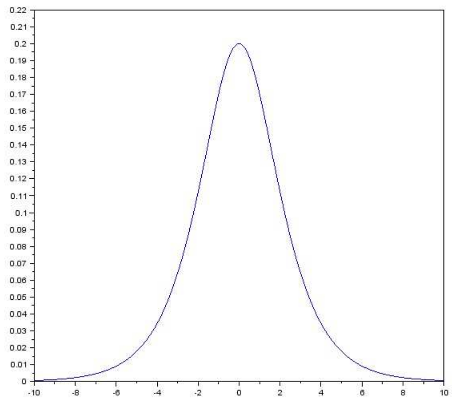
In the same way,

$$|\eta_c'| \leq \frac{dc}{(c-f(s_m))^2}|u_c'(x)| \leq K_2\exp(-\delta|x|),$$

$$|\eta_c''| \leq \frac{dc}{(c-f(s_m))^2}|u_c''(x)| + \frac{2dc}{(c-f(s_m))^3}|u_c'(x)|^2 \leq K_3\exp(-\delta|x|),$$

where $K_1,\ K_2,\ K_3$ are positive constants which ends the proof of Theorem 2, case 1.

In order to illustrate the use of Relation (1.16), we give an example of a solitary wave solution to the Beji-Nadaoka equations on Figure 1.



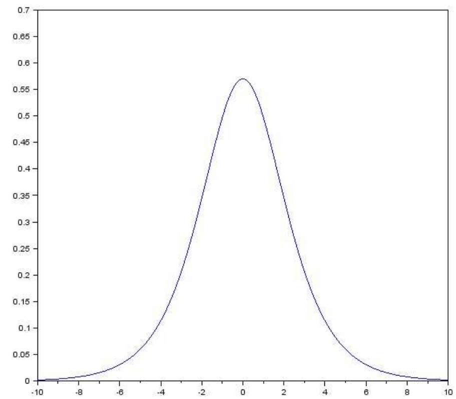


Figure 1: Solitary waves for Beji-Nadaoka equations (η_c is on the left and $\bar{u}_c$ on the right), with $d = 1$, $\eta_c^0 = 0.2$ and $\alpha_B = 1/15$.

3.2 Solitary waves solutions for Madsen and Sorensen equations

We recall the Madsen and Sorensen equations, assuming that the bathymetry d is constant,

$$\begin{cases} \eta_t + q_x = 0, & (3.16) \\ \bar{q}_t + gh\eta_x + \left(\dfrac{\bar{q}^2}{h}\right)_x - Bd^2\bar{q}_{txx} - \beta g d^3 \eta_{xxx} = 0, & (3.17) \end{cases}$$

where $B = \beta + 1/3$, $\beta \in \mathbb{R}$. We look for solitary waves under the form

$$\begin{cases} \eta(t,x) = \eta_c(x - ct) \\ \bar{q}(t,x) = \bar{q}_c(x - ct), \end{cases}$$

where $\eta_c, \bar{q}_c \in H^1(\mathbb{R})$. Plugging these relations into System (3.16), we obtain, after one integration, the following set of equations

$$\begin{cases} \bar{q}_c = c\eta_c, & (3.18) \\ -c\bar{q}_c + c_0^2\eta_c + \dfrac{g}{2}\eta_c^2 + \left(\dfrac{\bar{q}_c^2}{h}\right) + cBd^2\bar{q}_c'' - \beta g d^3 \eta_c'' = 0. & (3.19) \end{cases}$$

Plugging (3.18) into (3.19), one obtains the following equation on η_c

$$-c^2\eta_c + c_0^2\eta_c + \frac{g}{2}\eta_c^2 + c^2\frac{\eta_c^2}{d+\eta_c} + d^2(c^2B - \beta c_0^2)\eta_c^{''} = 0,$$

which can be rewritten into

$$-\eta'' = \frac{1}{K}\left(-c^2\eta + c_0^2\eta + \frac{g}{2}\eta^2 + c^2\frac{\eta^2}{d+\eta}\right), \tag{3.20}$$

where $K = d^2(c^2B - \beta c_0^2)$. Following Theorem 3, we define

$$g(s) = \frac{1}{K}\left(-c^2 s + c_0^2 s + \frac{g}{2}s^2 + c^2\frac{s^2}{d+s}\right),$$

$$G(s) := \int_0^s g(t)\,dt = \frac{1}{K}\left(-c^2 ds + c_0^2\frac{s^2}{2} + \frac{c_0^2}{6d}s^3 + c^2d^2\log(\frac{d+s}{d})\right),$$

and we investigate the behavior of G. Observe first that the equation $g(s) = 0$ has three solutions

$$s_0 = 0,\ s_1 = d\frac{-3 - \sqrt{1+8\frac{c^2}{c_0^2}}}{2},\ s_2 = d\frac{-3 + \sqrt{1+8\frac{c^2}{c_0^2}}}{2}.$$

Since $s_1 < 0$, we concentrate on s_2. If $c \le c_0$, we have $s_2 \le 0$ which implies that g doesn't vanish in $\mathbb{R}_*^+$ and that G is increasing on $\mathbb{R}_+$. We conclude by Theorem 3 that Equation (3.20) has no solution in this case.

Conversely, if $c > c_0$, K is positive, and $s_2 > 0$. Then g is negative on $(0, s_2)$ and positive on $(s_2, +\infty)$. Since $G(0) = 0$, we deduce that $G(s_2) < 0$. Moreover, it is easy to see that $\lim\limits_{s\to+\infty} G(s) = +\infty$. By the mean-value Theorem, we prove that G has a unique zero x_0 in $(s_2, +\infty)$ with $g(x_0) > 0$. By theorem 3, Equation (3.20) admits a unique solution $\eta_c \in H^1(\mathbb{R})$. Finally, denote $\bar{q}_c = c\eta_c$. It is clear that $(\bar{q}_c, \eta_c)$ is the unique solution to (3.18)-(3.19). Furthermore, the relation between the velocity c and the maximal amplitude $A > 0$ of η_c is given by $G(A) = 0$, that is

$$c^2 = c_0^2\frac{\frac{A^2}{2} + \frac{A^3}{6d}}{dA - d^2\log(\frac{d+A}{d})}. \tag{3.21}$$

Moreover, when $c > c_0$, one has

$$\frac{g(s)}{s} = \frac{1}{K}\left(c_0^2 - c^2 + \frac{g}{2}s + c^2\frac{s}{d+s}\right) \underset{s\to 0}{\longrightarrow} \frac{1}{K}\left(c_0^2 - c^2\right) < 0.$$

Applying the Theorem 3, we conclude that η_c (and hence $\bar{q}_c = c\eta_c$) decays exponentially at infinity. We end this section by given a solitary wave solution to Madsen and Sorensen equations using (3.21) (see Figure 2).

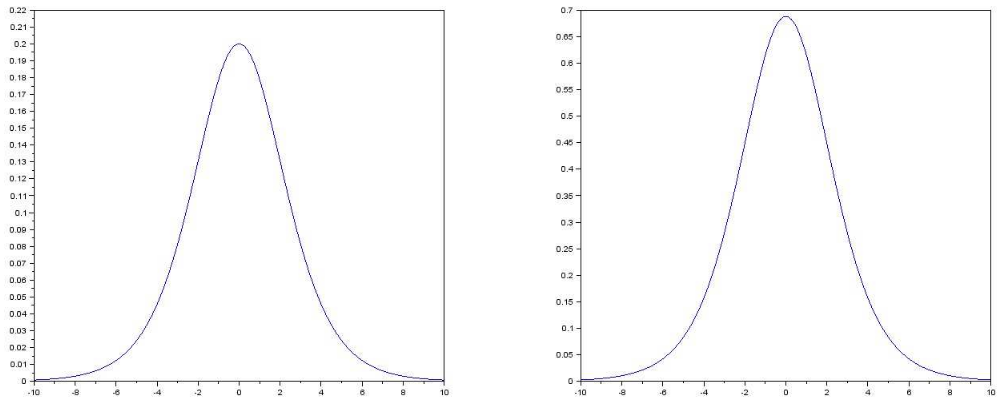


Figure 2: Solitary waves for Madsen and Sorensen equations (η_c is on the left and $\bar{q}_c$ on the right), with $d = 1$, $\eta_c^0 = 0.2$ and $\beta = 1/15$.

3.3 Solitary waves solutions for Nwogu equations

We recall the Nwogu equations with constant bathymetry d. Take $\alpha \in (-\frac{1}{2}, 0)$, $\beta = \alpha + 1/3$ (so that $\beta \in (-\frac{1}{6}, \frac{1}{3})$) and consider

$$\begin{cases} \eta_t + [(\eta + d)U]_x + \beta d^3 U_{xxx} = 0, & (3.22) \\ U_t + UU_x + g\eta_x + \alpha d^2 U_{txx} = 0. & (3.23) \end{cases}$$

Remark 5. If $\beta = 0$, the system corresponds to the so-called Peregrine equations. It can also be obtained by taking $\alpha_B = 0$ in the Beji-Nadaoka equations (3.4)-(3.5). According to Section 3.1, there exists a unique solitary waves solution to the Peregrine equations.

We look for solitary waves under the form

$$\begin{cases} \eta(t,x) = \eta_c(x - ct) \\ U(t,x) = U_c(x - ct), \end{cases}$$

where $\eta_c, U_c \in H^1(\mathbb{R})$. By plugging these quantities in Equations (3.22) and (3.23), we obtain after integration

$$\begin{cases} -c\eta_c + [(\eta_c + d)U_c] + \beta d^3 U_c'' = 0, & (3.24) \\ -cU_c + \dfrac{U_c^2}{2} + g\eta_c - c\alpha d^2 U_c'' = 0, & (3.25) \end{cases}$$

from which we derive, by elimination, the following relation between η_c and U_c

$$\eta_c = d\frac{cU_c/3 - \beta U_c^2/2}{c_0^2\beta - c^2\alpha + c\alpha U_c}. \tag{3.26}$$

Since we are looking for positive and finite solutions, the function U_c has to satisfied

$$0 \leq U_c < c - \frac{\beta}{\alpha}\frac{c_0^2}{c}, \;\; 0 \leq U_c < \frac{2c}{3\beta},$$

which leads to the following condition on parameters c, c_0, α and β

$$c^2 > \frac{\beta}{\alpha}c_0^2.$$

In order to obtain the equation satisfied by η_c, we multiply (3.24) by c and (3.25) by d, to obtain, by adding the resulting equations

$$-\frac{d^2}{3}U_c'' = \frac{(c_0^2 - c^2)\frac{U_c}{3} + \frac{cU_c^2}{2} - \frac{U_c^3}{6}}{\alpha(c_0^2 - c^2) + \frac{c_0^2}{3} + \alpha c U_c}.$$

As usual, taking into account the previous remarks, we introduce the function g on the interval $[0, c - \frac{\beta}{\alpha}\frac{c_0^2}{c})$

$$g(t) = \frac{(c_0^2 - c^2)\frac{t}{3} + \frac{ct^2}{2} - \frac{t^3}{6}}{\alpha(c_0^2 - c^2) + \frac{c_0^2}{3} + \alpha c t},$$

and

$$\begin{aligned} G(t) &:= \int_0^t g(x)\,dx, \\ &= -\frac{1}{18\alpha}t^3 + \left(\frac{c}{4\alpha} + \frac{\beta c_0^2 - \alpha c^2}{12c\alpha}\right)t^2 + \left(\frac{c_0^2 - c^2}{3\alpha} - \frac{\beta c_0^2 - \alpha c^2}{2\alpha^2} - \frac{(\beta c_0^2 - \alpha c^2)^2}{6\alpha^2 c^2}\right)t, \\ &+ \left(\frac{(\beta c_0^2 - \alpha c^2)^4}{6c^3\alpha^3} + \frac{(\beta c_0^2 - \alpha c^2)^3}{2c\alpha^3} - \frac{(c_0^2 - c^2)(\beta c_0^2 - \alpha c^2)^2}{3c\alpha^2}\right)\log(1 + \frac{c\alpha}{\beta c_0^2 - \alpha c^2}t). \end{aligned}$$

A quick study provides that the solutions to $g(t) = 0$ are

$$t_0 = 0, \; t_1 = c\frac{3 - \sqrt{1 + 8K}}{2}, \; t_2 = c\frac{3 + \sqrt{1 + 8K}}{2},$$

where $K = \frac{c_0^2}{c^2}$.

If $c \leq c_0$, g doesn't vanish in $[0, c - \frac{\beta}{\alpha}\frac{c_0^2}{c})$ from which we deduce that so does G. Then Equations (3.24)-(3.25) don't admit solutions in this case.

Conversely, if the condition

$$c > \max\left(c_0, \sqrt{\frac{\frac{\beta^2}{\alpha^2}}{2-\frac{\beta}{\alpha}}} c_0\right)$$

is satisfied, then t_1 is the only zero of g in the interval $[0, c - \frac{\beta}{\alpha}\frac{c_0^2}{c})$. Since in this case, $c - \frac{\beta}{\alpha}\frac{c_0^2}{c} \leq \frac{2c}{3\beta}$, we deduce that

$$g(t) \leq 0, \text{ for } t \in [0, c\frac{3-\sqrt{1+8K}}{2}],$$

$$g(t) > 0, \text{ for } t \in [c\frac{3-\sqrt{1+8K}}{2}, c - \frac{\beta}{\alpha}\frac{c_0^2}{c}),$$

and

$$G(c\frac{3-\sqrt{1+8K}}{2}) < 0,$$

since $G(0) = 0$. Recalling that $\alpha < 0$, it is obvious that

$$\lim_{t \to c - \frac{\beta}{\alpha}\frac{c_0^2}{c}} G(t) = +\infty,$$

and then the mean-value theorem provides that G admits a unique zero $t_m \in [c\frac{3-\sqrt{1+8K}}{2}, c - \frac{\beta}{\alpha}\frac{c_0^2}{c})$ satisfying $g(t_m) > 0$.
Again, Theorem 3 furnishes the existence of a unique solution U_c to Equation (3.26). It is then clear that U_c and η_c are the only solutions to the Nwogu equations (3.24)-(3.25) for all

$$c > \max\left(c_0, c_0\sqrt{\frac{\frac{\beta^2}{\alpha^2}}{2-\frac{\beta}{\alpha}}}\right). \tag{3.27}$$

Remark 6. If $\alpha \in (-\frac{1}{9}, 0)$, then

$$c_0 < c_0\sqrt{\frac{\frac{\beta^2}{\alpha^2}}{2-\frac{\beta}{\alpha}}}.$$

In this case t_1 is the only one zero of g on $[0, c - \frac{\beta}{\alpha}\frac{c_0^2}{c})$. Recalling that we look for a function U_c satisfying for all $t \in [0, c - \frac{\beta}{\alpha}\frac{c_0^2}{c})$

$$0 < U_c(t) < \frac{2c}{3\beta},$$

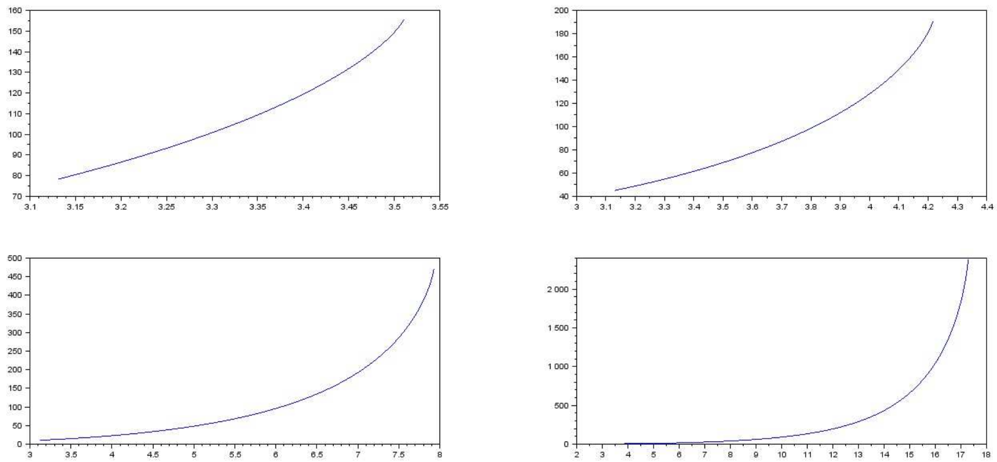


Figure 3: The values of $G(\frac{2c}{3\beta})$ with respect to $c_0 < c < c_0\sqrt{\frac{\frac{\beta^2}{\alpha^2}}{2-\frac{\beta}{\alpha}}}$ and for $\beta = 0.2333$ (top left), $\beta = 0.2593$ (top right), $\beta = 0.2963$ (bottom left), $\beta = -0.3233$ (bottom right).

we deduce that the existence of a solution to Equation (3.26) depends on the sign of $G(\frac{2c}{3\beta})$. More precisely, if $G(\frac{2c}{3\beta}) > 0$, Equation (3.26) admits a unique solution whereas in the opposite case, there is no solutions. Unfortunately, we are not able here to determine easily the sign of $G(\frac{2c}{3\beta})$ but as suggested by Figure 3, we conjecture that it is always positive which provides consequently the existence of U_c for all $c > c_0$ and $\alpha \in (-1/2, 0)$

Coming back to the case (3.27), we write the relation between the velocity c and the amplitude A of U_c

$$\begin{aligned}
&-\frac{1}{18\alpha}A^3 + \left(\frac{c}{4\alpha} + \frac{\beta c_0^2 - \alpha c^2}{12c\alpha}\right)A^2 \\
&+\left(\frac{c_0^2 - c^2}{3\alpha} - \frac{\beta c_0^2 - \alpha c^2}{2\alpha^2} - \frac{(\beta c_0^2 - \alpha c^2)^2}{6\alpha^2 c^2}\right)A \\
&+\left(\frac{(\beta c_0^2 - \alpha c^2)^4}{6c^3\alpha^3} + \frac{(\beta c_0^2 - \alpha c^2)^3}{2c\alpha^3}\right. \\
&\left.-\frac{(c_0^2 - c^2)(\beta c_0^2 - \alpha c^2)^2}{3c\alpha^2}\right)\log(1 + \frac{c\alpha}{\beta c_0^2 - \alpha c^2}A) = 0.
\end{aligned}$$

Furthermore,

$$\frac{g(t)}{t} = \frac{(c_0^2 - c^2)\frac{1}{3} + \frac{ct}{2} - \frac{t^2}{6}}{\alpha(c_0^2 - c^2) + \frac{c_0^2}{3} + \alpha ct} \underset{t \to 0}{\longrightarrow} \frac{(c_0^2 - c^2)\frac{1}{3}}{\alpha(c_0^2 - c^2) + \frac{c_0^2}{3}} < 0,$$

proving that U_c is exponentially decaying at infinity. Moreover, direct computations shows that

$$\begin{aligned} |\eta_c(x)| = |d\frac{cU_c(x)/3 - \beta U_c(x)^2/2}{c_0^2\beta - c^2\alpha + c\alpha U_c(x)}| &\leq d\frac{c/3 + |\beta|t_m/2}{c_0^2\beta - c^2\alpha + c\alpha t_m}|U_c(x)| \\ &\leq L_1 \exp(-\delta|x|), \end{aligned}$$

$$|\eta_c''(x)| \leq L_2 \exp(-\delta|x|),$$
$$|\eta_c''(x)| \leq L_3 \exp(-\delta|x|),$$

where $L_1,\ L_2,\ L_3$ are positive. This ends the proof of Theorem 2. □

We end this section by showing a solitary wave for Nwogu equations (see Figure 4).

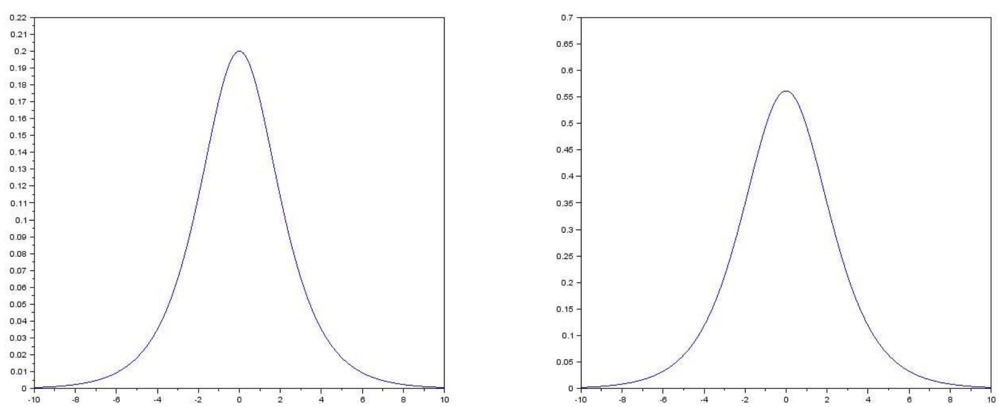


Figure 4: Solitary wave for Nwogu equations (η_c is on the left and U_c on the right), with $d = 1$, $\eta_c^0 = 0.2$ and $\beta = -1/15$.

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