Cauchy problem for the nonlinear Klein-Gordon equation coupled with the Maxwell equation

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Abstract

In this paper, we study the nonlinear Klein-Gordon equation coupled with a Maxwell equation. Using the energy method, we obtain a local existence result for the Cauchy problem.

Key words: Klein-Gordon-Maxwell system, Cauchy problem, symmetric hyperbolic system, energy method

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1 Introduction

In this paper, we consider the following nonlinear Klein-Gordon equation coupled with Maxwell equation:

$$\begin{aligned} \psi_{tt} - \Delta\psi = -2ie\phi\psi_t - ie\phi_t\psi + e^2|\phi|^2\psi - 2ie\nabla\psi\cdot\mathbf{A} \\ - e^2|\mathbf{A}|^2\psi - ie\psi\,\mathrm{div}\mathbf{A} - m^2\psi + W'(\psi). \end{aligned} \tag{1.1}$$

$$\mathbf{A}_{tt} - \Delta\mathbf{A} = e\,\mathrm{Im}(\bar{\psi}\nabla\psi) - e^2|\psi|^2\mathbf{A} - \nabla\phi_t - \nabla\mathrm{div}\mathbf{A}. \tag{1.2}$$

$$-\Delta\phi = e\,\mathrm{Im}(\psi\bar{\psi}_t) - e^2|\psi|^2\phi + \mathrm{div}\mathbf{A}_t. \tag{1.3}$$

where $\psi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$, $\mathbf{A} : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$, $\phi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$, $m > 0$, $e \in \mathbb{R}$ and i denotes the unit complex number, that is, $i^2 = -1$. Moreover, $W(s)$ is a regular real valued function which is extended to the complex plane by setting $W'(\psi) = W'(|\psi|)\frac{\psi}{|\psi|}$ for $\psi \in \mathbb{C}$.

In this system, ψ represents an electrically charged field and $(\phi, \mathbf{A})$ is a gauge potential of an electromagnetic field. System (1.1)-(1.3) describes the interaction of a particle with an electromagnetic field in the following way: the field ψ produces, on one hand, a current which acts as a force for the electromagnetic field and, on the other hand, carries an electric charge which is given by the electromagnetic field. (See Section 2 for the derivation.) We refer to [14], [16] for more physical backgrounds.

To our knowledge, there are few results concerning the Cauchy problem associated with System (1.1)-(1.3). In [8], [13], the Cauchy problem for the linear Klein-Gordon-Maxwell equation, that is the case $m = 0$ and $W' = 0$, has been studied. In [15], the author studied the Cauchy problem associated to (1.1)- (1.3) under several conditions on W. Our aim of this paper is to study the Cauchy problem corresponding to (1.1)-(1.3) and to obtain the local existence result by using the standard energy method. In this direction, one can look for particular global solutions of (1.1)-(1.3) under the form of standing waves of the type:

$$\psi(x,t) = u(x)e^{i\omega t}\ (\omega \in \mathbb{R}),\ \mathbf{A}(x,t) = \mathbf{0} \text{ and } \phi(x,t) = \phi(x). \tag{1.4}$$

Plugging (1.4) into (1.1)-(1.3) leads to the following elliptic system:

$$\begin{cases} -\Delta u + \big(m^2 - (\omega + e\phi)^2\big)u = W'(u). \\ -\Delta\phi + e^2u^2\phi = -e\omega u^2. \end{cases} \tag{1.5}$$

The existence of solutions to system (1.5) has been studied widely. (See [2], [3], [7], [9] and references therein.) The stability of standing waves has been also considered in [4], [5], [15]. Especially in [4] and [5], the authors showed that the standing wave is stable when the potential term is *positive*, that is, $\frac{m^2}{2}s^2 - W(s) \geq 0$ for $s \geq 0$. However some challenging problems, for example the (in)stability for large e, are still left open.

System (1.1)-(1.3) has a so-called *gauge ambiguity*, thus we need to choose a suitable gauge condition. In this paper, we impose the Coulomb condition, that is, we look for a solution $\mathbf{A}$ which satisfies

$$\mathrm{div}\mathbf{A} = 0. \tag{1.6}$$

In this setting, one has $|\,\mathrm{rot}\mathbf{A}|^2 = |\nabla\mathbf{A}|^2$ which seems to be useful for the stability analysis of the standing wave.

To state our main results, we introduce the following notations. First we impose the following initial conditions at $t = 0$:

$$\begin{cases} \psi(0,x) = \psi_{(0)}(x), & \psi_t(0,x) = \psi_{(1)}(x), \\ \mathbf{A}(0,x) = \mathbf{A}_{(0)}(x), & \mathbf{A}_t(0,x) = \mathbf{A}_{(1)}(x), \\ \mathrm{div}\mathbf{A}_{(0)} = 0, & \mathrm{div}\mathbf{A}_{(1)} = 0. \end{cases} \tag{1.7}$$

Moreover we assume that for some $m \in \mathbb{N}$ with $m \geq 2$,

$$\begin{aligned} \psi_{(0)} \in H^{m+1}(\mathbb{R}^3, \mathbb{C}), \quad \psi_{(1)} \in H^m(\mathbb{R}^3, \mathbb{C}), \\ \mathbf{A}_{(0)} \in H^{m+1}(\mathbb{R}^3, \mathbb{R}^3) \quad \text{and} \quad \mathbf{A}_{(1)} \in H^m(\mathbb{R}^3, \mathbb{R}^3), \end{aligned} \tag{1.8}$$

where $H^m(\mathbb{R}^3, \cdot)$ denotes the usual Sobolev space. We also introduce the space $D^{1,2}(\mathbb{R}^3)$ which denotes the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the inner product:

$$(u, v) := \int_{\mathbb{R}^3} \nabla u \cdot \nabla v \, dx.$$

We recall, by the Sobolev inequality, that the space $D^{1,2}(\mathbb{R}^3)$ is continuously embedded into $L^6(\mathbb{R}^3)$.

For the nonlinear term W, we assume that

$$(A) \qquad W \in C^{m+1}(\mathbb{C}, \mathbb{C}) \ \text{ and } W(0) = W'(0) = 0.$$

Some typical examples of the nonlinear term W are the power nonlinearity $W(s) = \pm\frac{1}{p}s^p$ with $[p] \geq m+1$ ($[p]$ denotes the integer part of p), or the *cubic-quintic* nonlinearity $W(s) = \frac{1}{3}s^3 - \frac{\lambda}{5}s^5$ for $\lambda > 0$, which frequently appears in the study of solitons in physical literatures. (See [12], [14], [16] for example.)

In this setting, we have the following result.

Theorem 1.1. *Suppose that (A) holds and* $(\psi_{(0)}, \psi_{(1)}, \mathbf{A}_{(0)}, \mathbf{A}_{(1)})$ *satisfies* (1.8). *Then there exists* $T^* > 0$ *such that system* (1.1)-(1.3) *with the initial condition* (1.7) *has a unique solution* $(\psi, \mathbf{A}, \phi)$ *satisfying the Coulomb condition* (1.6) *and*

$$\psi \in C((0, T^*), H^{m+1}) \cap C^1((0, T^*), H^m),$$

$$\mathbf{A} \in C((0, T^*), H^{m+1}) \cap C^1((0, T^*), H^m),$$

$$\phi \in C([0, T], D^{1,2}), \ \nabla\phi \in C((0, T^*), H^m), \ \phi_t \in C((0, T^*), H^m).$$

In [8], [13], [15], the authors used Strichartz estimates and space time estimates for null forms to obtain a local solution. In this paper, we adopt a different approach. More precisely, we apply the strategy developed in [6] and our proof is based on the standard energy method for symmetric hyperbolic systems. We emphasize that our approach is much elementary. We also expect that our method is applicable for the Cauchy problem associated with the nonlinear Klein-Gordon equation coupled with Born-Infeld equations. (See [10], [18].)

This paper is organized as follows. In Section 2, we introduce the derivation of system (1.1)-(1.3) and derive some conservation laws. In Section 3, we give several estimates for the elliptic equation (1.3). We prove Theorem 1.1 in Section 4. Firstly, we rewrite system (1.1)-(1.3) as a symmetric hyperbolic system in Section 4.1. Secondly, in Section 4.2, we prove the existence of an unique local solution by using the energy method.

2 Derivation and conservation laws

In this section, we briefly introduce the derivation of system (1.1)-(1.3) and derive some conservation laws. Now we consider the (complex) nonlinear Klein-Gordon equation:

$$\psi_{tt} - \Delta\psi = -m^2\psi + W'(\psi)$$

and the corresponding Lagrangian:

$$\mathcal{L}_0 = \frac{1}{2}\big(|\psi_t|^2 - |\nabla\psi|^2 - m^2|\psi|^2\big) + W(\psi) = -\frac{1}{2}\big(\overline{\partial_\alpha\psi}\,\partial^\alpha\psi + m^2|\psi|^2\big) + W(\psi), \tag{2.1}$$

where $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$, $\alpha = 0, 1, 2, 3$ and $x_0 = t$.

Suppose that ψ is an electrically charged field. Then ψ must interact with the Maxwell field. Let $\mathbf{E}$ and $\mathbf{H}$ be the electric and the magnetic fields respectively, and assume that they are described by the Gauge potential $(\phi, \mathbf{A})$, $\mathbf{A} = (A_1, A_2, A_3)$ as follows:

$$\mathbf{E} = \nabla\phi + \mathbf{A}_t \quad \text{and} \quad \mathbf{H} = \operatorname{rot}\mathbf{A}.$$

By the gauge invariance of the combined theory, the interaction between ψ and $(\phi, \mathbf{A})$ is given by exchanging the usual derivatives ∂_α with the gauge covariant derivative:

$$D_\alpha = \partial_\alpha - ie\mathbf{A}_\alpha, \;\; \mathbf{A}_\alpha = (-\phi, A_1, A_2, A_3).$$

Thus from (2.1), we obtain the following Lagrangian:

$$\mathcal{L}_0 = \frac{1}{2}\Big(\big|\psi_t + ie\phi\psi\big|^2 - \big|\nabla\psi - ie\mathbf{A}\psi\big|^2 - m^2|\psi|^2\Big) + W(\psi).$$

Moreover since the Lagrangian of $\mathbf{E}$ and $\mathbf{H}$ is described by

$$\mathcal{L}_1 = \frac{1}{2}\left(|\mathbf{E}|^2 - |\mathbf{H}|^2\right) = \frac{1}{2}\left(|\nabla\phi + \mathbf{A}_t|^2 - |\operatorname{rot}\mathbf{A}|^2\right),$$

the total Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ is given by

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\Big(\big|\psi_t + ie\phi\psi\big|^2 - \big|\nabla\psi - ie\mathbf{A}\psi\big|^2 - m^2|\psi|^2\Big) + W(\psi)\\ &+ \frac{1}{2}\big|\nabla\phi + \mathbf{A}_t\big|^2 - \frac{1}{2}\big|\operatorname{rot}\mathbf{A}\big|^2.\end{aligned} \tag{2.2}$$

Computing the Euler-Lagrange equations for $(\psi, \mathbf{A}, \phi)$, we can obtain system (1.1)-(1.3). Moreover since the electromagnetic current $\mathbf{J}^\mu = e\,\mathrm{Im}(\bar{\psi} D^\mu \psi)$ is conserved, we have the following conservation laws for the charge and the momentum:

$$\frac{d}{dt}\int_{\mathbb{R}^3}\Big(e\,\mathrm{Im}(\bar{\psi}\psi_t) + e^2|\psi|^2\phi\Big)\,dx = 0 \quad \text{(charge)},$$
$$\frac{d}{dt}\int_{\mathbb{R}^3}\Big(e\,\mathrm{Im}(\bar{\psi}\nabla\psi) - e^2|\psi|^2\mathbf{A}\Big)\,dx = 0 \quad \text{(momentum)}.$$

We refer to [11] for more details.

Finally we derive the energy conservation law which we will use later on. To this aim, we multiply $\bar{\psi}_t$ by (1.1), integrating it over $\mathbb{R}^3$ and taking the real part to obtain

$$\int_{\mathbb{R}^3}\Big\{\frac{\partial}{\partial t}\left(\frac{1}{2}|\psi_t|^2 + \frac{1}{2}|\nabla\psi|^2 + \frac{m^2}{2}|\psi|^2 - W(\psi)\right) - e^2|\phi|^2\,\mathrm{Re}(\psi\bar{\psi}_t)$$
$$+ e^2|\mathbf{A}|^2\,\mathrm{Re}(\psi\bar{\psi}_t) - \,\mathrm{Im}\Big(e\phi_t\psi\bar{\psi}_t + 2e\bar{\psi}_t\nabla\psi\cdot\mathbf{A} + \psi\bar{\psi}_t\mathrm{div}\,\mathbf{A}\Big)\Big\}\,dx = 0. \tag{2.3}$$

Next multiplying $\mathbf{A}_t$ by (1.2), we also have

$$\int_{\mathbb{R}^3}\Big\{\frac{\partial}{\partial t}\left(\frac{1}{2}|\mathbf{A}_t|^2 + \frac{1}{2}|\,\mathrm{rot}\mathbf{A}|^2\right) - e\,\mathrm{Im}(\bar{\psi}\nabla\psi)\cdot\mathbf{A}_t + e^2|\psi|^2\mathbf{A}\cdot\mathbf{A}_t + \nabla\phi_t\cdot\mathbf{A}_t\Big\}\,dx = 0, \tag{2.4}$$

where we used the fact

$$\mathrm{rot}\mathbf{A}\cdot\,\mathrm{rot}\mathbf{A}_t = \,\mathrm{div}(\mathbf{A}_t\times\,\mathrm{rot}\mathbf{A}) + \mathbf{A}_t\cdot\nabla(\,\mathrm{div}\mathbf{A}) - \mathbf{A}_t\cdot\Delta\mathbf{A}.$$

Finally multiplying $-\phi$ by (1.3), we get

$$\int_{\mathbb{R}^3}\Big\{\frac{\partial}{\partial t}\left(-\frac{1}{2}|\nabla\phi|^2\right) + e\,\mathrm{Im}(\psi\bar{\psi}_t)\phi_t - e^2|\psi|^2\phi\phi_t + \phi_t\,\mathrm{div}\mathbf{A}_t\Big\}\,dx = 0. \tag{2.5}$$

Summing (2.3)-(2.5) up and applying the integration by parts, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3}\Big\{\frac{1}{2}|\psi_t|^2 + \frac{1}{2}|\nabla\psi|^2 + \frac{m^2}{2}|\psi|^2 - W(\psi) + \frac{1}{2}|\,\mathrm{rot}\mathbf{A}|^2 + \frac{1}{2}|\mathbf{A}_t|^2 - \frac{1}{2}|\nabla\phi|^2$$
$$- \frac{e^2}{2}|\phi|^2|\psi|^2 + \frac{1}{2}|\mathbf{A}|^2|\psi|^2 + e\,\mathrm{Im}(\psi\nabla\bar{\psi})\cdot\mathbf{A} + \nabla\phi\cdot\mathbf{A}_t + \phi\,\mathrm{div}\mathbf{A}_t\Big\}\,dx = 0.$$

Using (1.3) again, we obtain the following energy conservation law:

$$\begin{aligned} 0 &= \frac{d}{dt}\mathcal{E}(\psi, \mathbf{A}, \phi)(t) \\ &= \frac{d}{dt}\int_{\mathbb{R}^3}\Big\{\frac{1}{2}|\psi_t + ie\phi\psi|^2 + \frac{1}{2}|\nabla\psi - ie\mathbf{A}\psi|^2 + \frac{m^2}{2}|\psi|^2 - W(\psi) \\ &\qquad + \frac{1}{2}|\operatorname{rot}\mathbf{A}|^2 + \frac{1}{2}|\mathbf{A}_t + \nabla\phi|^2\Big\}\,dx. \end{aligned} \tag{2.6}$$

(We can derive (2.6) by applying the Noether theorem to the Lagrangian $\mathcal{L}$ of (2.2). See [5] for this topics.)

3 Estimates for the elliptic part

In this section, we give several estimates for the elliptic equation:

$$-\Delta\phi + e^2|\psi|^2\phi = e\,\mathrm{Im}(\psi\bar{\psi}_t). \tag{3.1}$$

Throughout this section, we suppose that $\psi(t,\cdot)$ has a compact support for each $t \in [0,T]$ and $\psi \in C^\infty\big((0,T)\times\mathbb{R}^3\big)$. The estimates for general ψ can be obtained by a density argument.

Lemma 3.1. *Let ϕ be a solution of* (3.1). *Then for $m \in \mathbb{N}$ with $m \geq 2$, $\phi(t,\cdot)$ satisfies the following estimates for each $t \in [0,T]$.*

(i) $\|\nabla\phi(t,\cdot)\|_{H^m(\mathbb{R}^3)} \leq \tilde{C}\|\psi_t(t,\cdot)\|_{H^m(\mathbb{R}^3)}$.

(ii) $\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leq \tilde{C}\|\psi_t(t,\cdot)\|_{H^m(\mathbb{R}^3)}$.

(iii) $\|\phi_t(t,\cdot)\|_{H^m(\mathbb{R}^3)} \leq \tilde{C}\|\psi(t,\cdot)\|^2_{H^m(\mathbb{R}^3)}$.

Here $\tilde{C}$ is a constant depending on $\|\psi\|_{H^{m+1}}$, $\|\psi_t\|_{H^m}$ and $\|\nabla\mathbf{A}\|_{H^m}$.

Proof. (i) First we apply D^s to (3.1) and take the L^2-inner product with $D^s\phi$. Then by the integration by parts, we have

$$\|\nabla(D^s\phi)\|^2_{L^2} + e^2\int_{\mathbb{R}^3} D^s(|\psi|^2\phi)D^s\phi\,dx = e\int_{\mathbb{R}^3}\mathrm{Im}(D^s(\psi\bar{\psi}_t))D^s\phi\,dx.$$

Now we observe that, by Leibniz rule,

$$\begin{aligned} D^s(|\psi|^2\phi)D^s\phi &= 2\sum_{\alpha+\beta=s}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\mathrm{Re}(\psi D^\alpha\bar{\psi})D^\beta\phi D^s\phi \\ &= 2|\psi|^2|D^s\phi|^2 + 2\sum_{\alpha+\beta=s,\alpha\neq 0}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\mathrm{Re}(\psi D^\alpha\bar{\psi})D^\beta\phi D^s\phi. \end{aligned}$$

Thus one has

$$
\begin{aligned}
&\|\nabla(D^s\phi)\|_{L^2}^2 + e^2\int_{\mathbb{R}^3} |\psi|^2|D^s\phi|^2\,dx \\
&\le C\sum_{\alpha+\beta=s,\alpha\neq 0}\int_{\mathbb{R}^3}|\psi||D^\alpha\psi||D^\beta\phi||D^s\phi|\,dx + C\sum_{\alpha+\beta=s}\int_{\mathbb{R}^3}|D^\alpha\psi||D^\beta\psi_t||D^s\phi|\,dx \\
&\le C\sum_{\alpha+\beta=s,\alpha\neq 0}\|D^\alpha\psi D^\beta\phi\|_{L^2}\|\psi D^s\phi\|_{L^2} + C\sum_{\alpha+\beta=s}\|D^\alpha\psi\|_{L^3}\|D^\beta\psi_t\|_{L^2}\|D^s\phi\|_{L^6} \\
&\le C\sum_{\alpha+\beta=s,\alpha\neq 0}\|D^\alpha\psi D^\beta\|_{L^2}^2 + \frac{e^2}{2}\|\psi D^s\phi\|_{L^2}^2 \\
&\quad + C\sum_{\alpha+\beta=s}\|D^\alpha\psi\|_{L^3}^2\|D^\beta\psi_t\|_{L^2}^2 + \frac{1}{2}\|\nabla(D^s\phi)\|_{L^2}^2.
\end{aligned}
$$

By the Hölder and the Sobolev inequalities, it follows that

$$
\begin{aligned}
\sum_{\alpha+\beta=s,\alpha\neq 0}\|D^\alpha\psi D^\beta\phi\|_{L^2}^2 &\le \sum_{\alpha+\beta=s,\alpha\neq 0}\|D^\alpha\psi\|_{L^3}^2\|D^\beta\phi\|_{L^6}^2 \\
&\le C\sum_{\alpha+\beta=s,\alpha\neq 0}\|D^\alpha\psi\|_{H^1}^2\|\nabla(D^\beta\phi)\|_{L^2}^2 \\
&\le C\|\psi\|_{H^{s+1}}^2\|\nabla\phi\|_{H^{s-1}}, \\
\sum_{\alpha+\beta=s}\|D^\alpha\psi\|_{L^3}^2\|D^\beta\psi_t\|_{L^2}^2 &\le C\|\psi\|_{H^{s+1}}^2\|\psi_t\|_{H^s}^2.
\end{aligned}
$$

Thus we obtain

$$
\|\nabla\phi\|_{H^m}^2 \le C\Big(\|\psi\|_{H^{m+1}}^2\|\nabla\phi\|_{H^{m-1}}^2 + \|\psi\|_{H^{m+1}}^2\|\psi_t\|_{H^m}^2\Big). \tag{3.2}
$$

Next we multiply ϕ by (3.1) and integrate it over $\mathbb{R}^3$. Then by the integration by parts, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx + e^2\int_{\mathbb{R}^3}|\psi|^2|\phi|^2\,dx &\le |e|\int_{\mathbb{R}^3}|\psi||\psi_t||\phi|\,dx \\
&\le |e|\Big(\int_{\mathbb{R}^3}|\psi|^2|\phi|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^3}|\psi_t|^2\,dx\Big)^{\frac{1}{2}} \\
&\le \frac{e^2}{2}\int_{\mathbb{R}^3}|\psi|^2|\phi|^2\,dx + \frac{1}{2}\|\psi_t\|_{L^2(\mathbb{R}^3)}^2.
\end{aligned}
$$

This implies that $\|\nabla\phi\|_{L^2} \le C\|\psi_t\|_{L^2}$. Then by induction and from (3.2), one can see that (i) holds.

(ii) By the embedding $W^{1,6}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, one has

$$\|\phi\|_{L^\infty} \le C\left(\sum_{j=1}^{3}\left\|\frac{\partial\phi}{\partial x_j}\right\|_{L^6} + \|\phi\|_{L^6}\right).$$

Moreover by the Calderon-Zygmund inequality: $\left\|\frac{\partial^2\phi}{\partial x_j\partial x_k}\right\|_{L^2} \le C\|\Delta\phi\|_{L^2}$ and the Sobolev inequality, it follows that

$$\begin{aligned}\|\phi\|_{L^\infty} &\le C\left(\sum_{j=1}^{3}\left\|\nabla\left(\frac{\partial\phi}{\partial x_j}\right)\right\|_{L^2} + \|\nabla\phi\|_{L^2}\right)\\ &\le C(\|\Delta\phi\|_{L^2} + \|\nabla\phi\|_{L^2}).\end{aligned}$$

From (3.1), we have

$$\begin{aligned}\|\phi\|_{L^\infty} &\le C(\|\psi\psi_t\|_{L^2} + \||\psi|^2\phi\|_{L^2} + \|\nabla\phi\|_{L^2})\\ &\le C(\|\psi\|_{L^\infty}\|\psi_t\|_{L^2} + \|\psi\|_{L^3}^2\|\phi\|_{L^6} + \|\nabla\phi\|_{L^2}).\end{aligned}$$

Since $H^m(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ for $m \ge 2$, it follows that $\|\psi\|_{L^\infty} \le C\|\psi\|_{H^m}$. Thus from (i), we get

$$\|\phi\|_{L^\infty} \le C\Big(\|\psi\|_{H^m}\|\psi_t\|_{L^2} + \|\psi\|_{H^m}^2\|\psi_t\|_{L^2} + \|\psi_t\|_{L^2}\Big) \le \tilde{C}\|\psi_t\|_{L^2}.$$

This completes the proof of (ii).

To prove (iii), we have to derive the equation of ϕ_t. Now differentiating (3.1) with respect to t, we get

$$-\Delta\phi_t + e^2|\psi|^2\phi_t + 2e^2\operatorname{Re}(\psi\bar{\psi}_t)\phi = e\operatorname{Im}(\psi\bar{\psi}_{tt}). \tag{3.3}$$

On the other hand, taking the complex conjugate in (1.1), multiplying it by ψ and taking the imaginary part, we also have

$$\begin{aligned}\operatorname{Im}(\psi\bar{\psi}_{tt}) &= \operatorname{Im}\big(\psi\Delta\bar{\psi} + 2ie\psi\nabla\bar{\psi}\cdot\mathbf{A}\big) + 2e\operatorname{Re}(\psi\bar{\psi}_t)\phi + e|\psi|^2\phi_t\\ &= \operatorname{Im}\Big(\operatorname{div}(\psi\nabla\bar{\psi})\Big) + 2e\operatorname{Re}(\psi\nabla\bar{\psi})\cdot\mathbf{A} + 2e\operatorname{Re}(\psi\bar{\psi}_t)\phi + e|\psi|^2\phi_t\\ &= \operatorname{div}\Big(\operatorname{Im}(\psi\nabla\bar{\psi}) + |\psi|^2\mathbf{A}\Big) + 2e\operatorname{Re}(\psi\bar{\psi}_t)\phi + e|\psi|^2\phi_t. \qquad (3.4)\end{aligned}$$

Here we used the fact $\operatorname{div}\mathbf{A} = 0$. From (3.3) and (3.4), it follows that

$$-\Delta\phi_t = e\operatorname{div}\Big(\operatorname{Im}(\psi\nabla\bar{\psi}) + |\psi|^2\mathbf{A}\Big).$$

We observe that this equation can be written by the form:

$$\phi_t = (-\Delta)^{-\frac{1}{2}}\Big\{(-\Delta)^{-\frac{1}{2}} e\,\mathrm{div}\Big(\mathrm{Im}(\psi\nabla\bar{\psi}) + |\psi|^2\mathbf{A}\Big)\Big\}. \tag{3.5}$$

Now we recall the Hardy-Littlewood-Sobolev inequality (See [17], P. 119):

$$\|(-\Delta)^{-\frac{\alpha}{2}} f\|_{L^q(\mathbb{R}^N)} \le C\|f\|_{L^p(\mathbb{R}^N)}$$

for $0 < \alpha < N$, $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{N}$. Applying D^s to (3.5) and using the Hardy-Littlewood-Sobolev inequality with $N = 3$, $\alpha = 1$, $p = \frac{6}{5}$ and $q = 2$, we get

$$\begin{aligned}\|D^s\phi_t\|_{L^2} &\le C\Big\|(-\Delta)^{-\frac{1}{2}} e\,\mathrm{div}\Big(\mathrm{Im}(D^s(\psi\nabla\bar{\psi})) + D^s(|\psi|^2\mathbf{A})\Big)\Big\|_{L^{\frac{6}{5}}}\\ &\le C\Big(\|D^s(\psi\nabla\bar{\psi})\|_{L^{\frac{6}{5}}} + \|D^s(|\psi|^2\mathbf{A})\|_{L^{\frac{6}{5}}}\Big).\end{aligned}$$

Now we observe that

$$\begin{aligned}\|D^s(\psi\nabla\bar{\psi})\|_{L^{\frac{6}{5}}} &\le C\sum_{\alpha+\beta=s}\|D^\alpha\psi D^\beta(\nabla\bar{\psi})\|_{L^{\frac{6}{5}}} \le C\sum_{\alpha+\beta=s}\|D^\alpha\psi\|_{L^3}\|D^\beta(\nabla\bar{\psi})\|_{L^2}\\ &\le C\sum_{\alpha+\beta=s}\|D^\alpha\psi\|_{H^1}\|D^\beta(\nabla\bar{\psi})\|_{L^2} \le C\|\psi\|_{H^s}\|\psi\|_{H^{s+1}},\end{aligned}$$

$$\begin{aligned}\|D^s(|\psi|^2\mathbf{A})\|_{L^{\frac{6}{5}}} &\le C\sum_{\alpha+\beta=s}\|D^\alpha(|\psi|^2)D^\beta\mathbf{A}\|_{L^{\frac{6}{5}}}\\ &\le C\sum_{\alpha+\beta=s}\|D^\alpha(|\psi|^2)\|_{L^{\frac{3}{2}}}\|D^\beta\mathbf{A}\|_{L^6}\\ &\le C\sum_{\alpha+\beta=s}\|\psi\|_{L^3}\|D^\alpha\psi\|_{L^2}\|\nabla(D^\beta\mathbf{A})\|_{L^2}\\ &\le C\|\psi\|_{H^1}\|\psi\|_{H^{s+1}}\|\nabla\mathbf{A}\|_{H^s}.\end{aligned}$$

Thus we obtain $\|\psi_t\|_{H^m} \le \tilde{C}\|\psi\|_{H^m}$. □

Next let $(\psi, \mathbf{A})$ and $(\tilde{\psi}, \tilde{\mathbf{A}})$ be given such that

$$(\psi, \tilde{\psi}) \in \Big[C\big((0,T), H^1(\mathbb{R}^3)\big)\Big]^2, (\psi_t, \tilde{\psi}_t) \in \Big[C\big((0,T), L^2(\mathbb{R}^3)\big)\Big]^2,$$

$$(\nabla\mathbf{A}, \nabla\tilde{\mathbf{A}}) \in \Big[C\left((0,T), \left(L^2(\mathbb{R}^3)\right)^3\right)\Big]^2$$

and consider the unique solution ϕ and $\tilde{\phi}$ of (3.1) with ψ and $\tilde{\psi}$ respectively. Then we have the following estimate for the difference of $\phi - \tilde{\phi}$.

Lemma 3.2. *For each $t \in [0,T]$, the following estimates hold.*

(i) $\|\nabla(\phi-\tilde\phi)(t,\cdot)\|_{L^2(\mathbb{R}^3)} \le \tilde C\Big(\|\psi-\tilde\psi\|_{H^1(\mathbb{R}^3)} + \|\psi_t-\tilde\psi_t\|_{L^2(\mathbb{R}^3)}\Big)$.

(ii) $\|(\phi-\tilde\phi)(t,\cdot)\|_{L^6(\mathbb{R}^3)} \le \tilde C\Big(\|\psi-\tilde\psi\|_{H^1(\mathbb{R}^3)} + \|\psi_t-\tilde\psi_t\|_{L^2(\mathbb{R}^3)}\Big)$.

(iii) $\|(\phi_t-\tilde\phi_t)(t,\cdot)\|_{L^2(\mathbb{R})} \le \tilde C\Big(\|\psi-\tilde\psi\|_{H^1(\mathbb{R}^3)} + \|\nabla(\mathbf{A}-\tilde{\mathbf{A}})\|_{L^2(\mathbb{R}^3)}\Big)$.

Here $\tilde C$ is a constant depending on $\|\psi\|_{H^1}$, $\|\psi_t\|_{L^2}$, and $\|\nabla\mathbf{A}\|_{L^2}$.

Proof. Now taking the difference and using (3.1), we have

$$-\Delta(\phi-\tilde\phi) + e^2(|\psi|^2\phi - |\tilde\psi|^2\tilde\phi) = e\,\mathrm{Im}(\psi\bar\psi_t - \tilde\psi\bar{\tilde\psi}_t). \tag{3.6}$$

We observe that

$$(|\psi|^2\phi - |\tilde\psi|^2\tilde\phi)(\phi-\tilde\phi) = |\psi\phi-\tilde\psi\tilde\phi|^2 + \phi\tilde\phi(\bar\psi-\bar{\tilde\psi})(\tilde\psi-\psi)$$

and

$$(\psi\bar\psi_t-\tilde\psi\bar{\tilde\psi}_t)(\phi-\tilde\phi) = (\bar\psi_t-\bar{\tilde\psi}_t)(\psi\phi-\tilde\psi\tilde\phi)+\phi(\psi-\tilde\psi)(\bar{\tilde\psi}_t-\bar\psi_t)+\bar\psi_t(\psi-\tilde\psi)(\phi-\tilde\phi).$$

Thus multiplying $\phi-\tilde\phi$ by (3.6) and integrating it over $\mathbb{R}^3$, one has

$$\begin{aligned}
&\int_{\mathbb{R}^3}|\nabla(\phi-\tilde\phi)|^2\,dx + e^2\int_{\mathbb{R}^3}|\psi\phi-\tilde\psi\tilde\phi|^2\,dx \\
&\le e^2\int_{\mathbb{R}^3}|\phi||\tilde\phi||\psi-\tilde\psi|^2\,dx + e\int_{\mathbb{R}^3}|\psi_t-\tilde\psi_t||\psi\phi-\tilde\psi\tilde\phi|\,dx \\
&\quad + e\int_{\mathbb{R}^3}|\phi||\psi-\tilde\psi||\psi_t-\tilde\psi_t|\,dx + e\int_{\mathbb{R}}|\psi_t||\psi-\tilde\psi||\phi-\tilde\phi|\,dx \\
&\le C\|\phi\|_{L^6}\|\tilde\phi\|_{L^6}\|\psi-\tilde\psi\|_{L^3}^2 + \frac{e^2}{2}\|\psi_t-\tilde\psi_t\|_{L^2}^2 + \frac{e^2}{2}\int_{\mathbb{R}^3}|\psi\phi-\tilde\psi\tilde\phi|^2\,dx \\
&\quad + C\|\phi\|_{L^6}\|\psi-\tilde\psi\|_{L^3}\|\psi_t-\tilde\psi_t\|_{L^2} + C\|\psi_t\|_{L^2}\|\psi-\tilde\psi\|_{L^3}\|\phi-\tilde\phi\|_{L^6}.
\end{aligned}$$

By Lemma 3.1, it follows that

$$\begin{aligned}
&\|\nabla(\phi-\tilde\phi)\|_{L^2}^2 \\
&\le C\Big(\|\psi_t\|_{L^2}\|\tilde\psi_t\|_{L^2}\|\psi-\tilde\psi\|_{H^1}^2 + \|\psi_t-\tilde\psi_t\|_{L^2}^2 \\
&\qquad + \|\psi_t\|_{L^2}\|\psi-\tilde\psi\|_{H^1}\|\psi_t-\tilde\psi_t\|_{L^2} + \|\psi_t\|_{L^2}^2\|\psi-\tilde\psi\|_{H^1}^2\Big) \\
&= \tilde C\Big(\|\psi-\tilde\psi\|_{H^1}^2 + \|\psi_t-\tilde\psi_t\|_{L^2}^2\Big).
\end{aligned}$$

This completes the proof of (i), and (ii) follows from (i). We can also prove (iii) in a similar way as in Lemma 3.1. □

4 Solvability of the Cauchy problem

In this section, we prove Theorem 1.1. The proof is divided into two steps. Firstly, we reduce system (1.1)-(1.3) to a symmetric hyperbolic system. Secondly, we adopt the energy method of [6] to obtain the unique existence of the solution.

4.1 Reduction to the hyperbolic system

In this subsection, we rewrite system (1.1)-(1.3) as a hyperbolic form. First, in order to guarantee that the Coulomb condition holds for $t \in (0,T)$, we introduce the projection operator $\mathcal{P}$ on divergence free vector fields. More precisely, we define

$$\mathcal{P} : \Big(L^2(\mathbb{R}^3)\Big)^3 \longrightarrow \Big(L^2(\mathbb{R}^3)\Big)^3$$

by $\mathcal{P} = (-\Delta)^{-1}\,\mathrm{rot}\ \mathrm{rot}$. Then, by direct computations, one can see that if $\mathrm{div}\mathbf{A} = 0$, it follows that $\mathcal{P}\mathbf{A} = \mathbf{A}$. Applying $\mathcal{P}$ to equation (1.2), we obtain the following equation:

$$\Box\mathbf{A} = \mathcal{P}\Big(e\,\mathrm{Im}(\bar{\psi}\nabla\psi) - e^2|\psi|^2\mathbf{A}\Big). \tag{4.1}$$

Then we can see that when the initial data $\mathbf{A}_{(0)}$ and $\mathbf{A}_{(1)}$ are divergence free, the Coulomb condition holds for all $t \in (0,T)$. Indeed putting $\mathbf{B} = \mathrm{div}\mathbf{A}$, one has from (4.1) that $\Box\,\mathbf{B} = \mathbf{0}$, $\mathbf{B}(0,x) = \mathbf{0}$ and $\mathbf{B}_t(0,x) = \mathbf{0}$. Thus it follows that $\mathbf{B} \equiv \mathbf{0}$ for all $t > 0$.

Now we put $\psi = \psi_1 + i\psi_2$ and $\mathbf{A} = (\psi_3, \psi_4, \psi_5)$ with $\psi_i : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$. Then system (1.1), (1.3), (4.1) is reduced to the following set of equations:

$$\begin{aligned}
\Box\psi_1 &= 2e\phi(\psi_2)_t + e\phi_t\psi_2 + e^2\phi^2\psi_1 + 2e\Big((\psi_2)_{x_1}\psi_3 + (\psi_2)_{x_2}\psi_4 + (\psi_2)_{x_3}\psi_5\Big)\\
&\quad - e^2(\psi_3^2 + \psi_4^2 + \psi_5^2)\psi_1 - m^2\psi_1 + W'(\psi).\\
\Box\psi_2 &= -2e\phi(\psi_1)_t - e\phi_t\psi_1 + e^2\phi^2\psi_2 - 2e\Big((\psi_1)_{x_1}\psi_3 + (\psi_1)_{x_2}\psi_4 + (\psi_1)_{x_3}\psi_5\Big)\\
&\quad - e^2(\psi_3^2 + \psi_4^2 + \psi_5^2)\psi_2 - m^2\psi_2 + W'(\psi).
\end{aligned}$$

$$\Box\psi_3 = e\sum_{k=1}^{3}\mathcal{P}_{1k}\Big\{\Big(\psi_1(\psi_2)_{x_k} - \psi_2(\psi_1)_{x_k}\Big) - e(\psi_1^2 + \psi_2^2)\psi_{2+k}\Big\}.$$

$$\Box\psi_4 = e\sum_{k=1}^{3}\mathcal{P}_{2k}\Big\{\Big(\psi_1(\psi_2)_{x_k} - \psi_2(\psi_1)_{x_k}\Big) - e(\psi_1^2 + \psi_2^2)\psi_{2+k}\Big\}.$$

$$\Box\psi_5 = e\sum_{k=1}^{3}\mathcal{P}_{3k}\Big\{\Big(\psi_1(\psi_2)_{x_k} - \psi_2(\psi_1)_{x_k}\Big) - e(\psi_1^2 + \psi_2^2)\psi_{2+k}\Big\}.$$

Here $\mathcal{P} = (\mathcal{P}_{jk})_{1\le j,k\le 3}$, $\mathcal{P}_{jk} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is a linear operator defined by

$$\mathcal{P}_{jk} = \delta_{jk} + \mathcal{R}_j\mathcal{R}_k \quad (j, k = 1, 2, 3)$$

and $\mathcal{R}_j : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is the Riesz transform given by

$$\mathcal{R}_j = \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}.$$

By the Fourier transform, it follows that $\mathcal{R}_j u = \mathcal{F}^{-1}\left(i\frac{\xi_j}{|\xi|}\mathcal{F}[u]\right)$ for $u \in L^2(\mathbb{R}^3)$ and $\xi \in \mathbb{R}^3$. This implies that R_j is bounded from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$ and hence so does $\mathcal{P}_{jk}$.

Next we introduce $(\psi_i)_t = \psi_{i+5}$ $(i = 1, \cdots 5)$. Then we can see that ψ_i $(i = 6, \cdots 10)$ satisfy the following equations.

$$\begin{aligned}
(\psi_6)_t = (\psi_1)_{tt} &= \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\big((\psi_1)_{x_j}\big) + 2e\phi\psi_7 + e\phi_t\psi_2 + e^2\phi^2\psi_1 - m^2\psi_1 \\
&\quad + W'(\psi) + 2e\sum_{j=1}^{3}\frac{\partial\psi_2}{\partial x_j}\psi_{2+j} - e^2(\psi_3^2 + \psi_4^2 + \psi_5^2)\psi_1. \\
(\psi_7)_t = (\psi_2)_{tt} &= \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\big((\psi_2)_{x_j}\big) - 2e\phi\psi_6 - e\phi_t\psi_1 + e^2\phi^2\psi_2 - m^2\psi_2 \\
&\quad + W'(\psi) - 2e\sum_{j=1}^{3}\frac{\partial\psi_1}{\partial x_j}\psi_{2+j} - e^2(\psi_3^2 + \psi_4^2 + \psi_5^2)\psi_2. \\
(\psi_8)_t = (\psi_3)_{tt} &= \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\big((\psi_3)_{x_j}\big) \\
&\quad + e\sum_{k=1}^{3}\mathcal{P}_{1k}\Big\{\Big(\psi_1(\psi_2)_{x_k} - \psi_2(\psi_1)_{x_k}\Big) - e(\psi_1^2 + \psi_2^2)\psi_{2+k}\Big\}. \\
(\psi_9)_t = (\psi_4)_{tt} &= \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\big((\psi_4)_{x_j}\big) \\
&\quad + e\sum_{k=1}^{3}\mathcal{P}_{2k}\Big\{\Big(\psi_1(\psi_2)_{x_k} - \psi_2(\psi_1)_{x_k}\Big) - e(\psi_1^2 + \psi_2^2)\psi_{2+k}\Big\}. \\
(\psi_{10})_t = (\psi_5)_{tt} &= \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\big((\psi_5)_{x_j}\big) \\
&\quad + e\sum_{k=1}^{3}\mathcal{P}_{3k}\Big\{\Big(\psi_1(\psi_2)_{x_k} - \psi_2(\psi_1)_{x_k}\Big) - e(\psi_1^2 + \psi_2^2)\psi_{2+k}\Big\}.
\end{aligned}$$

Finally we put $(\psi_1)_{x_1} = \psi_{11},\ (\psi_1)_{x_2} = \psi_{12},\ \cdots, (\psi_5)_{x_3} = \psi_{25}$. Then we can write the equations of $\psi_6,\ \cdots, \psi_{10}$ as follows:

$$\begin{aligned}
(\psi_6)_t = & \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\psi_{10+j}) + 2e\phi\psi_7 + e\phi_t\psi_2 + e^2\phi^2\psi_1 - m^2\psi_1 + W'(\psi) \\
& + 2e\sum_{j=1}^{3} \psi_{13+j}\psi_{2+j} - e^2(\psi_3^2 + \psi_4^2 + \psi_5^2)\psi_1. \\
(\psi_7)_t = & \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\psi_{13+j}) - 2e\phi\psi_6 - e\phi_t\psi_1 + e^2\phi^2\psi_2 - m^2\psi_2 + W'(\psi) \\
& - 2e\sum_{j=1}^{3} \psi_{10+j}\psi_{2+j} - e^2(\psi_3^2 + \psi_4^2 + \psi_5^2)\psi_2.
\end{aligned}$$

$$(\psi_8)_t = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\psi_{16+j}) + e\sum_{k=1}^{3} \mathcal{P}_{1k}\big\{(\psi_1\psi_{13+k} - \psi_2\psi_{10+k}) - e(\psi_1^2 + \psi_2^2)\psi_{2+k}\big\}.$$

$$(\psi_9)_t = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\psi_{19+j}) + e\sum_{k=1}^{3} \mathcal{P}_{2k}\big\{(\psi_1\psi_{13+k} - \psi_2\psi_{10+k}) - e(\psi_1^2 + \psi_2^2)\psi_{2+k}\big\}.$$

$$(\psi_{10})_t = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\psi_{22+j}) + e\sum_{k=1}^{3} \mathcal{P}_{3k}\big\{(\psi_1\psi_{13+k} - \psi_2\psi_{10+k}) - e(\psi_1^2 + \psi_2^2)\psi_{2+k}\big\}.$$

Pointing out that $(\psi_{11})_t = \frac{\partial}{\partial x_1}\psi_6,\ (\psi_{12})_t = \frac{\partial}{\partial x_2}\psi_6, \cdots$ and $(\psi_{25})_t = \frac{\partial}{\partial x_3}\psi_{10}$ and defining $\mathbf{U} = {}^t(\psi_1, \cdots, \psi_{25})$, we can see that system (1.1), (1.3), (4.1) can be written into the following symmetric hyperbolic form

$$\frac{\partial \mathbf{U}}{\partial t} = \sum_{j=1}^{3} A_j \frac{\partial \mathbf{U}}{\partial x_j} + \mathbf{F}(\mathbf{U}, \phi, \phi_t). \tag{4.2}$$

Here A_j $(j = 1, 2, 3)$ is a symmetric 25×25 matrix which is defined by

$$A_j = \begin{array}{rl}
 & \begin{array}{cccccccccc} 6 & 7 & 8 & 9 & 10 & 10+j & 13+j & 16+j & 19+j & 22+j \end{array} \\
\begin{array}{c} 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 10+j \\ 13+j \\ 16+j \\ 19+j \\ 22+j \end{array} & \begin{pmatrix}
 & & & & & 1 & & & & \\
 & & & & & & 1 & & & \\
 & & & & & & & 1 & & \\
 & & & & & & & & 1 & \\
 & & & & & & & & & 1 \\
1 & & & & & & & & & \\
 & 1 & & & & & & & & \\
 & & 1 & & & & & & & \\
 & & & 1 & & & & & & \\
 & & & & 1 & & & & &
\end{pmatrix}
\end{array}$$

and the nonlinear term $\mathbf{F}(\mathbf{U},\phi,\phi_t)={}^t\Big(f_1,\ f_2,\ \cdots,f_{25}\Big)$ is given by

$$\begin{aligned}
&f_i=\psi_{5+i}\quad (i=1,\cdots 5).\\
&f_6=2e\phi\psi_7+e\phi_t\psi_2+e^2\phi^2\psi_1-m^2\psi_1+W'(\psi)\\
&\qquad +2e\sum_{j=1}^{3}\psi_{13+j}\psi_{2+j}-e^2(\psi_3^2+\psi_4^2+\psi_5^2)\psi_1.\\
&f_7=-2e\phi\psi_6-e\phi_t\psi_1+e^2\phi^2\psi_2-m^2\psi_2+W'(\psi)\\
&\qquad -2e\sum_{j=1}^{3}\psi_{10+j}\psi_{2+j}-e^2(\psi_3^2+\psi_4^2+\psi_5^2)\psi_2.\\
&f_{7+j}=e\sum_{k=1}^{3}\mathcal{P}_{jk}\Big\{(\psi_1\psi_{13+k}-\psi_2\psi_{10+k})-e(\psi_1^2+\psi_2^2)\psi_{2+k}\Big\}\quad (j=1,2,3).\\
&f_i=0\quad (i=11,\cdots,25).
\end{aligned}\tag{4.3}$$

Finally by using the vector $\mathbf{U}$, the Equation (3.1) can be rewritten in the following form:

$$-\Delta\phi+e^2(\psi_1^2+\psi_2^2)\phi=e(\psi_1\psi_7-\psi_2\psi_6).\tag{4.4}$$

4.2 Unique existence of the solution for the Cauchy problem

In this subsection, we show that the hyperbolic System (4.2) has a unique solution in a suitable function space. To this aim, we argue as in [6]. Firstly we consider a *linearized version* of (4.2) and prove the existence of an unique solution by the standard energy method. Secondly, we study the corresponding solution map S and show that S is a contraction mapping on a suitable ball provided that T is sufficiently small. We will see below that this procedure gives the unique solution of (4.2).

To this end, for $m\in\mathbb{N}$ with $m\geq 2$, we suppose that

$$\psi_{(0)}\in H^{m+1}(\mathbb{R}^3,\mathbb{C}),\ \psi_{(1)}\in H^m(\mathbb{R}^3,\mathbb{C}),$$
$$\mathbf{A}_{(0)}\in H^{m+1}(\mathbb{R}^3,\mathbb{R}^3)\text{ and }\mathbf{A}_{(1)}\in H^m(R^3,\mathbb{R}^3)$$

so that $\psi_{i\,(0)}$ $(i=1,\cdots,25)$ satisfy

$$\psi_{1\,(0)},\ \psi_{2\,(0)},\cdots,\psi_{5\,(0)}\in H^{m+1}(\mathbb{R}^3),\quad \psi_{6\,(0)},\ \psi_{7\,(0)},\cdots,\psi_{25\,(0)}\in H^m(\mathbb{R}^3).$$

Here we put

$$\psi_{(0)}=\psi_{1\,(0)}+i\psi_{2\,(0)},\ \mathbf{A}_{(0)}=(\psi_{3\,(0)},\psi_{4\,(0)},\psi_{5\,(0)}),$$

$$\psi_{(1)} = \psi_{6\,(0)} + i\psi_{7\,(0)},\ \mathbf{A}_{(1)} = (\psi_{8\,(0)}, \psi_{9\,(0)}, \psi_{10\,(0)}),$$
$$(\psi_{1\,(0)})_{x_1} = \psi_{11\,(0)},\ (\psi_{1\,(0)})_{x_2} = \psi_{12\,(0)} \cdots, (\psi_{5\,(0)})_{x_3} = \psi_{25\,(0)}.$$

We denote $X = \left(H^{m+1}(\mathbb{R}^3)\right)^5 \times \left(H^m(\mathbb{R}^3)\right)^{20}$ and $\mathbf{U}_{(0)} = {}^t(\psi_{1\,(0)}, \cdots, \psi_{25\,(0)})$. We take $\mathbf{U}_{(0)} \in X$ arbitrarily and put $R = 2\sum_{i=1}^{25} \|\psi_{i\,(0)}\|_{H^m}$. Finally for $T > 0$, we set

$$B_R := \Big\{\mathbf{U}(t,x) = {}^t\Big(\psi_i(t,x)\Big)_{i=1}^{25} \in C\big((0,T), X\big)\ ;\ \sup_{t\in(0,T)} \|\psi_i(t,\cdot)\|_{H^m} \leq R\Big\}.$$

We are going to find a solution of (4.2) by the following procedure. First for given $\mathbf{U} \in B_R$, we obtain ϕ by solving the elliptic equation (4.4). Then, we define the space

$$Y_T = C\big((0,T), X\big)$$

and the mapping

$$\begin{aligned} S : Y_T &\longrightarrow Y_T \\ \mathbf{U} &\longmapsto \mathbf{V}, \end{aligned}$$

where $\mathbf{V}$ is the solution to

$$\begin{cases} \dfrac{\partial \mathbf{V}}{\partial t} = \displaystyle\sum_{j=1}^{3} A_j \frac{\partial \mathbf{V}}{\partial x_j} + \mathbf{F}(\mathbf{U}, \phi, \phi_t), \\ \mathbf{V}(0,x) = \mathbf{U}_{(0)}(x). \end{cases} \tag{4.5}$$

Note that the existence of a solution $\mathbf{V}$ to (4.5) is straightforward (see Lemma 4.4 for more details). The idea is then to apply a fixed point theorem to S by getting estimates on $\mathbf{V}$ through the use of energy estimates.

To this end, we first apply Lemma 3.1 to obtain bounds on ϕ, which allows to perform estimates on the nonlinear terms $\mathbf{F}$ as it is proved in the next lemma.

Lemma 4.1. *Let $\mathbf{U} \in Y_T$ be given and ϕ be the corresponding solution of the elliptic equation (4.4). If $\mathbf{U} \in B_R$, then the nonlinear term $\mathbf{F} = {}^t(f_i)$ $(i = 1, \cdots, 25)$ defined in (4.3) satisfies*

$$\|f_i(\mathbf{U}, \phi, \phi_t)\|_{C((0,T),H^m)} \leq C(R),$$

where $C(R)$ is a constant depending on R.

Proof. First we observe by the definition of $\mathbf{F} = {}^t(f_i)$ that

$$\|f_i\|_{H^m} = \|\psi_{5+i}\|_{H^m} \leq R \quad (i = 1, \cdots, 5) \quad \text{and} \quad \|f_i\|_{H^m} = 0 \quad (i = 11, \cdots, 25).$$

Next we have

$$\|f_{7+j}\|_{H^m} \leq |e| \sum_{k=1}^{3} \Big(\|\mathcal{P}_{jk}\psi_1\psi_{13+k}\|_{H^m} + \|\mathcal{P}_{jk}\psi_2\psi_{10+k}\|_{H^m} \\ + \|e\mathcal{P}_{jk}\psi_1^2\psi_{2+k}\|_{H^m} + \|e\mathcal{P}_{jk}\psi_2^2\psi_{2+k}\|_{H^m} \Big) \quad (j = 1, 2, 3).$$

Since $\mathcal{P}_{jk}$ is a bounded operator from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$ and $H^m(\mathbb{R}^3)$ is a Banach algebra for $m \geq 2$, we get

$$\|\mathcal{P}_{jk}\psi_1\psi_{13+k}\|_{H^m} \leq C\|\psi_1\|_{H^m}\|\psi_{13+k}\|_{H^m} \leq CR^2,$$
$$\|\mathcal{P}_{jk}\psi_1^2\psi_{2+k}\|_{H^m} \leq C\|\psi_1\|_{H^m}^2\|\psi_{2+k}\|_{H^m} \leq CR^3.$$

Thus one has

$$\|f_{7+j}\|_{H^m} \leq C(R^2 + R^3) \quad (j = 1, 2, 3).$$

Next by the definition of f_6, it follows that

$$\|f_6\|_{H^m} \leq C\Big(\|\phi\psi_7\|_{H^m} + \|\phi_t\psi_2\|_{H^m} + \|\phi^2\psi_1\|_{H^m} + \|\psi_1\|_{H^m} \\ + \sum_{k=1}^{3} \big(\|\psi_{13+k}\psi_{2+k}\|_{H^m} + \|\psi_{2+k}^2\psi_1\|_{H^m} \big) + \|W'(\psi)\|_{H^m} \Big).$$

By Lemma 3.1, we have

$$\begin{aligned}
\|\phi\psi_7\|_{H^m} &= \sum_{s \leq m} \|D^s(\phi\psi_7)\|_{L^2} \leq C \sum_{\alpha+\beta \leq m} \|D^\alpha\phi D^\beta\psi_7\|_{L^2} \\
&= C \sum_{\beta \leq m} \|\phi D^\beta\psi_7\|_{L^2} + C \sum_{\alpha+\beta \leq m, \alpha \neq 0} \|D^\alpha\phi D^\beta\psi_7\|_{L^2} \\
&\leq C(\|\phi\|_{L^\infty}\|\psi_7\|_{H^m} + \|\nabla\phi\|_{H^m}\|\psi_7\|_{H^m}) \\
&\leq C\|\psi_t\|_{H^m}\|\psi_7\|_{H^m} \leq CR^2.
\end{aligned}$$

$$\begin{aligned}
\|\phi^2\psi_1\|_{H^m} &\leq C \sum_{\beta \leq m} \|\phi^2 D^\beta\psi_2\|_{L^2} + C \sum_{\alpha+\beta \leq m, \alpha \neq 0} \|\phi D^\alpha\phi D^\beta\psi_2\|_{L^2} \\
&\leq C\|\phi\|_{L^\infty}^2\|\psi_2\|_{H^m} + C\|\phi\|_{L^\infty}\|\nabla\phi\|_{H^m}\|\psi_2\|_{H^m} \\
&\leq C\|\psi_t\|_{H^m}^2\|\psi_2\|_{H^m} \leq CR^3,
\end{aligned}$$

$$\|\phi_t\psi_2\|_{H^m} \le \|\phi_t\|_{H^m}\|\psi_2\|_{H^m} \le C\|\psi\|_{H^m}\|\psi_2\|_{H^m} \le CR^2.$$

Moreover from (A), we also have

$$\|W'(\psi)\|_{H^m} \le C\|\psi\|_{H^m} \le CR.$$

(See [1].) Thus we obtain $\|f_6\|_{L^2} \le C(R)$. In a similar way, one can show that $\|f_7\|_{L^2} \le C(R)$. This completes the proof. □

Lemma 4.2. *Let $\mathbf{U}, \tilde{\mathbf{U}} \in Y_T$ be given and ϕ, $\tilde{\phi}$ be the corresponding solution of the elliptic equation (4.4) respectively. If $\mathbf{U} \in B_R$ and $\tilde{\mathbf{U}} \in B_R$, then it follows that*

$$\|\mathbf{F}(\mathbf{U}, \phi, \phi_t) - \mathbf{F}(\tilde{\mathbf{U}}, \tilde{\phi}, \tilde{\phi}_t)\|_{L^\infty((0,T),L^2)} \le C(R)\|\mathbf{U} - \tilde{\mathbf{U}}\|_{L^\infty((0,T),L^2)}.$$

Proof. By Lemma 3.2 and from the continuous embedding $H^m(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ for $m \ge 2$, we can see that the claim follows. □

Now we are ready to prove the existence of an unique solution to the Cauchy problem:

$$\begin{cases} \dfrac{\partial \mathbf{U}}{\partial t} = \displaystyle\sum_{j=1}^{3} A_j \frac{\partial \mathbf{U}}{\partial x_j} + \mathbf{F}(\mathbf{U}, \phi, \phi_t), \\ \mathbf{U}(0,x) = \mathbf{U}_{(0)}(x) \end{cases} \tag{4.6}$$

coupled with the elliptic equation:

$$-\Delta\phi + e^2(\psi_1^2 + \psi_2^2)\phi = e(\psi_1\psi_7 - \psi_2\psi_6). \tag{4.7}$$

The proof consists of four lemmas.

Lemma 4.3. *Let $\mathbf{U} \in B_R$ be given. Then there exists a unique solution $\phi = \phi(\mathbf{U}) \in C\big((0,T), D^{1,2}(\mathbb{R}^3)\big)$ of (4.7). Moreover ϕ satisfies the estimates in Lemmas 3.1, 3.2.*

Proof. Suppose that $\psi_1, \psi_2 \in H^1(\mathbb{R}^3)$ and $\psi_6, \psi_7 \in L^2(\mathbb{R}^3)$. We define a bilinear form $A : D^{1,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ by

$$A(u,v) := \int_{\mathbb{R}^3} \nabla u \cdot \nabla v + e^2(\psi_1^2 + \psi_2^2)uv \, dx.$$

Then by the Hölder and the Sobolev inequalities, one has

$$\begin{aligned} |A(u,v)| &\le \|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + |e|^2\left(\|\psi_1\|_{L^3}^2 + \|\psi_2\|_{L^3}^2\right)\|u\|_{L^6}\|v\|_{L^6} \\ &\le C\|u\|_{D^{1,2}}\|v\|_{D^{1,2}}, \\ \|u\|_{D^{1,2}}^2 &\le A(u,u). \end{aligned}$$

Moreover putting $g = e(\psi_1\psi_7 - \psi_2\psi_6)$, we also have

$$\begin{aligned}\|g\|_{L^{\frac{6}{5}}} &\le C\left(\|\psi_1\|_{L^3}\|\psi_7\|_{L^2} + \|\psi_2\|_{L^3}\|\psi_6\|_{L^2}\right)\\ &\le C\left(\|\psi_1\|_{H^1}\|\psi_7\|_{L^2} + \|\psi_2\|_{H^1}\|\psi_7\|_{L^2}\right).\end{aligned}$$

This implies that $g \in L^{\frac{6}{5}}(\mathbb{R}^3) \hookrightarrow \left(D^{1,2}(\mathbb{R}^3)\right)^*$. Thus by the Lax-Milgram theorem, there exists a unique solution of $A(\phi, v) = \langle g, v\rangle$ for all $v \in D^{1,2}$. $\square$

Next we consider the *linearized version* of (4.6):

$$\begin{cases} \dfrac{\partial \mathbf{V}}{\partial t} = \displaystyle\sum_{j=1}^{3} A_j \frac{\partial \mathbf{V}}{\partial x_j} + \mathbf{F}(\mathbf{U}, \phi, \phi_t), \\ \mathbf{V}(0, x) = \mathbf{U}_{(0)}(x). \end{cases} \tag{4.8}$$

Lemma 4.4. *For given* $\mathbf{U} \in B_R$, *the Cauchy problem* (4.8) *has a unique solution* $\mathbf{V} \in C\big((0,T), X\big)$.

Proof. The proof follows by the standard existence theory for the hyperbolic system. (See [1].) $\square$

Now by Lemmas 4.3 and 4.4, one can see that the mapping S is well defined.

Lemma 4.5. *For sufficiently small* $T^* > 0$, *one has* $S(B_R) \subset B_R$. *Furthermore, there exists* $\kappa \in (0,1)$ *such that for any* $\mathbf{U}, \tilde{\mathbf{U}} \in B_R$, *one can write*

$$\|S(\mathbf{U}) - S(\tilde{\mathbf{U}})\|_{L^\infty((0,T^*),L^2)} \le \kappa \|\mathbf{U} - \tilde{\mathbf{U}}\|_{L^\infty((0,T^*),L^2)}. \tag{4.9}$$

Proof. We apply the standard energy estimate method. To this aim, we apply D^s on (4.8) and take the L^2-inner product with $D^s\mathbf{V}$. Then one has

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} \frac{1}{2}|D^s\mathbf{V}|^2\,dx = \sum_{j=1}^{3}\int_{\mathbb{R}^3} D^s\left(A_j\frac{\partial \mathbf{V}}{\partial x_j}\right)\cdot D^s\mathbf{V}\,dx + \int_{\mathbb{R}^3} D^s\mathbf{F}\cdot D^s\mathbf{V}\,dx.$$

Since A_j is symmetric and consists of constant elements, it follows that

$$\begin{aligned}\int_{\mathbb{R}^3} D^s\left(A_j\frac{\partial \mathbf{V}}{\partial x_j}\right)\cdot D^s\mathbf{V}\,dx &= \int_{\mathbb{R}^3} A_j\frac{\partial}{\partial x_j}D^s\mathbf{V}\cdot D^s\mathbf{V}\,dx\\ &= \int_{\mathbb{R}^3}\frac{\partial}{\partial x_j}D^s\mathbf{V}\cdot(A_jD^s\mathbf{V})\,dx\\ &- \int_{\mathbb{R}^3} D^s\mathbf{V}\cdot\frac{\partial}{\partial x_j}(A_jD^s\mathbf{V})\,dx\\ &= -\int_{\mathbb{R}^3} D^s\mathbf{V}\cdot D^s\left(A_j\frac{\partial \mathbf{V}}{\partial x_j}\right)\,dx.\end{aligned}$$

Thus we obtain

$$\frac{\partial}{\partial t}\Big(\frac{1}{2}\|D^s\mathbf{V}(t,\cdot)\|_{L^2}^2\Big) \le \|D^s\mathbf{F}\|_{L^2}\|D^s\mathbf{V}\|_{L^2} \le \frac{1}{2}\|D^s\mathbf{V}(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\|D^s\mathbf{F}\|_{L^2}^2.$$

Now we put $y(t) = \frac{1}{2}\|D^s\mathbf{V}(t,\cdot)\|_{L^2}^2$. By Lemma 4.1, one has

$$y'(t) \le y(t) + C(R).$$

Then by the Gronwall inequality, we get

$$y(t) \le y(0)e^T + C(R)(e^T - 1) \text{ for all } t \in (0,T).$$

Choosing $T^* > 0$ small enough, we obtain

$$\begin{aligned}\sup_{t\in(0,T^*)} \|D^s\mathbf{V}(t,\cdot)\|_{L^2} &\le \Big(\|D^s\mathbf{U}_{(0)}\|_{L^2}^2 e^{T^*} + 2C(R)(e^{T^*}-1)\Big)^{\frac{1}{2}} \\ &\le \Big(\frac{R^2}{4}e^{T^*} + 2C(R)(e^{T^*}-1)\Big)^{\frac{1}{2}} \le R.\end{aligned}$$

This implies that $S(B_R) \subset B_R$.

Next we consider the difference $\mathbf{V} - \tilde{\mathbf{V}}$ to prove (4.9). Arguing as above, one has

$$\begin{aligned}&\frac{\partial}{\partial t}\Big(\frac{1}{2}\|(\mathbf{V}-\tilde{\mathbf{V}})(t,\cdot)\|_{L^2}^2\Big) \\ &\quad\le \frac{1}{2}\|\mathbf{V}-\tilde{\mathbf{V}}\|_{L^2}^2 + \frac{1}{2}\|\mathbf{F}(\mathbf{U},\phi,\phi_t) - \mathbf{F}(\tilde{\mathbf{U}},\tilde{\phi},\tilde{\phi}_t)\|_{L^2}^2.\end{aligned}$$

Again we put $z(t) = \frac{1}{2}\|(\mathbf{V}-\tilde{\mathbf{V}})(t,\cdot)\|_{L^2}^2$. Then by Lemma 4.2 and the Gronwall inequality, we obtain

$$z(t) \le z(0)e^T + C\|\mathbf{U}-\tilde{\mathbf{U}}\|_{L^\infty((0,T),L^2)}^2(e^T - 1).$$

Notice that $z(0) = \frac{1}{2}\|\mathbf{V}(0,\cdot) - \tilde{\mathbf{V}}(0,\cdot)\|_{L^2}^2 = \frac{1}{2}\|\mathbf{U}_{(0)} - \mathbf{U}_{(0)}\|_{L^2}^2 = 0$. Thus taking T^* smaller so that $\sqrt{2C(R)(e^{T^*}-1)} =: \kappa < 1$, we have

$$\|(\mathbf{V}-\tilde{\mathbf{V}})(t,\cdot)\|_{L^\infty((0,T^*),L^2)} \le \kappa\|(\mathbf{U}-\tilde{\mathbf{U}})(t,\cdot)\|_{L^\infty((0,T^*),L^2)}.$$

This completes the proof. $\square$

Now since S is a contraction mapping, there exists a unique $\mathbf{U} \in B_R$ such that $S(\mathbf{U}) = \mathbf{U}$. This implies that $\mathbf{U}$ is a solution of (4.6).

Lemma 4.6. $\mathbf{U}$ *is the unique solution of* (4.6).

Proof. The proof is a straightforward consequence of Estimate (4.9) and we omit the details. □

Proof of Theorem 1.1. Let ϕ and $\mathbf{U} = (U_j)_{1\leq j\leq 25}$ be the unique solution to Equations (4.4) and (4.6). We recall that $\phi \in C\big((0,T), D^{1,2}(\mathbb{R}^3)\big)$, $U_j \in C\big((0,T), H^{m+1}(\mathbb{R}^3)\big)$ for all $j = 1, \cdots, 5$ and $U_j \in C\big((0,T), H^m(\mathbb{R}^3)\big)$ for $j = 6, \cdots, 25$. Define $\tilde{\mathbf{U}} = (\tilde{U}_j)_{1\leq j\leq 25}$ by

$$\tilde{U}_j = U_j,\ \tilde{U}_{j+5} = (U_j)_t \ \text{ for } j = 1, \cdots, 5,$$

$$\big(\tilde{U}_{11}, \tilde{U}_{12}, \tilde{U}_{13}\big) = \nabla U_1,\ \big(\tilde{U}_{14}, \tilde{U}_{15}, \tilde{U}_{16}\big) = \nabla U_2,$$
$$\big(\tilde{U}_{17}, \tilde{U}_{18}, \tilde{U}_{19}\big) = \nabla U_3,\ \big(\tilde{U}_{20}, \tilde{U}_{21}, \tilde{U}_{22}\big) = \nabla U_4,$$
$$\big(\tilde{U}_{23}, \tilde{U}_{24}, \tilde{U}_{25}\big) = \nabla U_5.$$

Then it is obvious that $\tilde{U}$ satisfies the Cauchy Problem (4.4) and (4.8). By Estimate (4.9), one can write

$$\|(\mathbf{U} - \tilde{\mathbf{U}})(t,\cdot)\|_{L^\infty((0,T^*),L^2)} \leq \kappa \|(\mathbf{U} - \mathbf{U})(t,\cdot)\|_{L^\infty((0,T^*),L^2)} = 0.$$

which provides $\tilde{\mathbf{U}} = \mathbf{U}$. As a consequence, the functions

$$\psi = U_1 + iU_2,\ \mathbf{A} = (U_3, U_4, U_5) \text{ and } \phi$$

are the unique solutions to System (1.1)-(1.3). Furthermore, noticing that

$$\psi_t = U_6 + iU_7,\ \nabla\psi = \nabla U_1 + i\nabla U_2,$$

$$\mathbf{A}_t = (U_8, U_9, U_{10}) \text{ and } \nabla\mathbf{A} = (\nabla U_3, \nabla U_4, \nabla U_5),$$

one can prove that

$$\psi \in C((0,T^*), H^{m+1}) \cap C^1((0,T^*), H^m),$$

$$\mathbf{A} \in C((0,T^*), H^{m+1}) \cap C^1((0,T^*), H^m),$$

$$\phi \in C\big((0,T), D^{1,2}(\mathbb{R}^3)\big),\ \nabla\phi \in C((0,T^*), H^m),\ \phi_t \in C((0,T^*), H^m),$$

which ends the proof of Theorem 1.1. □

Remark 4.7. *Note that it seems for the moment out of reach to solve the Cauchy Problem* (1.1)-(1.3) *in the enery space defined by*

$$\psi \in C\big((0,T), H^1(\mathbb{R}^3,\mathbb{C})\big) \cap C^1\big((0,T), L^2(\mathbb{R}^3,\mathbb{C})\big), \phi \in C\big((0,T), D^{1,2}(\mathbb{R}^3)\big)$$
$$\mathbf{A} \in C\big((0,T), D^{1,2}(\mathbb{R}^3,\mathbb{R}^3)\big) \cap C^1\big((0,T), L^2(\mathbb{R}^3,\mathbb{R}^3)\big). \quad (4.10)$$

However, if we assume that W satisfies the following condition

(a3) There exists $\mu \in (0, m^2)$ such that

$$\frac{m^2}{2}s^2 - W(s) \geq \frac{\mu}{2}s^2 \text{ for all } s \geq 0,$$

then one can prove :

Proposition 4.8. *Assume (a1)-(a3). Then there exists $C > 0$ independent of $t > 0$ such that*

$$\sup_{t\in(0,T)} \Big(\|\psi(t,\cdot)\|_{H^1} + \|\psi_t(t,\cdot)\|_{L^2} + \|\nabla \mathbf{A}(t,\cdot)\|_{L^2} + \|\mathbf{A}_t(t,\cdot)\|_{L^2} \Big) \leq C.$$

Proof. First by the energy conservation law (2.6) and from (a3), we have

$$\|\nabla \mathbf{A}\|_{L^2} \leq 2\mathcal{E}(0), \quad \|\psi\|_{L^2} \leq \frac{2}{\mu}\mathcal{E}(0),$$
$$\|\nabla\psi - ie\mathbf{A}\psi\|_{L^2} + \|\psi_t + ie\phi\psi\|_{L^2} + \|\mathbf{A}_t + \nabla\phi\|_{L^2} \leq 2\mathcal{E}(0).$$

Now by the interpolation inequality, it follows that $\|\psi\|_{L^3} \leq \|\psi\|_{L^2}^{\frac{1}{2}} \|\psi\|_{L^6}^{\frac{1}{2}}$. Then by the Sobolev and the triangle inequalities, we get

$$\begin{aligned}
\|\psi\|_{L^3} &\leq C\|\nabla\psi\|_{L^2}^{\frac{1}{2}} \leq C\Big(\|\nabla\psi - ie\mathbf{A}\psi\|_{L^2} + \|e\mathbf{A}\psi\|_{L^2} \Big)^{\frac{1}{2}} \\
&\leq C\left(1 + \|\mathbf{A}\|_{L^6}\|\psi\|_{L^3}\right)^{\frac{1}{2}} \leq C\left(1 + \|\nabla\mathbf{A}\|_{L^2}\|\psi\|_{L^3}\right)^{\frac{1}{2}} \\
&\leq C\left(1 + \|\psi\|_{L^3}\right)^{\frac{1}{2}} \leq \frac{C^2+1}{2} + \frac{1}{2}\|\psi\|_{L^3}.
\end{aligned}$$

This implies that $\|\psi\|_{L^3} \leq C$ and hence $\|\nabla\psi\|_{L^2} \leq C$. Next we observe that (1.3) can be written as

$$-\Delta\phi = e\,\mathrm{Im}\left(\psi(\overline{\psi_t + ie\phi\psi})\right).$$

Multiplying this equation by ϕ and integrating the resulting equation over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
\|\nabla\phi\|_{L^2}^2 &\leq |e| \int_{\mathbb{R}^3} |\psi||\psi_t + ie\phi\psi||\phi|\,dx \\
&\leq |e|\|\psi\|_{L^3}\|\psi_t + ie\phi\psi\|_{L^2}\|\phi\|_{L^6} \\
&\leq C\|\nabla\phi\|_{L^2}.
\end{aligned}$$

This implies that $\|\nabla\phi\|_{L^2} \leq C$. Finally by the triangle inequality, one has

$$\|\mathbf{A}_t\|_{L^2} \leq \|\mathbf{A}_t + \nabla\phi\|_{L^2} + \|\nabla\phi\|_{L^2} \leq C.$$

This completes the proof. □

Owning Proposition 4.8 and the conservation laws (2.6)*, it is clear that every local solutions to* (1.1)-(1.3) *exists globally in time. To conclude, one can easily see that* $W(s) = \frac{1}{3}s^3 - \frac{\lambda}{4}s^4$ *satisfies (a3) for large* λ.

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