Approximation of a quasilinear Schrödinger equation by a Klein-Gordon equation in space dimension 2

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Abstract. In this article, we study the nonlinear plasma wave equation

$$-\varepsilon^2\frac{\partial^2 u_\varepsilon}{\partial t^2}+2i\frac{\partial u_\varepsilon}{\partial t}+\Delta u_\varepsilon=\left(\frac{1}{\sqrt{1+|u_\varepsilon|^2}}-1\right)u_\varepsilon+\frac{\Delta(\sqrt{1+|u_\varepsilon|^2})}{\sqrt{1+|u_\varepsilon|^2}}u_\varepsilon$$

with initial data $u_\varepsilon(\cdot,0)=u_0^\varepsilon(\cdot)\in H^8(\mathbb{R}^2)$, $\partial_t u_\varepsilon(\cdot,0)=u_1^\varepsilon(\cdot)\in H^7(\mathbb{R}^2)$. We show that the Cauchy problem is locally well-posed on an interval $[0,T]$ where the time T is independent of ε if u_1^ε is small enough. Then, we demonstrate the strong convergence of u_ε towards the solution u of a nonlinear relativistic Schrödinger equation as ε goes to 0.

1 Introduction

The nonlinear Schrödinger equation stated in $\mathbb{R}^3$

$$2i\frac{\partial u}{\partial t}+\Delta_\perp u-\frac{u}{\sqrt{1+|u|^2}}\Delta_\perp(\sqrt{1+|u|^2})+(1-\frac{1}{\sqrt{1+|u|^2}})u=0. \tag{1}$$

is known to describe properly the self-channeling of a high power ultra-short laser pulse in matter (see for example [3], [4], [5], [6], [13] and [7] for more references). Here, we denote $\Delta_\perp=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ and $u=u(x,y,z,t)$. The aim of this paper is to explain one of the approximations which lead to this equation. First from Maxwell system describing the propagation of high power ultra-short laser in gaseous medium we can derive the following wave equation (see [7])

$$\Box A=k_p^2\left(\frac{1+k_p^{-2}\Delta\gamma}{\gamma}\right)A, \tag{2}$$

where A is the electromagnetic vector field, $\gamma=\sqrt{1+|A|^2}$ is the Lorentz factor and $\Box=\Delta-\frac{\partial^2}{\partial t^2}$. Then, according to physical observations (see [4]), equation (2) can be simplified if we use the approximation in which the complex amplitude A is slowly varying over distances on the order of a wavelength in the direction of propagation z and over times on the order of the period of the high-frequency field oscillations. It is then natural to define

$$A=A_\varepsilon(x,y,z,t)=u_\varepsilon(x,y,\varepsilon z,\varepsilon t)e^{i(kz-\omega t)}$$

where

$$\varepsilon=\frac{\omega_{p,0}}{\omega},$$

depends on the parameters of the plasma ($\omega_{p,0}$ denotes the unperturbed plasma frequency).

Introducing A in (2) leads, after a rescaling in space, to equation

$$\Delta_\perp u_\varepsilon + \varepsilon^2 \frac{\partial^2 u_\varepsilon}{\partial z^2} - \varepsilon^2 \frac{\partial^2 u_\varepsilon}{\partial t^2} + 2i\sqrt{1-\varepsilon^2}\frac{\partial u_\varepsilon}{\partial z} + 2i\frac{\partial u_\varepsilon}{\partial t} = \left(\frac{1}{\gamma_\varepsilon} - 1\right) u_\varepsilon + \frac{(\Delta_\perp + \varepsilon^2\partial_z^2)\gamma_\varepsilon}{\gamma_\varepsilon} u_\varepsilon \tag{3}$$

where $\gamma_\varepsilon = \sqrt{1+|u_\varepsilon|^2}$. In the approximation of underdense plasmas, one can assume that $\varepsilon \ll 1$.

Then, taking formally $\varepsilon = 0$ in (3) and performing a change of variables give the nonlinear Schrödinger equation (1). It is then natural to think that the solutions u_ε of (3) could converge towards the solutions u of equation (1).

This kind of questions constitute an active domain of research in Mathematics. For example, in [2], T. Colin and L. Bergé investigated the so-called Langmuir wave envelope approximation which consists in taking the limit $\varepsilon \longrightarrow 0$ in the nonlinear plasma wave equation

$$\varepsilon^2 \frac{\partial^2 E_\varepsilon}{\partial t^2} - 2i\frac{\partial E_\varepsilon}{\partial t} - \Delta E_\varepsilon = f(|E_\varepsilon|^2)E_\varepsilon.$$

Under specific assumptions on f, the authors show the convergence of E_ε towards the solution E of the corresponding Schrödinger equation. In [1], the authors prove the same kind of results for a set of equations describing the inertial regime of the strong Langmuir turbulence, namely

$$\begin{aligned} &\varepsilon^2 \frac{\partial^2 E_\varepsilon}{\partial t^2} - 2i\frac{\partial E_\varepsilon}{\partial t} - \Delta E_\varepsilon = -n_\varepsilon E_\varepsilon \\ &\frac{1}{c^2}\frac{\partial^2 n_\varepsilon}{\partial t^2} - \Delta n_\varepsilon = \Delta |E_\varepsilon|^2. \end{aligned}$$

In [14], the authors show that the solutions of the nonlinear Klein-Gordon equation can be described by using a system of two coupled nonlinear Schrödinger equations as c tends to infinity.

In [9] , T. Gallay and G. Schneider obtain the KP equation

$$2\frac{\partial^2 A}{\partial x \partial t} + \frac{\partial^2 A^2}{\partial x^2} + \frac{\partial^4 A}{\partial x^4} + \frac{\partial^2 A}{\partial y^2} = 0$$

as an approximation for the Boussinesq equation

$$\frac{\partial^2 \phi}{\partial t^2} = \Delta\phi + \Delta(\phi^2) + \Delta\left(\frac{\partial^2 \phi}{\partial t^2}\right).$$

In order to justify rigorously the derivation of (1) from (3), we have to find a sequence u_ε solution of (3) which satisfies the following :

- u_ε exists on an interval $[0,T]$ where T is independent of ε
- u_ε is bounded in $H^s(\mathbb{R}^3)$ with a bound independent of ε
- the initial values do not tend to zero with ε.

Unfortunately, this program is too ambitious for the moment. That is why we will not study here equation (3). Indeed, the presence of ε^2 in front of ∂_z^2 gives rise to many problems. For example, in order to derive an energy estimate on equation (3), we have to consider the norm $||(1-\Delta_\varepsilon)^{\frac{s}{2}}||_{L^2(\mathbb{R}^3)}$ where $\Delta_\varepsilon = \partial_x^2 + \partial_y^2 + \varepsilon^2\partial_z^2$, which prevents from injecting the functions u_ε in $L^\infty(\mathbb{R}^3)$ with a constant independent of ε.

Then, we deal in this article with an intermediate model which is a modified equation in dimension 2 where all the terms depending on z have been dropped

$$-\varepsilon^2 \frac{\partial^2 u_\varepsilon}{\partial t^2} + 2i\frac{\partial u_\varepsilon}{\partial t} + \Delta u_\varepsilon = \left(\frac{1}{\gamma_\varepsilon} - 1\right) u_\varepsilon + \frac{\Delta \gamma_\varepsilon}{\gamma_\varepsilon} u_\varepsilon. \tag{4}$$

Here $\gamma_\varepsilon = \sqrt{1 + |u_\varepsilon|^2}$. Formally, taking $\varepsilon = 0$ in (4) gives the relativistic Schrödinger equation (1).

We first solve the Cauchy problem (4) with initial condition

$$u_\varepsilon(x,y,0) = u_0^\varepsilon(x,y), \;\; \partial_t u_\varepsilon(x,y,0) = u_1^\varepsilon(x,y), \tag{5}$$

keeping in view the three assertions of our program stated above. A first approach could be to treat (4) as a nonlinear Klein-Gordon equation. Indeed, if we denote

$$v_\varepsilon(x,y,t) = u_\varepsilon(x,y,t) e^{i\frac{1}{\varepsilon^2}t}$$

an easy calculation shows that v_ε is a solution of

$$-\varepsilon^2 \frac{\partial^2 v_\varepsilon}{\partial t^2} + \Delta v_\varepsilon - \frac{1}{\varepsilon^2} v_\varepsilon = \left(\frac{1}{\gamma_\varepsilon} - 1\right) v_\varepsilon + \frac{\Delta \gamma_\varepsilon}{\gamma_\varepsilon} v_\varepsilon \tag{6}$$

with initial condition

$$\begin{aligned} v_\varepsilon(x,y,0) &= u_0^\varepsilon(x,y) \\ \partial_t v_\varepsilon(x,y,0) &= u_1^\varepsilon(x,y,z) + i\frac{1}{\varepsilon^2} u_0^\varepsilon(x,y). \end{aligned}$$

Equation (6) satisfies the hypothesis of Klainerman and Ponce [12]. We can deduce that if the initial data are small enough in an adequate space, the Cauchy problem (6) admits a unique global solution $v_\varepsilon \in C([0,+\infty[; H^s(\mathbb{R}^2)) \cap C^1([0,+\infty[; H^{s-2}(\mathbb{R}^2))$ where s is an integer sufficiently large. Back to u_ε, we can show that (4) admits a unique solution $u_\varepsilon \in C([0,+\infty[; H^s(\mathbb{R}^2)) \cap C^1([0,+\infty[; H^{s-2}(\mathbb{R}^2))$ if the initial data is small enough. The problem is that, in doing so, it is easy to see that the presence of the term $\frac{1}{\varepsilon^2} v_\varepsilon$ imposes to v_ε and so to u_ε to tend to 0 in the space $L^\infty(0,T; H^s(\mathbb{R}^2))$ as ε goes to 0 which is in contradiction with our program. Thus, we have to solve the Cauchy problem (4) with initial condition (5) in an other way.

A second approach could be to apply the usual energy estimates (see [12]) to equation (4). Indeed, it is possible to prove the two following assertions

- The Cauchy problem (4) is locally well posed in $L^\infty(0, T_\varepsilon; H^s(\mathbb{R}^3))$ for any initial data u_0^ε, u_1^ε with $T_\varepsilon = \frac{\varepsilon^2}{(||u_0^\varepsilon||_{H^s(\mathbb{R}^3)} + ||u_1^\varepsilon||_{H^{s-1}(\mathbb{R}^3)})}$.
- The Cauchy problem (4) is locally well posed in $L^\infty(0, T_\varepsilon; H^s(\mathbb{R}^3))$ with $T = 1$ if $||u_0^\varepsilon||_{H^s(\mathbb{R}^3)} + ||u_1^\varepsilon||_{H^{s-1}(\mathbb{R}^3)} \leq h(\varepsilon)$ where $h(\varepsilon)$ tends to zero with ε.

These two assertions are not very useful to our problem. Indeed, it is clear that in the first one, the time T_ε tends to 0 with ε whereas in the second one, the initial data u_0^ε tends to zero with ε.

Then, we first prove the following theorem

Theorem 1.1 *Assume that $u_0^\varepsilon \in H^8(\mathbb{R}^2)$ and $u_1^\varepsilon \in H^7(\mathbb{R}^2)$. Then there exists $\delta_\varepsilon > 0$ ($\delta_\varepsilon \sim 2^{-\frac{1}{\varepsilon^2}}$) and $M > 0$ independent of ε and depending only on the initial datas such that if $||u_1^\varepsilon||_{H^7(\mathbb{R}^2)} \leq \delta_\varepsilon$, there exists a time T independent of ε such that (4) admits a unique solution u_ε satisfying*

$$u_\varepsilon \in L^\infty(0,T; H^8(\mathbb{R}^2)) \cap C([0,T]; L^2(\mathbb{R}^2)).$$

Furthermore, we have

$$\sup_{t\in[0,T]} \left(||u_\varepsilon||^2_{H^8(\mathbb{R}^2)} + \varepsilon^2 ||\partial_t u_\varepsilon||^2_{H^6(\mathbb{R}^2)} + \varepsilon^2 ||\partial_t^2 u_\varepsilon||^2_{H^6(\mathbb{R}^2)} \right) \leq M.$$

We have to notice that u_1^ε tends to 0 very rapidly with ε. In order to get estimates on u_ε which are independent of ε, we will consider (4) not as a wave equation but as a perturbation of a quasilinear Schrödinger equation. This means that in equation (1), the term $\varepsilon^2\partial_t^2 u_\varepsilon$ will be treated as a perturbative term. For that purpose, we will use the method introduced in [8], namely we rewrite equation (4) as a system in $(u_\varepsilon, \overline{u}_\varepsilon)^t$. Then, we differentiate the equation with respect to space and time to obtain a new system in u_0,..,u_5 (we drop for convenience the index ε) where

$$u_0 = u_\varepsilon,\ \ u_j = \partial_j u_\varepsilon\ \ \forall 1 \le j \le 2,\ \ u_3 = e^{q(|u_0|^2)}\Delta u_\varepsilon,\ \ u_4 = \partial_t \Delta u_\varepsilon,\ \ u_5 = \partial_t^2 \Delta u_\varepsilon.$$

The function q plays the role of a gauge transform which allows to treat the equations satisfied by u_0, u_1, u_2, and u_3 exactly as we have treated equations 2.9, 2.10 and 2.11 of [8]. Concerning u_4 and u_5 we do not use gauge transforms because their respective equations will be treated in a different way, which does not suppose to integrate by parts the terms including the first derivatives of u_4 and u_5. Indeed, to estimate u_4 and u_5, we use the classical energy estimates on wave equations, namely we multiply the equations respectively by $(\overline{u}_4)_t$ and $(\overline{u}_5)_t$ and we integrate the real part over $\mathbb{R}^2$. Finally, the proof of Theorem 1.1 will follow from a classical fixed-point Theorem.

Equipped with Theorem (1.1), we can study the convergence of u_ε to a solution u of (1) with initial data $u(x, y, 0) = u_0(x, y)$ under the condition that

$$u_0^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u_0 \quad \text{in} \quad H^6(\mathbb{R}^2) \quad \text{and} \quad u_1^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{in} \quad H^7(\mathbb{R}^2)$$

More precisely, we have the following

Theorem 1.2 *Let (u_0^ε) and (u_1^ε) be two bounded sequences respectively in $H^8(\mathbb{R}^2)$ and $H^7(\mathbb{R}^2)$ satisfying*

$$u_0^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u_0 \quad \textit{in} \quad H^6(\mathbb{R}^2) \quad \textit{and} \quad u_1^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \textit{in} \ H^7(\mathbb{R}^2).$$

We suppose furthermore that $||u_1^\varepsilon||_{H^7(\mathbb{R}^2)} \le 2^{-\frac{1}{\varepsilon^2}}$. Let u_ε be the solution of (4) with initial condition $u_\varepsilon(\cdot, 0) = u_0^\varepsilon(\cdot)$, $\partial_t u_\varepsilon(\cdot, 0) = u_1^\varepsilon(\cdot)$ given by Theorem 1.1. Let u the solution of (1) with initial condition $u(\cdot, 0) = u_0(\cdot)$ given by Theorem 1.1 of [8] Then, there exists a time $T > 0$ independent of ε such that u_ε and u exist on $[0, T]$ for any ε and

$$u_\varepsilon - u \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \textit{in} \quad L^\infty(0, T; H^6(\mathbb{R}^2)).$$

The proof uses an energy estimate which reads exactly as the one of [8].

Notation and function spaces : As usual, we denote by $L^p(\mathbb{R}^2)$ the Lebesgue space

$$L^p(\mathbb{R}^2) = \left\{ u \in \mathcal{S}'(\mathbb{R}^2) \ \ / \ \ ||u||_p < +\infty \right\}$$

where

$$||u||_p = \left(\int_{\mathbb{R}^N} |u(x)|^p dx \right)^{\frac{1}{p}} \ \text{if } 1 \le p < +\infty$$

and

$$||u||_\infty = \text{ess.sup}\left\{ |u(x)|; x \in \mathbb{R}^N \right\}.$$

We define the Sobolev space $H^s(\mathbb{R}^2)$ as follows

$$H^s(\mathbb{R}^2) = \left\{ u \in \mathcal{S}'(\mathbb{R}^2) \ \ / \ \ ||u||^2_{H^s(\mathbb{R}^2)} = \int_{\mathbb{R}^2} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi < +\infty \right\}$$

where $\widehat{u}(\xi)$ is the Fourier transform of u. Let $C(I, E)$ be the space of continuous functions from an interval I of $\mathbb{R}$ to a Banach space E. For $1 \le j \le 2$, we set $\partial_j = \partial_{x_j}$ and by convention $\partial_0 u = u$. For $k \in \mathbb{N}^2$,

$k=(k_1,k_2)$, we denote $\partial^k u = \partial_1^{k_1}\partial_2^{k_2}u$ and $|k| = k_1+k_2$. As usual, the coefficients of a matrix $\mathcal{D}$ will be denoted by $\mathcal{D}^{ij}$. The notation $\mathcal{D}(D^r V)$ means that the matrix $\mathcal{D}$ depends on $\partial^k V$ for $|k|\leq r$. Different positive constants might be denoted by the same letter C. We also denote by $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$ the real part and the imaginary part of u and $[a]$ is used to denote the integer part of a.

In section 2, we will transform equation (4) into a system to which we can apply the appropriate energy method. Then, in section 3, we prove Theorem 1.1 concerning the Cauchy problem (4). Finally, in section 4, we prove Theorem 1.2.

2 Transformation of equation (4)

In this section, we transform equation (4) into a system to which we can apply our energy method. We fix $a_0^\varepsilon \in H^8(\mathbb{R}^2)$ and $a_1^\varepsilon \in H^7(\mathbb{R}^2)$. Following [8], equation (4) can be rewritten as a system in $(u_\varepsilon, \overline{u}_\varepsilon)^t$

$$\begin{array}{l} -\varepsilon^2 \begin{pmatrix} (u_\varepsilon)_{tt} \\ (\overline{u}_\varepsilon)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (u_\varepsilon)_t \\ (\overline{u}_\varepsilon)_t \end{pmatrix} + \mathcal{A}(u_\varepsilon) \begin{pmatrix} \Delta u_\varepsilon \\ \Delta \overline{u}_\varepsilon \end{pmatrix} \\ - \begin{pmatrix} \frac{u_\varepsilon}{\gamma_\varepsilon^2}|\nabla u_\varepsilon|^2 - \frac{u_\varepsilon}{4\gamma_\varepsilon^4}|\nabla |u_\varepsilon|^2|^2 + u_\varepsilon g(|u_\varepsilon|^2) \\ \frac{-\overline{u_\varepsilon}}{\gamma_\varepsilon^2}|\nabla u_\varepsilon|^2 + \frac{\overline{u}_\varepsilon}{4\gamma_\varepsilon^4}|\nabla |u_\varepsilon|^2|^2 - \overline{u_\varepsilon} g(|u_\varepsilon|^2) \end{pmatrix} = 0 \end{array} \tag{7}$$

where

$$g(s) = \frac{1}{\sqrt{1+s}} - 1, \;\; \gamma_\varepsilon = \sqrt{1+|u_\varepsilon|^2},$$

$$\mathcal{A}(u_\varepsilon) = \frac{1}{2\gamma_\varepsilon^2} \begin{pmatrix} 2+|u_\varepsilon|^2 & -u_\varepsilon^2 \\ \overline{u_\varepsilon}^2 & -(2+|u_\varepsilon|^2) \end{pmatrix}.$$

We now drop for convenience the parameter ε. We set $u_0 = u_\varepsilon$, $u_1 = \partial_1 u_\varepsilon$, $u_2 = \partial_2 u_\varepsilon$, $u_3 = e^{q(|u_0|^2)}\Delta u_\varepsilon$, $u_4 = \partial_t \Delta u_\varepsilon$ and $u_5 = \partial_t^2 \Delta u_\varepsilon$. Here, u_0 does not denote an initial condition. The function q will play the role of a gauge transform and will be chosen later (see [8] for more details). We also set $U^* = (u_j)_{j=0}^2$ and $U = (u_j)_{j=0}^5$. Then, equation (7) can be rewritten as

$$2i \begin{pmatrix} \partial_t u_0 \\ \partial_t \overline{u}_0 \end{pmatrix} + \mathcal{A}(u_0) \begin{pmatrix} \Delta u_0 \\ \Delta \overline{u}_0 \end{pmatrix} - \varepsilon^2 \begin{pmatrix} \Delta^{-1} u_5 \\ \Delta^{-1} \overline{u}_5 \end{pmatrix} + \mathcal{F}_0(U^*) = 0 \tag{8}$$

where

$$\mathcal{F}_0(U^*) = \begin{pmatrix} \frac{-u_0}{\gamma^2(u_0)}(\sum_{k=1}^2 |u_k|^2) + \frac{u_0}{4\gamma^4(u_0)}(\sum_{k=1}^2 (u_0\overline{u}_k + \overline{u}_0 u_k)^2) + u_0 g(|u_0|^2) \\ \frac{\overline{u}_0}{\gamma^2(u_0)}(\sum_{k=1}^2 |u_k|^2) - \frac{\overline{u}_0}{4\gamma^4(u_0)}(\sum_{k=1}^2 (u_0\overline{u}_k + \overline{u}_0 u_k)^2) - \overline{u}_0 g(|u_0|^2) \end{pmatrix}.$$

Differentiating equation (7) with respect to x_j for $j=1,2$, we obtain

$$\begin{array}{l} 2i \begin{pmatrix} (u_j)_t \\ (\overline{u}_j)_t \end{pmatrix} + \mathcal{A}(u_0) \begin{pmatrix} \Delta u_j \\ \Delta \overline{u}_j \end{pmatrix} + \sum_{k=1}^2 B(u_0, u_k) \begin{pmatrix} T_{kj} u_3 \\ T_{kj} \overline{u}_3 \end{pmatrix} \\ + C(u_0, u_j) \begin{pmatrix} (e^{-q(|u_0|^2)} u_3) \\ (e^{-q(|u_0|^2)} \overline{u}_3) \end{pmatrix} - \varepsilon^2 \begin{pmatrix} \partial_j \Delta^{-1} u_5 \\ \partial_j \Delta^{-1} \overline{u}_5 \end{pmatrix} + \begin{pmatrix} F(U^*, u_j) \\ -\overline{F}(U^*, u_j) \end{pmatrix} = 0 \end{array}$$

where

$$B(v,w)=\begin{pmatrix} \frac{v}{4\gamma^4(v)}(2|v|^2\overline{w}+2\overline{v}^2w)-\frac{v}{\gamma^2(v)}\overline{w} & \frac{v}{4\gamma^4(v)}(2|v|^2w+2v^2\overline{w})-\frac{v}{\gamma^2(v)}w \\ \frac{-\overline{v}}{4\gamma^4(v)}(2|v|^2\overline{w}+2\overline{v}^2w)+\frac{\overline{v}}{\gamma^2(v)}\overline{w} & \frac{-\overline{v}}{4\gamma^4(v)}(2|v|^2w+2v^2\overline{w})+\frac{\overline{v}}{\gamma^2(v)}w \end{pmatrix}$$

$$C(v,w)=\begin{pmatrix} (\frac{1}{2\gamma^2(v)}-\frac{2+|v|^2}{2\gamma^4(v)})(v\overline{w}+\overline{v}w) & -\frac{vw}{\gamma^2(v)}+\frac{v^2}{4\gamma^4(v)}(v\overline{w}+\overline{v}w) \\ \frac{\overline{vw}}{\gamma^2(v)}-\frac{\overline{v}^2}{2\gamma^4(v)}(v\overline{w}+\overline{v}w) & (\frac{-1}{2\gamma^2(v)}+\frac{2+|v|^2}{2\gamma^4(v)})(v\overline{w}+\overline{v}w) \end{pmatrix}$$

$$\begin{aligned} F(U^*,w)=&\left(\frac{u_0\overline{w}+\overline{u}_0w}{\gamma^4(u_0)}-\frac{w}{\gamma^2(u_0)}\right)\left(\sum_{k=1}^N|u_k|^2\right)+u_0g^{'}(|u_0|^2)(u_0\overline{w}+\overline{u}_0w)\\ &+\left(\frac{w}{4\gamma^4(u_0)}-\frac{(u_0\overline{w}+\overline{u}_0w)}{2\gamma^6(u_0)}\right)\left(\sum_{k=1}^N(u_0\overline{u}_k+\overline{u}_0u_k)^2+\right)\\ &+\frac{u_0}{4\gamma^4(u_0)}\left(2(u_0\overline{w}+\overline{u}_0w)(\sum_{k=1}^N|u_k|^2)+2\overline{u}_0\overline{w}(\sum_{k=1}^Nu_k^2)\right.\\ &\left.+2u_0w(\sum_{k=1}^N\overline{u}_k^2)\right)+wg(|u_0|^2) \end{aligned}$$

and for $i,j=1,2$, $T_{i,j}$ is the following operator of order 0

$$T_{i,j}v=\partial_i\partial_j\Delta^{-1}(e^{-q(|u_0|^2)}v).$$

Furthermore, this operator is continuous from L^2 to L^2 with a norm independent of ε. We can rewrite these equations as follows

$$2i\begin{pmatrix}(u_j)_t\\(\overline{u}_j)_t\end{pmatrix}+\mathcal{A}(u_0)\begin{pmatrix}\Delta u_j\\\Delta\overline{u}_j\end{pmatrix}-\varepsilon^2\begin{pmatrix}\partial_j\Delta^{-1}u_5\\\partial_j\Delta^{-1}\overline{u}_5\end{pmatrix}+\mathcal{F}_j(U^*,u_3,Tu_3)=0 \tag{9}$$

where

$$\begin{aligned}\mathcal{F}_j(U^*,u_3,Tu_3)=&\begin{pmatrix}F(U^*,u_j)\\-\overline{F}(U^*,u_j)\end{pmatrix}+C(u_0,u_j)\begin{pmatrix}(e^{-q(|u_0|^2)}u_3)\\(e^{-q(|u_0|^2)}\overline{u}_3)\end{pmatrix}\\&+\sum_{k=1}^2B(u_0,u_k)\begin{pmatrix}T_{kj}u_3\\T_{kj}\overline{u}_3\end{pmatrix}.\end{aligned}$$

In order to derive an equation for u_3, we differentiate equation (7) twice with respect to x_j to obtain

$$\begin{aligned}&2i\begin{pmatrix}(\partial_j^2u_0)_t\\(\partial_j^2\overline{u}_0)_t\end{pmatrix}+\mathcal{A}(u_0)\begin{pmatrix}\Delta\partial_j^2u_0\\\Delta\partial_j^2\overline{u}_0\end{pmatrix}+\sum_{k=1}^2B(u_0,u_k)\begin{pmatrix}\partial_k\partial_j^2u_0\\\partial_k\partial_j^2\overline{u}_0\end{pmatrix}\\&+2C(u_0,u_j)\begin{pmatrix}\partial_j\Delta u_0\\\partial_j\Delta\overline{u}_0\end{pmatrix}+\partial_jC(u_0,u_j)\begin{pmatrix}\Delta u_0\\\Delta\overline{u}_0\end{pmatrix}-\varepsilon^2\begin{pmatrix}(\partial_j^2u_0)_{tt}\\(\partial_j^2\overline{u}_0)_{tt}\end{pmatrix}\\&+\sum_{k=1}^2\partial_jB(u_0,u_k)\begin{pmatrix}\partial_k\partial_ju_0\\\partial_k\partial_j\overline{u}_0\end{pmatrix}+\begin{pmatrix}\partial_jF(U^*,u_j)\\-\partial_j\overline{F}(U^*,u_j)\end{pmatrix}=0.\end{aligned}\tag{10}$$

Summing the two equations and multiplying by $\mathcal{A}^{-1}(u_0)$, we obtain

$$
\begin{aligned}
&2i\mathcal{A}^{-1}(u_0)\begin{pmatrix} (\Delta u_0)_t \\ (\Delta \overline{u}_0)_t \end{pmatrix} + \begin{pmatrix} \Delta\Delta u_0 \\ \Delta\Delta \overline{u}_0 \end{pmatrix} + \sum_{k=1}^{2} \mathcal{A}^{-1}(u_0)B(u_0,u_k)\begin{pmatrix} \partial_k \Delta u_0 \\ \partial_k \Delta \overline{u}_0 \end{pmatrix} \\
&+2\sum_{j=1}^{2}\mathcal{A}^{-1}(u_0)C(u_0,u_j)\begin{pmatrix} \partial_j \Delta u_0 \\ \partial_j \Delta \overline{u}_0 \end{pmatrix} + \sum_{j=1}^{2}\mathcal{A}^{-1}(u_0)\partial_j C(u_0,u_j)\begin{pmatrix} \Delta u_0 \\ \Delta \overline{u}_0 \end{pmatrix} \\
&-\varepsilon^2\mathcal{A}^{-1}(u_0)\begin{pmatrix} (\Delta u_0)_{tt} \\ (\Delta \overline{u}_0)_{tt} \end{pmatrix} + \sum_{j=1}^{2}\sum_{k=1}^{2}\mathcal{A}^{-1}(u_0)\partial_j B(u_0,u_k)\begin{pmatrix} \partial_k\partial_j u_0 \\ \partial_k\partial_j \overline{u}_0 \end{pmatrix} \\
&+\sum_{j=1}^{2}\mathcal{A}^{-1}(u_0)\begin{pmatrix} \partial_j F(U^*,u_j) \\ -\partial_j \overline{F}(U^*,u_j) \end{pmatrix} = 0.
\end{aligned}
\tag{11}
$$

which can be rewritten, after the multiplication by $e^{q(|u_0|^2)}$, in the following way

$$
\begin{aligned}
&2i\begin{pmatrix} (u_3)_t \\ (\overline{u}_3)_t \end{pmatrix} + \mathcal{A}(u_0)\begin{pmatrix} \Delta u_3 \\ \Delta \overline{u}_3 \end{pmatrix} + \sum_{k=1}^{2}\mathcal{E}(u_0,u_k)\begin{pmatrix} \partial_k u_3 \\ \partial_k \overline{u}_3 \end{pmatrix} - \varepsilon^2\begin{pmatrix} u_5 \\ \overline{u}_5 \end{pmatrix} \\
&+\mathcal{I}(U,Tu_3,\Delta^{-1}u_4) = 0
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
\mathcal{E}(u_0,u_k) =& B(u_0,u_k) + 2C(u_0,u_k) \\
& - 2\mathcal{A}(u_0)q^{'}(|u_0|^2)\begin{pmatrix} u_0\overline{u}_k + \overline{u}_0 u_k & 0 \\ 0 & u_0\overline{u}_k + \overline{u}_0 u_k \end{pmatrix}
\end{aligned}
$$

and $\mathcal{I}$ is a matrix depending on U, $T_{ij}u_3$ for $i,j = 1,2$ and $\Delta^{-1}u_4$ in the following way

$$
\begin{aligned}
&\mathcal{I}(U,Tu_3,\Delta^{-1}u_4) = \\
&\sum_{m=1}^{2} e^{q(|u_0|^2)}\begin{pmatrix} \partial_m F(U^*,u_j) \\ -\partial_m \overline{F}(U^*,u_j) \end{pmatrix} - 2iq^{'}(|u_0|^2)\begin{pmatrix} u_0(\Delta^{-1}\overline{u}_4)u_3 + \overline{u}_0(\Delta^{-1}u_4)u_3 \\ u_0(\Delta^{-1}\overline{u}_4)\overline{u}_3 + \overline{u}_0(\Delta^{-1}u_4)\overline{u}_3 \end{pmatrix} \\
&-\sum_{k=1}^{2}(2C(u_0,u_k) + B(u_0,u_k))q^{'}(|u_0|^2)\begin{pmatrix} (u_0\overline{u}_k + \overline{u}_0 u_k)u_3 \\ (u_0\overline{u}_k + \overline{u}_0 u_k)\overline{u}_3 \end{pmatrix} \\
&+\sum_{k=1}^{2}\partial_k C(u_0,u_k)\begin{pmatrix} u_3 \\ \overline{u}_3 \end{pmatrix} + \sum_{j=1}^{2}\sum_{k=1}^{2} e^{q(|u_0|^2)}\partial_k B(u_0,u_k)\begin{pmatrix} T_{kj}u_3 \\ T_{kj}\overline{u}_3 \end{pmatrix} \\
&+\mathcal{A}(u_0)\begin{pmatrix} (q^{'2}(|u_0|^2) - q^{''}(|u_0|^2))(\sum_{k=1}^{2}(\overline{u}_0 u_k + u_0\overline{u}_k)^2)u_3 \\ (q^{'2}(|u_0|^2) - q^{''}(|u_0|^2))(\sum_{k=1}^{2}(\overline{u}_0 u_k + u_0\overline{u}_k)^2)\overline{u}_3 \end{pmatrix} \\
&-\mathcal{A}(u_0)\begin{pmatrix} q^{'}(|u_0|^2)(2\sum_{k=1}^{2}|u_k|^2 + \overline{u}_0(\sum_{k=1}^{2}T_{kk}u_3)u_3 \\ q^{'}(|u_0|^2)(2\sum_{k=1}^{2}|u_k|^2 + \overline{u}_0(\sum_{k=1}^{2}T_{kk}u_3))\overline{u}_3 \end{pmatrix} - \mathcal{A}(u_0)\begin{pmatrix} q^{'}(|u_0|^2)u_0(\sum_{k=1}^{2}T_{kk}\overline{u}_3)u_3 \\ q^{'}(|u_0|^2)u_0(\sum_{k=1}^{2}T_{kk}\overline{u}_3)\overline{u}_3 \end{pmatrix}
\end{aligned}
$$

We have to notice here that the matrix $\mathcal{I}$ depends on $\Delta^{-1}u_4$, which is due to the gauge transform $u_3 = e^{q(|u_0|^2)}\Delta u_0$. Indeed, in equation (11), we have to replace Δu_0 by $e^{q(|u_0|^2)}u_3$ in the term $(\Delta u_0)_t$ which gives the contribution $e^{q(|u_0|^2)}(q(|u_0|^2))_t u_3$. Then, recalling that $(u_0)_t = \Delta^{-1}u_4$, we include this last term in $\mathcal{I}$.

Differentiating equation (11) with respect to t, we obtain

$$
\begin{aligned}
&-\varepsilon^2\mathcal{A}^{-1}(u_0)\begin{pmatrix}(u_4)_{tt}\\(\overline{u}_4)_{tt}\end{pmatrix}-\varepsilon^2\partial_t\left(\mathcal{A}^{-1}(u_0)\right)\begin{pmatrix}u_5\\\overline{u}_5\end{pmatrix}+2i\mathcal{A}^{-1}(u_0)\begin{pmatrix}(u_4)_t\\(\overline{u}_4)_t\end{pmatrix}+2i\partial_t\left(\mathcal{A}^{-1}(u_0)\right)\begin{pmatrix}u_4\\\overline{u}_4\end{pmatrix}\\
&+\begin{pmatrix}\Delta u_4\\\Delta\overline{u}_4\end{pmatrix}+\sum_{k=1}^{2}\mathcal{A}^{-1}(u_0)B(u_0,u_k)\begin{pmatrix}\partial_k u_4\\\partial_k\overline{u}_4\end{pmatrix}+\sum_{k=1}^{2}\partial_t\left(\mathcal{A}^{-1}(u_0)B(u_0,u_k)\right)\begin{pmatrix}\partial_k(e^{-q(|u_0|^2)}u_3)\\\partial_k(e^{-q(|u_0|^2)}\overline{u}_3)\end{pmatrix}\\
&+2\sum_{k=1}^{2}\mathcal{A}^{-1}(u_0)C(u_0,u_k)\begin{pmatrix}\partial_k u_4\\\partial_k\overline{u}_4\end{pmatrix}+2\sum_{j=1}^{2}\partial_t\left(\mathcal{A}^{-1}(u_0)C(u_0,u_j)\right)\begin{pmatrix}\partial_j(e^{-q(|u_0|^2)}u_3)\\\partial_j(e^{-q(|u_0|^2)}\overline{u}_3)\end{pmatrix}\\
&+\sum_{j=1}^{2}\mathcal{A}^{-1}(u_0)\partial_j C(u_0,u_j)\begin{pmatrix}u_4\\\overline{u}_4\end{pmatrix}+\sum_{j=1}^{2}\partial_t\left(\mathcal{A}^{-1}(u_0)\partial_j C(u_0,u_j)\right)\begin{pmatrix}e^{-q(|u_0|^2)}u_3\\e^{-q(|u_0|^2)}\overline{u}_3\end{pmatrix}\\
&+\sum_{j=1}^{2}\sum_{k=1}^{2}\mathcal{A}^{-1}(u_0)\partial_j B(u_0,u_k)\begin{pmatrix}\partial_k\partial_j\Delta^{-1}u_4\\\partial_k\partial_j\Delta^{-1}\overline{u}_4\end{pmatrix}+\sum_{j=1}^{2}\sum_{k=1}^{2}\partial_t\left(\mathcal{A}^{-1}(u_0)\partial_j B(u_0,u_k)\right)\begin{pmatrix}T_{kj}u_3\\T_{kj}\overline{u}_3\end{pmatrix}\\
&+\sum_{j=1}^{2}\partial_t\left(\mathcal{A}^{-1}(u_0)\begin{pmatrix}\partial_j F(U^*,u_j)\\-\partial_j\overline{F}(U^*,u_j)\end{pmatrix}\right)=0.
\end{aligned}
\tag{13}
$$

Keeping in view that we have $\partial_t u_0=\Delta^{-1}u_4$, we can rewrite the last equation in the following way

$$
\begin{aligned}
&-\varepsilon^2\mathcal{A}^{-1}(u_0)\begin{pmatrix}(u_4)_{tt}\\(\overline{u}_4)_{tt}\end{pmatrix}+2i\mathcal{A}^{-1}(u_0)\begin{pmatrix}(u_4)_t\\(\overline{u}_4)_t\end{pmatrix}+\begin{pmatrix}\Delta u_4\\\Delta\overline{u}_4\end{pmatrix}+\sum_{k=1}^{2}\mathcal{D}_k(u_0,u_k)\begin{pmatrix}\partial_k u_4\\\partial_k\overline{u}_4\end{pmatrix}\\
&+\mathcal{G}(DU,Tu_3)\begin{pmatrix}u_4\\\overline{u}_4\end{pmatrix}+\sum_{j=0}^{2}\sum_{k=0}^{2}\mathcal{H}_{jk}(DU,Tu_3)\begin{pmatrix}\partial_j\partial_k\Delta^{-1}u_4\\\partial_j\partial_k\Delta^{-1}\overline{u}_4\end{pmatrix}=0
\end{aligned}
\tag{14}
$$

where for $k=1,2$

$$\mathcal{D}_k(u_0,u_k)=\mathcal{A}^{-1}(u_0)\left(B(u_0,u_k)+2C(u_0,u_k)\right),$$

and for $j,k=1,2$, $\mathcal{H}_{jk}$ is a matrix depending on U and its first derivatives and on $T_{mn}u_3$ $(m,n=1,2)$. We have to notice here that in equation (14), all the terms of equation (13) which contains $\partial_t u_0$ and its first and second space derivatives are included in the last terms

$$\sum_{j=0}^{2}\sum_{k=0}^{2}\mathcal{H}_{jk}(DU,Tu_3)\begin{pmatrix}\partial_j\partial_k\Delta^{-1}u_4\\\partial_j\partial_k\Delta^{-1}\overline{u}_4\end{pmatrix}.$$

Indeed, this sum include the values $j=0$ and $k=0$. Recalling that we have adopted the notation $\partial_0 u=u$, then it is possible to write for $j=0,..,2$ and $k=0,..,2$

$$\partial_t\partial_j\partial_k u_0=\partial_j\partial_k\Delta^{-1}u_4.$$

Finally, in order to obtain an equation satisfied by u_5, we differentiate twice with respect to t equation

(11) to obtain

$$
\begin{aligned}
&-\varepsilon^2\mathcal{A}^{-1}(u_0)\begin{pmatrix}(u_5)_{tt}\\(\overline{u}_5)_{tt}\end{pmatrix}-2\varepsilon^2\partial_t\left(\mathcal{A}^{-1}(u_0)\right)\begin{pmatrix}(u_5)_t\\(\overline{u}_5)_t\end{pmatrix}-\varepsilon^2\partial_t^2\left(\mathcal{A}^{-1}(u_0)\right)\begin{pmatrix}u_5\\\overline{u}_5\end{pmatrix}\\
&+2i\mathcal{A}^{-1}(u_0)\begin{pmatrix}(u_5)_t\\(\overline{u}_5)_t\end{pmatrix}+4i\partial_t\left(\mathcal{A}^{-1}(u_0)\right)\begin{pmatrix}u_5\\\overline{u}_5\end{pmatrix}+2i\partial_t^2\left(\mathcal{A}^{-1}(u_0)\right)\begin{pmatrix}u_4\\\overline{u}_4\end{pmatrix}\\
&+\begin{pmatrix}\Delta u_5\\\Delta\overline{u}_5\end{pmatrix}+\sum_{k=1}^{2}\mathcal{A}^{-1}(u_0)B(u_0,u_k)\begin{pmatrix}\partial_k u_5\\\partial_k\overline{u}_5\end{pmatrix}+2\sum_{k=1}^{2}\partial_t\left(\mathcal{A}^{-1}(u_0)B(u_0,u_k)\right)\begin{pmatrix}\partial_k u_4\\\partial_k\overline{u}_4\end{pmatrix}\\
&+\sum_{k=1}^{2}\partial_t^2\left(\mathcal{A}^{-1}(u_0)B(u_0,u_k)\right)\begin{pmatrix}\partial_k(e^{-q(|u_0|^2)}u_3)\\\partial_k(e^{-q(|u_0|^2)}\overline{u}_3)\end{pmatrix}+2\sum_{j=1}^{2}\mathcal{A}^{-1}(u_0)C(u_0,u_j)\begin{pmatrix}\partial_j u_5\\\partial_j\overline{u}_5\end{pmatrix}\\
&+4\sum_{j=1}^{2}\partial_t\left(\mathcal{A}^{-1}(u_0)C(u_0,u_j)\right)\begin{pmatrix}\partial_j u_4\\\partial_j\overline{u}_4\end{pmatrix}+2\sum_{j=1}^{2}\partial_t^2\left(\mathcal{A}^{-1}(u_0)C(u_0,u_j)\right)\begin{pmatrix}\partial_j(e^{-q(|u_0|^2)}u_3)\\\partial_j(e^{-q(|u_0|^2)}\overline{u}_3)\end{pmatrix}\\
&+\sum_{j=1}^{2}\mathcal{A}^{-1}(u_0)\partial_jC(u_0,u_j)\begin{pmatrix}u_5\\\overline{u}_5\end{pmatrix}+2\sum_{j=1}^{2}\partial_t\left(\mathcal{A}^{-1}(u_0)\partial_jC(u_0,u_j)\right)\begin{pmatrix}u_4\\\overline{u}_4\end{pmatrix}\\
&+\sum_{j=1}^{2}\partial_t^2\left(\mathcal{A}^{-1}(u_0)\partial_jC(u_0,u_j)\right)\begin{pmatrix}(e^{-q(|u_0|^2)}u_3)\\(e^{-q(|u_0|^2)}\overline{u}_3)\end{pmatrix}+\sum_{j=1}^{2}\sum_{k=1}^{2}\mathcal{A}^{-1}(u_0)\partial_jB(u_0,u_k)\begin{pmatrix}\partial_k\partial_j\Delta^{-1}u_5\\\partial_k\partial_j\Delta^{-1}\overline{u}_5\end{pmatrix}\\
&+2\sum_{j=1}^{2}\sum_{k=1}^{2}\partial_t\left(\mathcal{A}^{-1}(u_0)\partial_jB(u_0,u_k)\right)\begin{pmatrix}\partial_k\partial_j\Delta^{-1}u_4\\\partial_k\partial_j\Delta^{-1}\overline{u}_4\end{pmatrix}+\sum_{j=1}^{2}\sum_{k=1}^{2}\partial_t^2\left(\mathcal{A}^{-1}(u_0)\partial_jB(u_0,u_k)\right)\begin{pmatrix}T_{kj}u_3\\T_{kj}\overline{u}_3\end{pmatrix}\\
&+\partial_t^2\left(\sum_{j=1}^{2}\mathcal{A}^{-1}(u_0)\begin{pmatrix}\partial_jF(U^*,u_j)\\\partial_j\overline{F}(U^*,u_j)\end{pmatrix}\right)=0.
\end{aligned}
$$

This equation takes the form

$$
\begin{aligned}
&-\varepsilon^2\mathcal{A}^{-1}(u_0)\begin{pmatrix}(u_5)_{tt}\\(\overline{u}_5)_{tt}\end{pmatrix}+\mathcal{C}(u_0,\Delta^{-1}u_4)\begin{pmatrix}(u_5)_t\\(\overline{u}_5)_t\end{pmatrix}+\begin{pmatrix}\Delta u_5\\\Delta\overline{u}_5\end{pmatrix}+\sum_{k=1}^{2}\mathcal{D}_k(u_0,u_k)\begin{pmatrix}\partial_k u_5\\\partial_k\overline{u}_5\end{pmatrix}\\
&+\mathcal{I}(DU,Tu_3)\begin{pmatrix}u_5\\\overline{u}_5\end{pmatrix}+\sum_{j=0}^{2}\sum_{k=0}^{2}\mathcal{J}_{jk}(DU,Tu_3)\begin{pmatrix}\partial_j\partial_k\Delta^{-1}u_5\\\partial_j\partial_k\Delta^{-1}\overline{u}_5\end{pmatrix}\\
&+\sum_{j=0}^{2}\sum_{k=0}^{2}\mathcal{K}_{jk}(DU,Tu_3,D^2\Delta^{-1}u_4)\begin{pmatrix}\partial_j\partial_k\Delta^{-1}u_4\\\partial_j\partial_k\Delta^{-1}\overline{u}_4\end{pmatrix}=0.
\end{aligned}
\tag{15}
$$

where

$$\mathcal{C}(u_0,\Delta^{-1}u_4)=2i\mathcal{A}^{-1}(u_0)-2\varepsilon^2\partial_t\left(\mathcal{A}^{-1}(u_0)\right).$$

For $j,k=0,..,2$, the matrices $\mathcal{J}_{jk}$ depend on j, k, on U and its first derivatives and on $T_{mn}u_3$ for $m,m=1,2$. For $j,k=0,..,2$, the matrices $\mathcal{K}_{jk}$ depend on j, k, on U and its first derivatives, on $T_{mn}u_3$ for $m,m=1,2$ and on $\Delta^{-1}u_4$ and its first and second derivatives. In order to simplify the notations, we now introduce for $j,k=0,..,2$ the operator R_{jk}

$$R_{jk}=\partial_j\partial_k\Delta^{-1}.$$

Once again, we have included the terms which contain $\partial_t u_0$ and its first and second space derivatives in

$$\sum_{j=0}^{2}\sum_{k=0}^{2}\mathcal{K}_{jk}(DU,Tu_3,D^2\Delta^{-1}u_4)\begin{pmatrix}\partial_j\partial_k\Delta^{-1}u_4\\\partial_j\partial_k\Delta^{-1}\overline{u}_4\end{pmatrix}$$

and the terms which contain $\partial_t^2 u_0$ and its first and second space derivatives in

$$\sum_{j=0}^{2}\sum_{k=0}^{2}\mathcal{J}_{jk}(DU,Tu_3)\begin{pmatrix}\partial_j\partial_k\Delta^{-1}u_5\\ \partial_j\partial_k\Delta^{-1}\overline{u}_5\end{pmatrix}.$$

In conclusion, we have transformed equation (7) into the following system

$$2i\begin{pmatrix}\partial_t u_0\\ \partial_t\overline{u}_0\end{pmatrix}+\mathcal{A}(u_0)\begin{pmatrix}\Delta u_0\\ \Delta\overline{u}_0\end{pmatrix}-\varepsilon^2\begin{pmatrix}\Delta^{-1}u_5\\ \Delta^{-1}\overline{u}_5\end{pmatrix}+\mathcal{F}_0(U^*)=0 \tag{16}$$

for $j=1,2$,

$$2i\begin{pmatrix}(u_j)_t\\ (\overline{u}_j)_t\end{pmatrix}+\mathcal{A}(u_0)\begin{pmatrix}\Delta u_j\\ \Delta\overline{u}_j\end{pmatrix}-\varepsilon^2\begin{pmatrix}\partial_j\Delta^{-1}u_5\\ \partial_j\Delta^{-1}\overline{u}_5\end{pmatrix}+\mathcal{F}_j(U^*,u_3,Tu_3)=0 \tag{17}$$

$$\begin{aligned}&2i\begin{pmatrix}(u_3)_t\\ (\overline{u}_3)_t\end{pmatrix}+\mathcal{A}(u_0)\begin{pmatrix}\Delta u_3\\ \Delta\overline{u}_3\end{pmatrix}+\sum_{k=1}^{2}\mathcal{E}(u_0,u_k)\begin{pmatrix}\partial_k u_3\\ \partial_k\overline{u}_3\end{pmatrix}-\varepsilon^2\begin{pmatrix}u_5\\ \overline{u}_5\end{pmatrix}\\ &+\mathcal{I}(U,Tu_3,\Delta^{-1}u_4)=0\end{aligned} \tag{18}$$

$$\begin{aligned}&-\varepsilon^2\mathcal{A}^{-1}(u_0)\begin{pmatrix}(u_4)_{tt}\\ (\overline{u}_4)_{tt}\end{pmatrix}+2i\mathcal{A}^{-1}(u_0)\begin{pmatrix}(u_4)_t\\ (\overline{u}_4)_t\end{pmatrix}+\begin{pmatrix}\Delta u_4\\ \Delta\overline{u}_4\end{pmatrix}+\sum_{k=1}^{2}\mathcal{D}_k(u_0,u_k)\begin{pmatrix}\partial_k u_4\\ \partial_k\overline{u}_4\end{pmatrix}\\ &+\sum_{j=0}^{2}\sum_{k=0}^{2}\mathcal{H}_{jk}(DU,Tu_3)\begin{pmatrix}R_{jk}u_4\\ R_{jk}\overline{u}_4\end{pmatrix}+\mathcal{G}(DU,Tu_3)\begin{pmatrix}u_4\\ \overline{u}_4\end{pmatrix}=0\end{aligned} \tag{19}$$

$$\begin{aligned}&-\varepsilon^2\mathcal{A}^{-1}(u_0)\begin{pmatrix}(u_5)_{tt}\\ (\overline{u}_5)_{tt}\end{pmatrix}+\mathcal{C}(u_0,\Delta^{-1}u_4)\begin{pmatrix}(u_5)_t\\ (\overline{u}_5)_t\end{pmatrix}+\begin{pmatrix}\Delta u_5\\ \Delta\overline{u}_5\end{pmatrix}+\sum_{k=1}^{2}\mathcal{D}_k(u_0,u_k)\begin{pmatrix}\partial_k u_5\\ \partial_k\overline{u}_5\end{pmatrix}\\ &+\sum_{j=0}^{2}\sum_{k=0}^{2}\mathcal{J}_{jk}(DU,Tu_3)\begin{pmatrix}R_{jk}u_5\\ R_{jk}\overline{u}_5\end{pmatrix}+\sum_{j=0}^{2}\sum_{k=0}^{2}\mathcal{K}_{jk}(DU,Tu_3,D^2\Delta^{-1}u_4)\begin{pmatrix}R_{jk}u_4\\ R_{jk}\overline{u}_4\end{pmatrix}\\ &+\mathcal{I}(DU,Tu_3)\begin{pmatrix}u_5\\ \overline{u}_5\end{pmatrix}=0.\end{aligned} \tag{20}$$

3 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. We study the system (16), (17), (18), (19) and (20) in the following function space

$$X_T=\left\{\begin{array}{l}U=(u_j)_{j=0}^5:u_j\in C([0,T];L^2(\mathbb{R}^2))\cap L^\infty(0,T;H^4(\mathbb{R}^2)),\\ ||U||_{X_T}=\sum_{j=0}^{5}\sup_{0\le t\le T}||u_j(t)||_{H^4(\mathbb{R}^2)}<\infty\end{array}\right\}$$

For $M=(m_j)_{j=0}^5\in\mathbb{R}_+^6$ and $r\in\mathbb{R}_+$, we denote

$$X_T(M,r)=\left\{\begin{array}{l}U=(u_j)_{j=0}^5\in X_T:\forall j=0,..,5\ \ ||u_j||_{L^\infty(0,T;H^4(\mathbb{R}^2))}\le m_j\\ ||(u_0)_t||_{L^\infty(0,T;H^2(\mathbb{R}^2))}\le r\text{ and }u_0(\cdot,0)=u_0^\varepsilon(\cdot)\end{array}\right\}$$

and we let $V=(v_j)_{j=0}^5 \in X_T(M,r)$. We also set $V^*=(v_j)_{j=0}^2$. Consider the linearized version of equations (16), (17), (18), (19) and (20)

$$2i\begin{pmatrix}\partial_t u_0\\ \partial_t \overline{u}_0\end{pmatrix}+\mathcal{A}(v_0)\begin{pmatrix}\Delta u_0\\ \Delta\overline{u}_0\end{pmatrix}-\varepsilon^2\begin{pmatrix}\Delta^{-1}v_5\\ \Delta^{-1}\overline{v}_5\end{pmatrix}+\mathcal{F}_0(V^*)=0 \tag{21}$$

for $j=1,2$,

$$2i\begin{pmatrix}(u_j)_t\\ (\overline{u}_j)_t\end{pmatrix}+\mathcal{A}(v_0)\begin{pmatrix}\Delta u_j\\ \Delta\overline{u}_j\end{pmatrix}-\varepsilon^2\begin{pmatrix}\partial_j\Delta^{-1}v_5\\ \partial_j\Delta^{-1}\overline{v}_5\end{pmatrix}+\mathcal{F}_j(V^*,v_3,Tv_3)=0 \tag{22}$$

$$\begin{aligned}&2i\begin{pmatrix}(u_3)_t\\ (\overline{u}_3)_t\end{pmatrix}+\mathcal{A}(v_0)\begin{pmatrix}\Delta u_3\\ \Delta\overline{u}_3\end{pmatrix}+\sum_{k=1}^{2}\mathcal{E}(v_0,v_k)\begin{pmatrix}\partial_k u_3\\ \partial_k\overline{u}_3\end{pmatrix}-\varepsilon^2\begin{pmatrix}v_5\\ \overline{v}_5\end{pmatrix}\\ &+\mathcal{I}(V,Tv_3,\Delta^{-1}v_4)=0\end{aligned} \tag{23}$$

$$\begin{aligned}&-\varepsilon^2\mathcal{A}^{-1}(v_0)\begin{pmatrix}(u_4)_{tt}\\ (\overline{u}_4)_{tt}\end{pmatrix}+2i\mathcal{A}^{-1}(v_0)\begin{pmatrix}(u_4)_t\\ (\overline{u}_4)_t\end{pmatrix}+\begin{pmatrix}\Delta u_4\\ \Delta\overline{u}_4\end{pmatrix}+\sum_{k=1}^{2}\mathcal{D}_k(v_0,v_k)\begin{pmatrix}\partial_k u_4\\ \partial_k\overline{u}_4\end{pmatrix}\\ &+\sum_{j=0}^{2}\sum_{k=0}^{2}\mathcal{H}_{jk}(DV,Tv_3)\begin{pmatrix}R_{jk}u_4\\ R_{jk}\overline{u}_4\end{pmatrix}+\mathcal{G}(DV,Tv_3)\begin{pmatrix}u_4\\ \overline{u}_4\end{pmatrix}=0\end{aligned} \tag{24}$$

$$\begin{aligned}&-\varepsilon^2\mathcal{A}^{-1}(v_0)\begin{pmatrix}(u_5)_{tt}\\ (\overline{u}_5)_{tt}\end{pmatrix}+\mathcal{C}(v_0,\Delta^{-1}v_4)\begin{pmatrix}(u_5)_t\\ (\overline{u}_5)_t\end{pmatrix}+\begin{pmatrix}\Delta u_5\\ \Delta\overline{u}_5\end{pmatrix}+\sum_{k=1}^{2}\mathcal{D}_k(v_0,v_k)\begin{pmatrix}\partial_k u_5\\ \partial_k\overline{u}_5\end{pmatrix}\\ &+\sum_{j=0}^{2}\sum_{k=0}^{2}\mathcal{J}_{jk}(DV,Tv_3)\begin{pmatrix}R_{jk}u_5\\ R_{jk}\overline{u}_5\end{pmatrix}+\sum_{j=0}^{2}\sum_{k=0}^{2}\mathcal{K}_{jk}(DV,Tv_3,D^2\Delta^{-1}v_4)\begin{pmatrix}R_{jk}u_4\\ R_{jk}\overline{u}_4\end{pmatrix}\\ &+\mathcal{I}(DV,Tv_3)\begin{pmatrix}u_5\\ \overline{u}_5\end{pmatrix}=0.\end{aligned} \tag{25}$$

Let $\mathcal{Z}=\left[L^\infty(0,T;H^4(\mathbb{R}^2))\cap C([0,T];L^2(\mathbb{R}^2))\right]^6$. Then, it is not difficult to see that the linear inhomogeneous Cauchy problem (21), (22), (23), (24) and (25) with initial condition

$$\begin{aligned}&u_0(\cdot,0)=u_0^\varepsilon(\cdot),\ \ u_1(\cdot,0)=\partial_1 u_0^\varepsilon(\cdot),\ \ u_2(\cdot,0)=\partial_2 u_0^\varepsilon(\cdot),\\ &u_3(\cdot,0)=e^{q(|u_0^\varepsilon|^2)}\Delta u_0^\varepsilon(\cdot),\ \ u_4(\cdot,0)=\Delta u_1^\varepsilon(\cdot),\ \ u_5(\cdot,0)=\partial_t u_4(\cdot,0),\\ &\partial_t u_4(\cdot,0)=\frac{1}{\varepsilon^2}\Delta\ \Big(\Delta u_0^\varepsilon+2iu_1^\varepsilon-\big(\tfrac{1}{\sqrt{1+|u_0^\varepsilon|^2}}-1\big)u_0^\varepsilon\\ &\qquad\qquad -\tfrac{1}{\sqrt{1+|u_0^\varepsilon|^2}}\Delta\big(\sqrt{1+|u_0^\varepsilon|^2}\big)u_0^\varepsilon\Big),\\ &\partial_t u_5(\cdot,0)=\tfrac{1}{\varepsilon^2}\Delta\ \Big(\Delta u_1^\varepsilon-\big(\tfrac{1}{\sqrt{1+|u_0^\varepsilon|^2}}-1\big)u_1^\varepsilon+\tfrac{u_0^\varepsilon\overline{u}_1^\varepsilon+\overline{u}_0^\varepsilon u_1^\varepsilon}{2(1+|u_0^\varepsilon|^2)^{\frac{3}{2}}}u_0^\varepsilon\\ &\qquad\qquad -\tfrac{1}{\sqrt{1+|u_0^\varepsilon|^2}}\Delta\big(\sqrt{1+|u_0^\varepsilon|^2}\big)u_1^\varepsilon+\tfrac{u_0^\varepsilon\overline{u}_1^\varepsilon+\overline{u}_0^\varepsilon u_1^\varepsilon}{2(1+|u_0^\varepsilon|^2)^{\frac{3}{2}}}\Delta\big(\sqrt{1+|u_0^\varepsilon|^2}\big)u_0^\varepsilon\\ &\qquad\qquad +\tfrac{u_0^\varepsilon}{\sqrt{1+|u_0^\varepsilon|^2}}\Delta\left(\tfrac{u_0^\varepsilon\overline{u}_1^\varepsilon+\overline{u}_0^\varepsilon u_1^\varepsilon}{2(1+|u_0^\varepsilon|^2)^{\frac{3}{2}}}\right)+\tfrac{2i}{\varepsilon^2}\big(\Delta u_0^\varepsilon+2iu_1^\varepsilon\\ &\qquad\qquad -\big(\tfrac{1}{\sqrt{1+|u_0^\varepsilon|^2}}-1\big)u_0^\varepsilon-\tfrac{1}{\sqrt{1+|u_0^\varepsilon|^2}}\Delta\big(\sqrt{1+|u_0^\varepsilon|^2}\big)u_0^\varepsilon\big)\Big).\end{aligned} \tag{26}$$

defines a mapping $\mathcal{S}$

$$\begin{aligned}\mathcal{S}:\ &\mathcal{Z}\longrightarrow\mathcal{Z}\\ &V\longmapsto U.\end{aligned}$$

The initial conditions (26) are derived from equations (16), (17), (18), (19) and (20). To prove Theorem 1.1, we have to find a time T and constants M and r such that $\mathcal{S}$ maps the closed ball $X_T(M,r)$ into itself and is a contracting mapping in the norm

$$\sup_{t\in[0,T]} \sum_{j=0}^{5} ||v_j||_2.$$

The first thing to do is to estimate u_4 by using equation (24). As we have already said in the introduction of this article, we will treat (24) as a wave equation. Multiplying equation (24) by $\mathcal{A}(v_0)$ and applying the operator $(1-\Delta)$, we obtain denoting $\phi_4 = (1-\Delta)u_4$

$$\begin{aligned}
&-\varepsilon^2 \begin{pmatrix} (\phi_4)_{tt} \\ (\overline{\phi}_4)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (\phi_4)_t \\ (\overline{\phi}_4)_t \end{pmatrix} + \mathcal{A}(v_0) \begin{pmatrix} \Delta\phi_4 \\ \Delta\overline{\phi}_4 \end{pmatrix} + \sum_{k=1}^{2} \mathcal{L}(v_0, v_k, \partial_k v_0) \begin{pmatrix} \partial_k\phi_4 \\ \partial_k\overline{\phi}_4 \end{pmatrix} \\
&\sum_{|\alpha|\le 2} \mathcal{M}_\alpha(D^{h_1-1}V, D^{h_2-1}Tv_3) \begin{pmatrix} \partial_1^{\alpha_1}\partial_2^{\alpha_2} u_4 \\ \partial_1^{\alpha_1}\partial_2^{\alpha_2}\overline{u}_4 \end{pmatrix} \\
&+ \sum_{|\alpha|\le 2} \sum_{j,k=0}^{2} \mathcal{N}_\alpha(D^{h_2}V, D^{h_2-1}Tv_3) \begin{pmatrix} \partial_1^{\alpha_1}\partial_2^{\alpha_2} R_{jk} u_4 \\ \partial_1^{\alpha_1}\partial_2^{\alpha_2} R_{jk}\overline{u}_4 \end{pmatrix} = 0.
\end{aligned} \tag{27}$$

where the matrices $\mathcal{M}_\alpha$ depend on the multi-index $\alpha \in \mathbb{N}^2$, $\mathcal{N}_\alpha$ on α, j and k

$$\mathcal{L}(v_0, v_k, \partial_k v_0) = \mathcal{A}(v_0)\mathcal{D}_k(v_0, v_k) + 2\partial_k \mathcal{A}(v_0),$$

$$h_1 = 3 \quad \text{if } |\alpha| = 1, 2, \quad h_1 = 4 \quad \text{if } |\alpha| = 0,$$

$$h_2 = 3 - |\alpha|.$$

Finally, differentiating (27) with respect to x_m for $m = 1, 2$ and denoting $\phi_4^m = \partial_m \phi_4$, we obtain

$$\begin{aligned}
&-\varepsilon^2 \begin{pmatrix} (\phi_4^m)_{tt} \\ (\overline{\phi}_4^m)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (\phi_4^m)_t \\ (\overline{\phi}_4^m)_t \end{pmatrix} + \mathcal{A}(v_0) \begin{pmatrix} \Delta\phi_4^m \\ \Delta\overline{\phi}_4^m \end{pmatrix} + \sum_{k=1}^{2} \mathcal{L}(v_0, v_k, \partial_k v_0) \begin{pmatrix} \partial_k\phi_4^m \\ \partial_k\overline{\phi}_4^m \end{pmatrix} \\
&+ \sum_{k=1}^{2} \partial_m \mathcal{A}(v_0) \begin{pmatrix} \partial_k\phi_4^k \\ \partial_k\overline{\phi}_4^k \end{pmatrix} + \sum_{|\alpha|\le 3} \widetilde{\mathcal{M}}_\alpha(D^{h_1}V, D^{h_2}Tv_3) \begin{pmatrix} \partial_1^{\alpha_1}\partial_2^{\alpha_2} u_4 \\ \partial_1^{\alpha_1}\partial_2^{\alpha_2}\overline{u}_4 \end{pmatrix} \\
&+ \sum_{|\alpha|\le 3} \sum_{j,k=0}^{2} \widetilde{\mathcal{N}}_\alpha(D^{h_2+1}V, D^{h_2}Tv_3) \begin{pmatrix} \partial_1^{\alpha_1}\partial_2^{\alpha_2} R_{jk} u_4 \\ \partial_1^{\alpha_1}\partial_2^{\alpha_2} R_{jk}\overline{u}_4 \end{pmatrix} = 0.
\end{aligned} \tag{28}$$

where

$$h_1 = 2, \quad h_2 = 0 \quad \text{if } |\alpha| = 3.$$

The matrices $\widetilde{\mathcal{M}}_\alpha$ and $\widetilde{\mathcal{N}}_\alpha$ depend on α for $|\alpha| \le 3$. They also depend on respectively $D^{h_1}V$, $D^{h_2}Tv_3$ and $D^{h_2+1}V$, $D^{h_2}Tv_3$. We have to notice that in (27), the differentiation of

$$\mathcal{A}(v_0) \begin{pmatrix} \Delta\phi_4 \\ \Delta\overline{\phi}_4 \end{pmatrix}$$

with respect to x_m for a fixed m gives the term

$$\partial_m \left(\mathcal{A}(v_0)\right) \begin{pmatrix} \Delta\phi_4 \\ \Delta\overline{\phi}_4 \end{pmatrix}$$

which is rewritten in (28) as

$$\sum_{k=1}^{2} \partial_m \mathcal{A}(v_0) \begin{pmatrix} \partial_k\phi_4^k \\ \partial_k\overline{\phi}_4^k \end{pmatrix}.$$

We also recall here the expression of $\mathcal{A}^{-1}(v_0)$

$$\mathcal{A}^{-1}(v_0) = \frac{1}{2}\begin{pmatrix} 2+|v_0|^2 & -v_0^2 \\ v_0^2 & -(2+|v_0|^2) \end{pmatrix}.$$

We multiply equation (28) by $\mathcal{A}^{-1}(v_0)$ and we consider the first line of the resulting equation. Furthermore, we add to both side of the equation the term $-\phi_4^m$. We can now perform the usual energy estimate for wave equations, namely we multiply the equation by $(\overline{\phi}_4^m)_t$ and we integrate over $\mathbb{R}^2$.

$$\begin{aligned}
&\varepsilon^2 \int_{\mathbb{R}^2} \frac{(2+|v_0|^2)}{2} (\phi_4^m)_{tt} (\overline{\phi}_4^m)_t dx - \varepsilon^2 \int_{\mathbb{R}^2} \frac{v_0^2}{2} (\overline{\phi}_4^m)_{tt} (\overline{\phi}_4^m)_t dx \\
&-i \int_{\mathbb{R}^2} (2+|v_0|^2) |(\phi_4^m)_t|^2 dx + i \int_{\mathbb{R}^2} v_0^2 (\overline{\phi}_4^m)_t (\overline{\phi}_4^m)_t dx \\
&-\int_{\mathbb{R}^2} \Delta \phi_4^m (\overline{\phi}_4^m)_t dx - \sum_{k=1}^2 \int_{\mathbb{R}^2} \mathcal{O}^{11}(v_0, v_k, \partial_k v_0) \partial_k \phi_4^m (\overline{\phi}_4^m)_t dx \\
&-\sum_{k=1}^2 \int_{\mathbb{R}^2} \mathcal{O}^{12}(v_0, v_k, \partial_k v_0) \partial_k \overline{\phi}_4^m (\overline{\phi}_4^m)_t dx + \int_{\mathbb{R}^2} \phi_4^m (\overline{\phi}_4^m)_t dx \\
&-\sum_{k=1}^2 \int_{\mathbb{R}^2} \left(\mathcal{A}^{-1}(v_0) \partial_m \mathcal{A}(v_0)\right)^{11} \partial_k \phi_4^k (\overline{\phi}_4^m)_t dx - \int_{\mathbb{R}^2} \phi_4^m (\overline{\phi}_4^m)_t dx \\
&-\sum_{k=1}^2 \int_{\mathbb{R}^2} \left(\mathcal{A}^{-1}(v_0) \partial_m \mathcal{A}(v_0)\right)^{12} \partial_k \overline{\phi}_4^k (\overline{\phi}_4^m)_t dx \\
&-\sum_{|\alpha|\le 3} \int_{\mathbb{R}^2} \mathcal{P}_\alpha^{11}(D^{h_1}V, D^{h_2}Tv_3)(\partial_1^{\alpha_1}\partial_2^{\alpha_2} u_4)(\overline{\phi}_4^m)_t dx \\
&-\sum_{|\alpha|\le 3} \int_{\mathbb{R}^2} \mathcal{P}_\alpha^{12}(D^{h_1}V, D^{h_2}Tv_3)(\partial_1^{\alpha_1}\partial_2^{\alpha_2} \overline{u}_4)(\overline{\phi}_4^m)_t dx \\
&-\sum_{|\alpha|\le 3} \sum_{j,k=0}^2 \int_{\mathbb{R}^2} \mathcal{Q}_\alpha^{11}(D^{h_2+1}V, D^{h_2}Tv_3)\partial_1^{\alpha_1}\partial_2^{\alpha_2} R_{jk} u_4 (\overline{\phi}_4^m)_t dx \\
&-\sum_{|\alpha|\le 3} \sum_{j,k=0}^2 \int_{\mathbb{R}^2} \mathcal{Q}_\alpha^{12}(D^{h_2+1}V, D^{h_2}Tv_3)\partial_1^{\alpha_1}\partial_2^{\alpha_2} R_{jk} \overline{u}_4 (\overline{\phi}_4^m)_t dx = 0.
\end{aligned} \tag{29}$$

where $\mathcal{O} = \mathcal{A}^{-1}(v_0)\mathcal{L}$, $\mathcal{P}_\alpha = \mathcal{A}^{-1}(v_0)\widetilde{\mathcal{M}}_\alpha$ and $\mathcal{Q}_\alpha = \mathcal{A}^{-1}(v_0)\widetilde{\mathcal{N}}_\alpha$.

We now take the real part of equation (29). The first line of the resulting expression gives

$$\begin{aligned}
&\mathrm{Re}\left(\varepsilon^2 \int_{\mathbb{R}^2} \frac{(2+|v_0|^2)}{2} (\phi_4^m)_{tt} (\overline{\phi}_4^m)_t dx - \varepsilon^2 \int_{\mathbb{R}^2} \frac{v_0^2}{2} (\overline{\phi}_4^m)_{tt} (\overline{\phi}_4^m)_t dx\right) \\
&= \varepsilon^2 \int_{\mathbb{R}^2} \frac{(2+|v_0|^2)}{4} |(\phi_4^m)_t|_t^2 dx - \varepsilon^2 \int_{\mathbb{R}^2} \left(\frac{v_0^2}{8}\frac{d}{dt}(\overline{\phi}_4^m)_t^2 + \frac{\overline{v}_0^2}{8}\frac{d}{dt}(\phi_4^m)_t^2\right) dx \\
&= \frac{d}{dt}\left(\varepsilon^2 \int_{\mathbb{R}^2} \frac{(2+|v_0|^2)}{4} |(\phi_4^m)_t|^2 dx - \varepsilon^2 \int_{\mathbb{R}^2} \left(\frac{v_0^2}{8}(\overline{\phi}_4^m)_t^2 + \frac{\overline{v}_0^2}{8}(\phi_4^m)_t^2\right) dx\right) \\
&- \varepsilon^2 \int_{\mathbb{R}^2} \frac{(2+|v_0|^2)_t}{4} |(\phi_4^m)_t|^2 dx + \varepsilon^2 \int_{\mathbb{R}^2} \left(\frac{(v_0^2)_t}{8}(\overline{\phi}_4^m)_t^2 + \frac{(\overline{v}_0^2)_t}{8}(\phi_4^m)_t^2\right) dx.
\end{aligned}$$

For the second line, since $V \in X_T(M,r)$ we have, using the continuous embedding of $H^2(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$

$$\left| -i \int_{\mathbb{R}^2} (2+|v_0|^2)(\phi_4^m)_t (\overline{\phi}_4^m)_t dx + i \int_{\mathbb{R}^2} v_0^2 (\overline{\phi}_4^m)_t (\overline{\phi}_4^m)_t dx \right| \le C(M) \int_{\mathbb{R}^2} |(\phi_4^m)_t|^2 dx.$$

The first term of the third line is,

$$-\mathrm{Re}\left(\int_{\mathbb{R}^2}\Delta\phi_4^m(\overline{\phi}_4^m)_t dx\right)=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla\phi_4^m|^2dx.$$

Using again the fact that $V\in X_T(M,r)$ and that the matrix $\mathcal{O}_m$ depends only on V and $\partial_k v_0$, we obtain by Cauchy-Schwarz inequality

$$\begin{aligned}&\left|-\sum_{k=1}^{2}\left(\int_{\mathbb{R}^2}\mathcal{O}^{11}(v_0,v_k,\partial_k v_0)\partial_k\phi_4^m(\overline{\phi}_4^m)_t dx+\int_{\mathbb{R}^2}\mathcal{O}^{12}(v_0,v_k,\partial_k v_0)\partial_k\overline{\phi}_4^m(\overline{\phi}_4^m)_t dx\right)\right|\\&\leq C(M)\sum_{k=1}^{2}||\partial_k\phi_4^m||_2||(\phi_4^m)_t||_2\\&\leq C(M)\left(\sum_{k=1}^{2}||\partial_k\phi_4^m||_2^2+||(\phi_4^m)_t||_2^2\right).\end{aligned}$$

In the same way, we have

$$\left|\sum_{k=1}^{2}\int_{\mathbb{R}^2}\left(\mathcal{A}^{-1}(v_0)\partial_m\mathcal{A}(v_0)\right)^{11}\partial_k\phi_4^k(\overline{\phi}_4^m)_t dx\right|\leq C(M)\left(\sum_{k=1}^{2}||\partial_k\phi_4^k||_2^2+||(\phi_4^m)_t||_2^2\right),$$

$$\left|\sum_{k=1}^{2}\int_{\mathbb{R}^2}\left(\mathcal{A}^{-1}(v_0)\partial_m\mathcal{A}(v_0)\right)^{12}\partial_k\overline{\phi}_4^k(\overline{\phi}_4^m)_t dx\right|\leq C(M)\left(\sum_{k=1}^{2}||\partial_k\phi_4^k||_2^2+||(\phi_4^m)_t||_2^2\right).$$

We now treat the two terms containing $\phi_4^m(\overline{\phi}_4^m)_t$ by two different methods. On one hand, one can write

$$\mathrm{Re}\left(\int_{\mathbb{R}^2}\phi_4^m(\overline{\phi}_4^m)_t dx\right)=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|\phi_4^m|^2dx.$$

On the other hand, we derive from Cauchy-Schwarz inequality

$$\left|\int_{\mathbb{R}^2}\phi_4^m(\overline{\phi}_4^m)_t dx\right|\leq C(M)\left(||\phi_4^m||_2^2+||(\phi_4^m)_t||_2^2\right).$$

For the next term, we have to be more careful. If $|\alpha|=3$, $\mathcal{P}_\alpha$ depends on derivatives of V of order less than or equal to 2 and on Tv_3 and its first order derivatives. Thus, it can be estimated in $L^\infty(\mathbb{R}^2)$. The two other terms are estimated by Cauchy-Schwarz inequality. As a consequence, one can find a positive $C(M)$ such that

$$\begin{aligned}\left|\sum_{|\alpha|=3}\int_{\mathbb{R}^2}\mathcal{P}_\alpha^{11}(D^{h_1}V,D^{h_2}Tv_3)(\partial_1^{\alpha_1}\partial_2^{\alpha_2}u_4)(\overline{\phi}_4^m)_t dx\right|&\leq C(M)\sum_{|\alpha|=3}||\partial_1^{\alpha_1}\partial_2^{\alpha_2}u_4||_2||(\phi_4^m)_t||_2\\&\leq C(M)\left(\sum_{k=1}^{2}||\phi_4^k||_2^2+||(\phi_4^m)_t||_2^2\right).\end{aligned}$$

If $|\alpha|=0,..,2$, $\mathcal{P}_\alpha$ depends on derivatives of V of order less or equal to 4 and on derivatives of Tv_3 of order less or equal to 3. Thus it has to be estimated in $L^2(\mathbb{R}^2)$ together with $(\overline{\phi}_4^m)_t$ by Cauchy-Schwarz inequality whereas $\partial_1^\alpha \partial_2^\beta u_4$ is estimated in $L^\infty(\mathbb{R}^2)$.

$$\left| \sum_{|\alpha|=0}^{2} \int_{\mathbb{R}^2} \mathcal{P}_\alpha^{11}(D^{h_1}V, D^{h_2}Tv_3)(\partial_1^{\alpha_1}\partial_2^{\alpha_2}u_4)(\overline{\phi}_4^m)_t dx \right|$$
$$\leq C(M) \sum_{|\alpha|=0}^{2} ||\partial_1^{\alpha_1}\partial_2^{\alpha_2}u_4||_{L^\infty(\mathbb{R}^2)}||(\phi_4^m)_t||_2$$
$$\leq C(M) \left(||\phi_4||_2^2 + \sum_{k=1}^{2}(||\partial_k \phi_4^k||_2^2 + ||\phi_4^k||_2^2) + ||(\phi_4^m)_t||_2^2 \right).$$

In definitive, we have proved that

$$\left| \sum_{|\alpha|\leq 3} \int_{\mathbb{R}^2} \mathcal{P}_\alpha^{11}(D^{h_1}V, D^{h_2}Tv_3)(\partial_1^{\alpha_1}\partial_2^{\alpha_2}u_4)(\overline{\phi}_4^m)_t dx \right|$$
$$\leq C(M) \left(||\phi_4||_2^2 + \sum_{k=1}^{2}(||\phi_4^k||_2^2 + ||\partial_k \phi_4^k||_2^2) + ||(\phi_4^m)_t||_2^2 \right).$$

In the same way, one can also obtain

$$\left| \sum_{|\alpha|\leq 3} \int_{\mathbb{R}^2} \mathcal{P}_\alpha^{12}(D^{h_1}V, D^{h_2}Tv_3)(\partial_1^{\alpha_1}\partial_2^{\alpha_2}\overline{u}_4)(\overline{\phi}_4^m)_t dx \right|$$
$$\leq C(M) \left(||\phi_4||_2^2 + \sum_{k=1}^{2}(||\phi_4^k||_2^2 + ||\partial_k \phi_4^k||_2^2) + ||(\phi_4^m)_t||_2^2 \right).$$

The last two terms are treated exactly in the same way, the only thing to notice is that the operator R_{jk} is continuous from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$. Thus we can write

$$\left| \sum_{|\alpha|\leq 3} \sum_{j,k=0}^{2} \int_{\mathbb{R}^2} \mathcal{Q}_\alpha^{11}(D^{h_2+1}V, D^{h_2}Tv_3)\partial_1^{\alpha_1}\partial_2^{\alpha_2}R_{jk}u_4(\overline{\phi}_4^m)_t dx \right|$$
$$\leq \left(||\phi_4||_2^2 + \sum_{k=1}^{2}(||\phi_4^k||_2^2 + ||\partial_k \phi_4^k||_2^2) + ||(\phi_4^m)_t||_2^2 \right),$$

$$\left| \sum_{|\alpha|\leq 3} \sum_{j,k=0}^{2} \int_{\mathbb{R}^2} \mathcal{Q}_\alpha^{12}(D^{h_2+1}V, D^{h_2}Tv_3)\partial_1^{\alpha_1}\partial_2^{\alpha_2}R_{jk}\overline{u}_4(\overline{\phi}_4^m)_t dx \right|$$
$$\leq \left(||\phi_4||_2^2 + \sum_{k=1}^{2}(||\phi_4^k||_2^2 + ||\partial_k \phi_4^k||_2^2) + ||(\phi_4^m)_t||_2^2 \right).$$

Collecting all these results, we derive from (29)

$$
\begin{aligned}
&\frac{d}{dt}\left(\varepsilon^2\int_{\mathbb{R}^2}\frac{(2+|v_0|^2)}{4}|(\phi_4^m)_t|^2dx-\varepsilon^2\int_{\mathbb{R}^2}\left(\frac{v_0^2}{8}(\overline{\phi}_4^m)_t^2+\frac{\overline{v}_0^2}{8}(\phi_4^m)_t^2\right)dx+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla\phi_4^m|^2dx\right)\\
&+\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}^2}|\phi_4^m|^2dx\right)-\varepsilon^2\int_{\mathbb{R}^2}\frac{|v_0|_t^2}{4}|(\phi_4^m)_t|^2dx+\varepsilon^2\int_{\mathbb{R}^2}\left(\frac{(v_0^2)_t}{8}(\overline{\phi}_4^m)_t^2+\frac{(\overline{v}_0^2)_t}{8}(\phi_4^m)_t^2\right)dx \qquad (30)\\
&\leq C(M)\left(||\phi_4||_2^2+\sum_{k=1}^{2}\left(||\partial_k\phi_4^k||_2^2+||\phi_4^k||_2^2\right)+||(\phi_4^m)_t||_2^2\right).
\end{aligned}
$$

Since $\phi_4=(1-\Delta)u_4$, it is not difficult to derive from (24)

$$
||\phi_4||_2^2\leq C(M)\left(\sum_{k=1}^{2}\left(||\partial_k\phi_4^k||_2^2+||\phi_4^k||_2^2\right)+||(\phi_4^m)_t||_2^2\right).
$$

Then, integrating inequality (30) from 0 to t, one obtains

$$
\begin{aligned}
&\frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}|(\phi_4^m(t))_t|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla\phi_4^m(t)|^2dx+\frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}\frac{|v_0(t)|^2}{2}|(\phi_4^m(t))_t|^2dx\\
&+\frac{1}{2}\int_{\mathbb{R}^2}|\phi_4^m|^2dx-\frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}\left(\frac{v_0^2(t)}{4}(\overline{\phi}_4^m(t))_t^2-\frac{\overline{v}_0^2(t)}{4}(\phi_4^m(t))_t^2\right)dx\\
&\leq\varepsilon^2\int_{\mathbb{R}^2}\frac{(2+|v_0(0)|^2)}{4}|(\phi_4^m)_t(0)|^2dx-\varepsilon^2\int_{\mathbb{R}^2}\left(\frac{v_0^2(0)}{8}(\overline{\phi}_4^m)_t^2(0)+\frac{\overline{v}_0^2(0)}{8}(\phi_4^m)_t^2(0)\right)dx\\
&+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla\phi_4^m(0)|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\phi_4^m(0)|^2dx+\varepsilon^2\int_0^t\int_{\mathbb{R}^2}\frac{(|v_0(s)|^2)_s}{4}|(\phi_4^m(s))_s|^2dxds \qquad (31)\\
&+\varepsilon^2\int_0^t\int_{\mathbb{R}^2}\left(\frac{|(v_0^2(s))_s|}{8}|(\overline{\phi}_4^m(s))_s^2|+\frac{|(\overline{v}_0^2(s))_s|}{8}|(\phi_4^m(s))_s^2|\right)dxds\\
&+C(M)\int_0^t\left(\sum_{k=1}^{2}\left(||\partial_k\phi_4^k(s)||_2^2+||\phi_4^k(s)||_2^2\right)+||(\phi_4^m)_s(s)||_2^2\right)ds.
\end{aligned}
$$

Using the fact for all $s\in[0,t]$, $\frac{|v_0(s)|^2}{2}|(\phi_4(s))_s|^2-\frac{v_0^2(s)}{4}(\overline{\phi}_4^m(s))_s^2-\frac{\overline{v}_0^2(s)}{4}(\phi_4^m(s))_s^2\geq 0$, that $||(u_0)_t||_{L^\infty(\mathbb{R}^2)}\leq r$, and denoting

$$
\begin{aligned}
f^m(0)=&\frac{1}{2}\int_{\mathbb{R}^2}|\nabla_\varepsilon\phi_4^m(0)|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\phi_4^m(0)|^2dx\\
&+\varepsilon^2\int_{\mathbb{R}^2}\frac{(2+|v_0(0)|^2)}{4}|(\phi_4^m)_t(0)|^2dx\\
&-\varepsilon^2\int_{\mathbb{R}^2}\left(\frac{v_0^2(0)}{8}(\overline{\phi}_4^m)_t^2(0)+\frac{\overline{v}_0^2(0)}{8}(\phi_4^m)_t^2(0)\right)dx,
\end{aligned}
$$

we obtain from (31)

$$\begin{aligned}&\sum_{m=1}^{2}\left(\frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}|(\phi_4^m(t))_t|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla\phi_4^m(t)|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\phi_4^m(t)|^2dx\right)\\&\le\sum_{m=1}^{2}f^m(0)+C(M,r)\sum_{m=1}^{2}\int_0^t\left(\sum_{k=1}^{2}\left(||\partial_k\phi_4^m(s)||_2^2+||\phi_4^k(s)||_2^2\right)+||(\phi_4^m(s))_s||_2^2\right)ds.\end{aligned}\tag{32}$$

This implies,

$$\begin{aligned}&\sum_{m=1}^{2}\left(\frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}|(\phi_4^m(t))_t|^2dx+\sum_{k=1}^{2}\int_{\mathbb{R}^2}|\partial_k\phi_4^m(t)|^2dx+\int_{\mathbb{R}^2}|\phi_4^m(t)|^2dx\right)\\&\le\sum_{m=1}^{2}f^m(0)+C(M,r)\sum_{m=1}^{2}\int_0^t\left(\sum_{k=1}^{2}\left(||\partial_k\phi_4^m(s)||_2^2+||\phi_4^k(s)||_2^2\right)+||(\phi_4^m(s))_s||_2^2\right)ds.\end{aligned}\tag{33}$$

Applying Gronwall lemma to (33), we get

$$\begin{aligned}&\sup_{t\in[0,T]}\sum_{m=1}^{2}\left(\frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}|(\phi_4^m(t))_t|^2dx+\sum_{k=1}^{2}\int_{\mathbb{R}^2}|\partial_k\phi_4^m(t)|^2dx+\int_{\mathbb{R}^2}|\phi_4^m(t)|^2dx\right)\\&\le e^{C(M,r)\frac{T}{\varepsilon^2}}\sum_{m=1}^{2}f^m(0).\end{aligned}\tag{34}$$

Then, choosing $C(M,r)T\le 1$ and $u_4(0)$ such that

$$\sum_{m=1}^{2}f^m(0)\le e^{-\frac{T}{\varepsilon^2}}m_4\tag{35}$$

gives

$$\sup_{t\in[0,T]}||u_4(t)||_{H^4(\mathbb{R}^2)}\le m_4.\tag{36}$$

It is important to notice here that (35) implies that

$$||u_4(\cdot,0)||_{H^4(\mathbb{R}^2)}\le e^{-\frac{T}{\varepsilon^2}}m_4.$$

Furthermore, we also have

$$\sup_{t\in[0,T]}\varepsilon||\partial_t u_4(t)||_{H^3(\mathbb{R}^2)}\le m_4.\tag{37}$$

We now study equation (25) satisfied by u_5. This equation is of the same type as equation (24) and it will be treated with the same method; we multiply (25) by $\mathcal{A}(v_0)$ and we apply the operator $\partial_m(1-\Delta)$ for

$m = 1, 2$. Denoting $\phi_5 = (1-\Delta)u_5$ and $\phi_5^m = \partial_m(1-\Delta)u_5$, we obtain

$$
\begin{aligned}
&-\varepsilon^2 \begin{pmatrix} (\phi_5^m)_{tt} \\ (\overline{\phi}_5^m)_{tt} \end{pmatrix} + C(v_0, \Delta^{-1}v_4) \begin{pmatrix} (\phi_5^m)_t \\ (\overline{\phi}_5^m)_t \end{pmatrix} + \mathcal{A}(u_0) \begin{pmatrix} \Delta\phi_5^m \\ \Delta\overline{\phi}_5^m \end{pmatrix} + \sum_{k=1}^{2} \mathcal{L}_k(v_0, v_k, \partial_k v_0) \begin{pmatrix} \partial_k \phi_5^m \\ \partial_k \overline{\phi}_5^m \end{pmatrix} \\
&+ \sum_{k=1}^{2} \partial_m \mathcal{A}(v_0) \begin{pmatrix} \partial_k \phi_5^k \\ \partial_k \overline{\phi}_5^k \end{pmatrix} + \sum_{|\alpha|\leq 3} \mathcal{R}_\alpha(D^{h_1}V, D^{h_2}Tv_3) \begin{pmatrix} \partial_1^{\alpha_1}\partial_2^{\alpha_2} u_5 \\ \partial_1^{\alpha_1}\partial_2^{\alpha_2} \overline{u}_5 \end{pmatrix} \\
&+ \sum_{|\alpha|\leq 3} \sum_{j,k=0}^{2} \mathcal{T}_\alpha(D^{h_2+1}V, D^{h_2}Tv_3) \begin{pmatrix} \partial_1^{\alpha_1}\partial_2^{\alpha_2} R_{jk} u_5 \\ \partial_1^{\alpha_1}\partial_2^{\alpha_2} R_{jk} \overline{u}_5 \end{pmatrix} \\
&+ \sum_{|\alpha|\leq 3} \sum_{j,k=0}^{2} \mathcal{U}_\alpha(D^{h_2+1}V, D^{h_2}Tv_3) \begin{pmatrix} \partial_1^{\alpha_1}\partial_2^{\alpha_2} R_{jk} u_4 \\ \partial_1^{\alpha_1}\partial_2^{\alpha_2} R_{jk} \overline{u}_4 \end{pmatrix} \\
&+ \sum_{|\alpha|\leq 2} \mathcal{V}_\alpha(D^{h_2}V, D^{h_2}\Delta^{-1}u_4) \begin{pmatrix} \partial_1^{\alpha_1}\partial_2^{\alpha_2} (u_5)_t \\ \partial_1^{\alpha_1}\partial_2^{\alpha_2} (\overline{u}_5)_t \end{pmatrix} = 0.
\end{aligned}
\tag{38}
$$

We now multiply equation (38) by $\mathcal{A}^{-1}(v_0)$, and then we take the real part of the first line of the equation multiplied by $(\overline{\phi}_5^m)_t$. The difference with equation (29) is due to the presence of the terms

$$
\sum_{|\alpha|\leq 3} \sum_{j,k=0}^{2} \mathcal{U}_\alpha(D^{h_2+1}V, D^{h_2}Tv_3) \begin{pmatrix} \partial_1^{\alpha_1}\partial_2^{\alpha_2} R_{jk} u_4 \\ \partial_1^{\alpha_1}\partial_2^{\alpha_2} R_{jk} \overline{u}_4 \end{pmatrix}
$$

and

$$
\sum_{|\alpha|\leq 2} \mathcal{V}_\alpha(D^{h_2}V, D^{h_2}\Delta^{-1}u_4) \begin{pmatrix} \partial_1^{\alpha_1}\partial_2^{\alpha_2} (u_5)_t \\ \partial_1^{\alpha_1}\partial_2^{\alpha_2} (\overline{u}_5)_t \end{pmatrix}.
$$

Anyway, these terms do not yield any new difficulty. Indeed, each term of the first sum above can be estimated in the following way

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^2} \mathcal{A}^{-1}(v_0)\mathcal{U}_\alpha(D^{h_2+1}V, D^{h_2}Tv_3)\partial_1^\alpha \partial_2^\beta R_{jk} u_4 (\overline{\phi}_5^m)_t dx \right| \\
&\leq C(M) \left(||\phi_4||_2^2 + \sum_{k=1}^{2} (||\phi_4^k||_2^2 + ||\partial_k \phi_4^k||_2^2) + ||(\phi_5^m)_t||_2^2 \right)
\end{aligned}
$$

and the terms of the second sum

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^2} \mathcal{V}_\alpha(D^{h_2}V, D^{h_2}\Delta^{-1}u_4)\partial_1^\alpha \partial_2^\beta (u_5)_t (\overline{\phi}_5^m)_t dx \right| \\
&\leq C(M) \left(||(\phi_5)_t||_2^2 + \sum_{k=1}^{2} ||(\phi_5^k)_t||_2^2 + ||(\phi_5^m)_t||_2^2 \right).
\end{aligned}
$$

It is then possible to prove, using the same estimate as above, that there exists a positive $C(M,r)$ such that denoting

$$
\begin{aligned}
g^m(0) = &\frac{1}{2} \int_{\mathbb{R}^2} |\nabla \phi_5^m(0)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} |\phi_5^m(0)|^2 dx \\
&+ \varepsilon^2 \int_{\mathbb{R}^2} \frac{(2+|v_0(0)|^2)}{4} |(\phi_5^m)_t(0)|^2 dx \\
&- \varepsilon^2 \int_{\mathbb{R}^2} \left(\frac{v_0^2(0)}{8} (\overline{\phi}_5^m)_t^2(0) + \frac{\overline{v}_0^2(0)}{8} (\phi_5^m)_t^2(0) \right) dx.
\end{aligned}
$$

we have

$$\sum_{m=1}^{2}\left(\frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}|(\phi_5^m(t))_t|^2dx+\frac{1}{2}\sum_{k=1}^{2}\int_{\mathbb{R}^2}|\partial_k\phi_5^m(t)|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\phi_5^m(t)|^2dx\right)$$
$$\leq\sum_{m=1}^{2}g^m(0)+\int_0^t\sum_{m=1}^{2}\left(||\phi_4^m(s)||_2^2+\sum_{k=1}^{2}||\partial_k\phi_4^4(s)||_2^2\right)ds$$
$$+C(M,r)\int_0^t\sum_{m=1}^{2}\left(\frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}|(\phi_5^m(s))_s|^2dx+\frac{1}{2}\sum_{k=1}^{2}\int_{\mathbb{R}^2}|\partial_k\phi_5^m(s)|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\phi_5^m(s)|^2dx\right).$$

Then, using again a Gronwall lemma, we obtain

$$\sup_{t\in[0,T]}||u_5(t)||_{H^4(\mathbb{R}^2)}\leq e^{C(M,r)\frac{T}{\varepsilon^2}}\sum_{m=1}^{2}(g^m(0)+Tf^m(0)). \tag{39}$$

Finally, if $C(M,r)T\leq 1$ and

$$\sum_{m=1}^{2}g^m(0)\leq e^{-\frac{T}{\varepsilon^2}}m_5-T\sum_{m=1}^{2}f^m(0)\quad\text{and}\quad\sum_{m=1}^{2}f^m(0)\leq\frac{e^{-\frac{T}{\varepsilon^2}}}{T}m_5, \tag{40}$$

we obtain

$$\sup_{t\in[0,T]}||u_5(t)||_{H^4(\mathbb{R}^2)}\leq m_5. \tag{41}$$

We can notice that (40) implies that

$$||u_5(\cdot,0)||_{H^4(\mathbb{R}^2)}\leq e^{-\frac{T}{\varepsilon^2}}m_5.$$

We have also proved

$$\sup_{t\in[0,T]}\varepsilon||\partial_t u_5(t)||_{H^3(\mathbb{R}^2)}\leq m_5. \tag{42}$$

In order to obtain an estimate on u_3, we now work on equation (23). As we have already mentionned in the introduction, (23) can be considered as a Schrödinger equation. Indeed, since no terms of the form $(u_3)_{tt}$ appear in (23), it is possible to estimate u_3 in the same way as we did in [8] to estimate 3.5. For this purpose, we set $\chi_3=(1-\Delta)u_3$ and we first apply the operateur $(1-\Delta)$ to (23)

$$\begin{aligned}&2i\begin{pmatrix}(\chi_3)_t\\(\overline{\chi}_3)_t\end{pmatrix}+\mathcal{A}(v_0)\begin{pmatrix}\Delta\chi_3\\\Delta\overline{\chi}_3\end{pmatrix}+\sum_{k=1}^{2}H(v_0,v_k,\partial_k v_0)\begin{pmatrix}\partial_k\chi_3\\\partial_k\overline{\chi}_3\end{pmatrix}\\&-\varepsilon^2\begin{pmatrix}(1-\Delta)v_5\\(1-\Delta)\overline{v}_5\end{pmatrix}+K(D^2V,D^2u_3,D^2Tv_3D^2\Delta^{-1}v_4)=0\end{aligned} \tag{43}$$

where

$$H(v_0,v_k,\partial_k)=\mathcal{E}(v_0,v_k)+2\partial_k\mathcal{A}(v_0),$$

$$
\begin{aligned}
K(D^2V, D^2u_3, D^2Tv_3, D^2\Delta^{-1}v_4) =& \mathcal{I}(V, Tv_3, \Delta^{-1}v_4) - 2\sum_{k=1}^{2} \partial_k \mathcal{A}(v_0) \begin{pmatrix} \partial_k u_3 \\ \partial_k \overline{u}_3 \end{pmatrix} \\
& - \sum_{j=1}^{2} \left(\partial_j^2 \mathcal{I}(V, Tv_3, \Delta^{-1}v_4) + \partial_j^2 \mathcal{A}(v_0) \begin{pmatrix} \Delta u_3 \\ \Delta \overline{u}_3 \end{pmatrix} \right) \\
& - 2\sum_{j=1}^{2}\sum_{k=1}^{2} \partial_j \mathcal{E}(v_0, v_k) \begin{pmatrix} \partial_k\partial_j u_3 \\ \partial_k\partial_j \overline{u}_3 \end{pmatrix} \\
& - \sum_{j=1}^{2}\sum_{k=1}^{2} \partial_j^2 \mathcal{E}(v_0, v_k) \begin{pmatrix} \partial_k u_3 \\ \partial_k u_3 \end{pmatrix}
\end{aligned}
$$

Applying again the operator $(1-\Delta)$ to (43) and setting $\phi_3 = (1-\Delta)\chi_3$, we get

$$
\begin{aligned}
& 2i \begin{pmatrix} (\phi_3)_t \\ (\overline{\phi}_3)_t \end{pmatrix} + \mathcal{A}(v_0) \begin{pmatrix} \Delta\phi_3 \\ \Delta\overline{\phi}_3 \end{pmatrix} + \sum_{k=1}^{2} \mathcal{W}(v_0, v_k, \partial_k v_0) \begin{pmatrix} \partial_k\phi_3 \\ \partial_k\overline{\phi}_3 \end{pmatrix} \\
& -\varepsilon^2 \begin{pmatrix} (1-\Delta)^2 v_5 \\ (1-\Delta)^2 \overline{v}_5 \end{pmatrix} + \mathcal{X}(D^4V, D^4u_3, D^4Tv_3, D^4\Delta^{-1}v_4) = 0
\end{aligned} \tag{44}
$$

where

$$
\mathcal{W}(v_0, v_k, \partial_k) = H(v_0, v_k) + 2\partial_k \mathcal{A}(v_0),
$$

$$
\begin{aligned}
& \mathcal{X}(D^4V, D^4u_3, D^4Tv_3, D^4\Delta^{-1}v_4) \\
& = K(D^2V, D^2u_3, D^2Tv_3, D^2\Delta^{-1}v_4) - 2\sum_{k=1}^{2} \partial_k \mathcal{A}(v_0) \begin{pmatrix} \partial_k\chi_3 \\ \partial_k\overline{\chi}_3 \end{pmatrix} \\
& - \sum_{j=1}^{2} \left(\partial_j^2 K(D^2V, D^2u_3, D^2Tv_3, D^2\Delta^{-1}v_4) + \partial_j^2 \mathcal{A}(v_0) \begin{pmatrix} \Delta\chi_3 \\ \Delta\overline{\chi}_3 \end{pmatrix} \right) \\
& - 2\sum_{j=1}^{2}\sum_{k=1}^{2} \partial_j H(v_0, v_k, \partial_k v_0) \begin{pmatrix} \partial_j\partial_k\chi_3 \\ \partial_j\partial_k\overline{\chi}_3 \end{pmatrix} \\
& - \sum_{j=1}^{2}\sum_{k=1}^{2} \partial_j^2 H(v_0, v_k, \partial_k v_0) \begin{pmatrix} \partial_k\chi_3 \\ \partial_k\overline{\chi}_3 \end{pmatrix}
\end{aligned}
$$

Following the energy method intoduced in [5], we first choose q such that $\mathrm{Re}(\mathcal{E}^{11}(v_0, v_k)) = 0$. The energy method supposes to integrate by parts the terms involving the first space derivatives of ϕ_3. This is why we need the diagonal terms of the matrices $\mathcal{W}$ to be purely imaginary. For this purpose, we have to use a second gauge transform (see [8] for more details). Following this idea, we denote $\psi_3 = e^{p(v_0)}\phi_3$ and we derive from (44)

$$
\begin{aligned}
& 2i \begin{pmatrix} (\psi_3)_t \\ (\overline{\psi}_3)_t \end{pmatrix} + \mathcal{A}(v_0) \begin{pmatrix} \Delta\psi_3 \\ \Delta\overline{\psi}_3 \end{pmatrix} + \sum_{k=1}^{2} \mathcal{Y}(v_0, v_k, \partial_k v_0) \begin{pmatrix} \partial_k\psi_3 \\ \partial_k\overline{\psi}_3 \end{pmatrix} \\
& -\varepsilon^2 \begin{pmatrix} (1-\Delta)^2 v_5 \\ (1-\Delta)^2 \overline{v}_5 \end{pmatrix} + \widetilde{\mathcal{X}}(D^4V, D^5u_3, D^4Tv_3, D^4\Delta^{-1}v_4, (v_0)_t) = 0
\end{aligned} \tag{45}
$$

where

$$
\mathcal{Y}(v_0, v_k, \partial_k v_0) = \mathcal{W}(v_0, v_k, \partial_k v_0) - 2 \begin{pmatrix} \partial_k p(v_0) & 0 \\ 0 & \partial_k p(v_0) \end{pmatrix}
$$

and

$$\widetilde{\mathcal{X}}(D^4V, D^4u_3, D^4Tv_3, D^4\Delta^{-1}v_4, (v_0)_t) = \mathcal{X}(D^4V, D^4u_4, D^4Tv_3, D^4\Delta^{-1}v_4) + 2i(e^{-p(v_0)})_t \begin{pmatrix} \psi_3 \\ \overline{\psi}_3 \end{pmatrix} + \mathcal{A}(v_0)(\Delta e^{-p(v_0)}) \begin{pmatrix} \psi_3 \\ \overline{\psi}_3 \end{pmatrix} + \sum_{k=1}^{2} \mathcal{W}(v_0, v_k, \partial_k v_0)\partial_k(e^{-p(v_0)}) \begin{pmatrix} \phi_3 \\ \overline{\phi}_3 \end{pmatrix}.$$

Once again, an easy calculation shows that there exists a function p such that

$$\mathrm{Re}^{11}\left(\mathcal{Y}(v_0, v_k, \partial_k v_0)\right) = 0.$$

With such q and p, it is possible to make our energy estimates. We multiply (45) by $\mathcal{A}^{-1}(v_0)$ and we integrate over $\mathbb{R}^2$ the first line after having multiplied by $\overline{\psi}_3$

$$\begin{aligned} & i\int_{\mathbb{R}^2} (2+|v_0|^2)(\psi_3)_t\overline{\psi}_3 dx - i\int_{\mathbb{R}^2} v_0^2(\overline{\psi}_3)_t\overline{\psi}_3 dx + \int_{\mathbb{R}^2} \Delta\psi_3\overline{\psi}_3 dx \\ & + \sum_{k=1}^{2}\int_{\mathbb{R}^2} \mathcal{Y}^{11}(v_0, v_k, \partial_k v_0)(\partial_k\psi_3)\overline{\psi}_3 dx + \sum_{k=1}^{2}\int_{\mathbb{R}^2} \mathcal{Y}^{12}(v_0, v_k, \partial_k v_0)(\partial_k\overline{\psi}_3)\overline{\psi}_3 dx \\ & + \int_{\mathbb{R}^2} \widetilde{\mathcal{X}}^1(D^4V, D^4u_3, D^4Tv_3, D^4\Delta^{-1}v_4, (v_0)_t)\overline{\psi}_3 dx - \varepsilon^2\int_{\mathbb{R}^2} \frac{(2+|v_0|^2)}{2}(1-\Delta)^2 v_5\overline{\psi}_3 dx \\ & -\varepsilon^2\int_{\mathbb{R}^2} \frac{v_0^2}{2}(1-\Delta)^2\overline{v}_5\overline{\psi}_3 dx = 0 \end{aligned} \tag{46}$$

where all the matrices $\mathcal{Y}$, $\mathcal{V}$ and $\mathcal{X}$ have been multiplied by $\mathcal{A}^{-1}(v_0)$. We take the imaginary part of equation (46).

$$\begin{aligned} & \mathrm{Im}\left(i\int_{\mathbb{R}^2}(2+|v_0|^2)(\psi_3)_t\overline{\psi}_3 dx - i\int_{\mathbb{R}^2} v_0^2(\overline{\psi}_3)_t\overline{\psi}_3 dx\right) \\ & = \int_{\mathbb{R}^2}\left(\frac{2+|v_0|^2}{2}\right)|\psi_3|_t^2 - \int_{\mathbb{R}^2}\left(\frac{v_0^2}{4}(\overline{\psi}_3^2)_t + \frac{\overline{v_0}^2}{4}(\psi_3^2)_t\right)dx \\ & = \frac{d}{dt}\left(\int_{\mathbb{R}^2}\left(\frac{2+|v_0|^2}{2}\right)|\psi_3|^2 - \int_{\mathbb{R}^2}\left(\frac{v_0^2}{4}(\overline{\psi_3}^2) + \frac{\overline{v_0}^2}{4}(\psi_3^2)\right)dx\right) \\ & - \int_{\mathbb{R}^2}\frac{|v_0|_t^2}{2}|\psi_3|^2 dx + \int_{\mathbb{R}^2}\left(\frac{(v_0^2)_t}{4}(\overline{\psi}_3^2) + \frac{(\overline{v_0}^2)_t}{4}(\psi_3^2)\right)dx. \end{aligned}$$

$$\mathrm{Im}\left(\int_{\mathbb{R}^2}\Delta\psi_3\overline{\psi}_3 dx\right) = -\mathrm{Im}\left(\int_{\mathbb{R}^2}|\nabla\psi_3|^2 dx\right) = 0.$$

Since $\mathcal{Y}^{11}(v_0, v_k, \partial_k v_0) \in i\mathbb{R}$, we have

$$\begin{aligned} \mathrm{Im}\left(\sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathcal{Y}^{11}(v_0, v_k, \partial_k v_0)(\partial_k\psi_3)\overline{\psi}_3 dx\right) & = \sum_{k=1}^{2}\int_{\mathbb{R}^2}\mathrm{Im}(\mathcal{Y}^{11}(v_0, v_k, \partial_k v_0))\partial_k\frac{|\psi_3|^2}{2}dx \\ & = -\sum_{k=1}^{2}\int_{\mathbb{R}^2}\partial_k(\mathrm{Im}(\mathcal{Y}^{11}(v_0, v_k, \partial_k v_0)))\frac{|\psi_3|^2}{2}dx \end{aligned}$$

Using the continuous embedding of $H^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$, we obtain

$$\left| \mathrm{Im} \left(\sum_{k=1}^{2} \int_{\mathbb{R}^2} \mathcal{Y}^{11}(v_0, v_k, \partial_k v_0)(\partial_k \psi_3)\overline{\psi}_3 dx \right) \right| \leq C(M)||\psi_3||_2^2.$$

Since $\partial_k \overline{\psi}_3 \overline{\psi}_3 = \frac{1}{2}\partial_k(\overline{\psi}_3^2)$ an integration by parts gives directly

$$\left| \mathrm{Im} \left(\sum_{k=1}^{2} \int_{\mathbb{R}^2} \mathcal{Y}^{12}(v_0, v_k, \partial_k v_0)(\partial_k \overline{\psi}_3)\overline{\psi}_3 dx \right) \right| \leq C(M)||\psi_3||_2^2.$$

The last terms do not involve space derivatives of ϕ_3. As a consequence, they can be estimated directly by the Cauchy-Schwarz inequality and by the preceding continuous embedding. The only thing to notice is that since $V \in X_T(M, r)$

$$\int_{\mathbb{R}^2} |(1-\Delta)^2 v_5|^2 dx \leq m_5.$$

Thus, recalling that $||(v_0)_t||_{H^2(\mathbb{R}^2)} \leq r$ and using the same computations we did in [8], one can find two constants $C(M, r)$ and $C(M)$ such that

$$\left| \mathrm{Im} \left(\int_{\mathbb{R}^3} \widetilde{\mathcal{X}}^1(D^4 V, D^4 u_3, D^4 T v_3, D^4 \Delta^{-1} v_4, (v_0)_t)\overline{\psi}_3 dx \right) \right| \leq C(M, r)||\psi_3||_2,$$

$$\left| \mathrm{Im} \left(\varepsilon^2 \left(\int_{\mathbb{R}^3} \frac{(2+|v_0|^2)}{2}(1-\Delta)^3 v_5 \overline{\psi}_3 dx - \int_{\mathbb{R}^2} \frac{v_0^2}{2}(1-\Delta)^2 \overline{v}_5 \overline{\psi}_3 dx \right) \right) \right| \leq C(M)||\psi_3||_2.$$

Collecting all these inequalities, we derive from (46)

$$\frac{d}{dt} \left(\int_{\mathbb{R}^2} \left(\frac{2+|v_0|^2}{2} \right) |\psi_3|^2 - \int_{\mathbb{R}^2} \left(\frac{v_0^2}{4}(\overline{\psi}_3^2) + \frac{\overline{v_0}^2}{4}(\psi_3^2) \right) dx \right)$$

$$\leq \int_{\mathbb{R}^2} \frac{|v_0|_t^2}{2}|\psi_3|^2 dx - \int_{\mathbb{R}^2} \left(\frac{(v_0^2)_t}{4}(\overline{\psi}_3^2) + \frac{(\overline{v_0}^2)_t}{4}(\psi_2^2) \right) dx.$$

$$+ C(M)||\psi_3||_2^2 + C(M, r)||\psi_3||_2.$$

Integrating this last inequality from 0 to t, one obtains keeping in view that $||(v_0)_t||_{L^\infty(\mathbb{R}^2)} \leq r$,

$$\begin{aligned} &\int_{\mathbb{R}^2} \left(\frac{2+|v_0(t)|^2}{2} \right) |\psi_3(t)|^2 - \int_{\mathbb{R}^2} \left(\frac{v_0^2(t)}{4}(\overline{\psi}_3^2(t)) + \frac{\overline{v_0}^2(t)}{4}(\psi_3^2(t)) \right) dx \\ &\leq \int_{\mathbb{R}^2} \left(\frac{2+|v_0(0)|^2}{2} \right) |\psi_3(0)|^2 - \int_{\mathbb{R}^2} \left(\frac{v_0^2(0)}{4}(\overline{\psi}_3^2(0)) + \frac{\overline{v_0}^2(0)}{4}(\psi_3^2(0)) \right) dx \\ &\quad + C(M, r)T \left(\sup_{t \in [0,T]} ||\psi_3(t)||_2^2 + 1 \right). \end{aligned}$$

Since for all $t \in [0, T]$,

$$\int_{\mathbb{R}^2} \frac{|v_0(t)|^2}{2}|\psi_3(t)|^2 - \int_{\mathbb{R}^2} \left(\frac{v_0^2(t)}{4}(\overline{\psi}_3^2(t)) + \frac{\overline{v_0}^2(t)}{4}(\psi_3^2(t)) \right) dx \geq 0,$$

we obtain

$$\sup_{t\in[0,T]} ||\psi_3(t)||_2^2 \le \int_{\mathbb{R}^2} (1+|v_0(0)|^2)|\psi_3(0)|^2 dx + C(M,r)T\left(\sup_{t\in[0,T]} ||\psi_3(t)||_2^2 + 1\right),$$

which implies if $C(M,r)T \le \frac{1}{2}$ that

$$\sup_{t\in[0,T]} ||\psi_3(t)||_2^2 \le 2\int_{\mathbb{R}^2} (1+|v_0(0)|^2)|\psi_3(0)|^2 dx + 1.$$

Recalling that p only depends on v_0, there is a constant $C(m_0)$ depending only on m_0 such that

$$||e^{-p(v_0)}||^2_{L^\infty(0,T;H^4(\mathbb{R}^2))} \le C(m_0).$$

Then, since $\phi_3 = e^{-p(v_0)}\psi_3$, one has

$$\begin{aligned}\sup_{t\in[0,T]} ||\phi_3(t)||_2^2 &\le C(m_0) \sup_{t\in[0,T]} ||\psi_3(t)||_2^2 \\ &\le 2C(m_0)\int_{\mathbb{R}^2} (1+|v_0(0)|^2)|\psi_3(0)|^2 dx + C(m_0).\end{aligned}$$

Choosing m_3 such that

$$m_3^2 \ge 4C(m_0)\int_{\mathbb{R}^2} (1+|v_0(0)|^2)|e^{p(v_0(0))}\phi_3(0)|^2 dx + 1, \tag{47}$$

gives finally

$$\sup_{t\in[0,T]} ||\phi_3||^2_{H^4(\mathbb{R}^2)} \le m_3^2. \tag{48}$$

We now have to prove that there exists a time T and constant m_0, m_1 and m_2 such that for all $j = 0,..,2$

$$\sup_{t\in[0,T]} ||u_j(t)||_{H^4(\mathbb{R}^2)} \le m_j, \tag{49}$$

which is done exactly as in [8].

It remains to estimate $(u_0)_t$ in $H^2(\mathbb{R}^2)$ which is made by applying the operator $(1-\Delta)$ to (21) and by doing a direct estimate on the resulting equation. Indeed, doing so, we get

$$\sup_{t\in[0,T]} ||(u_0)_t||_{H^2(\mathbb{R}^2)} \le C(M).$$

It is important to notice that this constant $C(M)$ does not depend on T if we make the estimate before estimating u_3, u_2, u_1 and u_0. Doing so, it is then enough to choose $r \ge C(M)$ to obtain

$$\sup_{t\in[0,T]} ||(u_0)_t||_{H^2(\mathbb{R}^2)} \le r. \tag{50}$$

In conclusion, we have proved that $\mathcal{S}(X_T(M,r)) \subset X_T(M,r)$.

We now show that the mapping $\mathcal{S}$ is a contraction mapping in the ball $X_T(M,r)$ endowed with the metric of Y_T where

$$Y_T = \left\{ \begin{array}{l} U = (u_0,...,u_5) : u_j \in C([0,T]; L^2(\mathbb{R}^2)), \quad j = 0,..,5 \\ ||U||_{Y_T} = \sum_{j=0}^{5} \sup_{t\in[0,T]} ||u_j(t)||_2 < \infty \end{array} \right\}.$$

Let $V^1 = (v_0^1, ..., v_5^1) \in X_T(M,r)$ and $V^2 = (v_0^2, ..., v_5^2) \in X_T(M,r)$. We put

$$U^1 = (u_0^1, ..., u_5^1) = \mathcal{S}V_1$$

and

$$U^2 = (u_0^2, ..., u_5^2) = \mathcal{S}V_2,$$

the solutions of (21), (22), (23), (24) and (25) with initial condition (26). For $j = 0, ..., 5$, we put $w_j = u_j^1 - u_j^2$. The equation satisfied by w_4 is

$$\begin{aligned}
&-\varepsilon^2 \begin{pmatrix} (w_4)_{tt} \\ (\overline{w}_4)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (w_4)_t \\ (\overline{w}_4)_t \end{pmatrix} + \mathcal{A}(v_0^1) \begin{pmatrix} \Delta w_4 \\ \Delta \overline{w}_4 \end{pmatrix} + \left(\mathcal{A}(v_0^1) - \mathcal{A}(v_0^2)\right) \begin{pmatrix} \Delta u_4^2 \\ \Delta \overline{u}_4^2 \end{pmatrix} \\
&+ \sum_{k=1}^{2} \mathcal{D}_k(v_0^1, v_k^1) \begin{pmatrix} \partial_k w_4 \\ \partial_k \overline{w}_4 \end{pmatrix} + \sum_{k=1}^{2} \left(\mathcal{D}_k(v_0^1, v_k^1) - \mathcal{D}_k(v_0^2, v_k^2)\right) \begin{pmatrix} \partial_k u_4^2 \\ \partial_k \overline{u}_4^2 \end{pmatrix} \\
&+ \sum_{j,k=0}^{2} \mathcal{H}_{jk}(DV^1, Tv_3^1) \begin{pmatrix} R_{jk} w_4 \\ R_{jk} \overline{w}_4 \end{pmatrix} + \sum_{j,k=0}^{2} \left(\mathcal{H}_{jk}(DV^1, Tv_3^1) - \mathcal{H}_{jk}(DV^2, Tv_4^2)\right) \begin{pmatrix} R_{jk} u_4^2 \\ R_{jk} \overline{u}_4^2 \end{pmatrix} \\
&+ \mathcal{G}(DV^1, Tv_3^1) \begin{pmatrix} w_4 \\ \overline{w}_4 \end{pmatrix} + \left(\mathcal{G}(DV^1, Tv_3^1) - \mathcal{G}(DV^2, Tv_3^2)\right) \begin{pmatrix} u_4^2 \\ \overline{u}_4^2 \end{pmatrix} = 0
\end{aligned} \tag{51}$$

Treating (51) exactly as we have treated (24) and using the fact that the matrices $\mathcal{D}$, $\mathcal{H}$ and $\mathcal{G}$ are of class C^1 with respect to their arguments, we easily get if T is small enough

$$\sup_{t \in [0,T]} ||w_4||_2 \leq \frac{1}{2} \sup_{t \in [0,T]} \sum_{j=0}^{5} ||v_j^1 - v_j^2||_2. \tag{52}$$

Using the same kind of arguments, it is clear that the following is also true

$$\sup_{t \in [0,T]} ||w_5||_2 \leq \frac{1}{2} \sup_{t \in [0,T]} \sum_{j=0}^{5} ||v_j^1 - v_j^2||_2. \tag{53}$$

Next we write the equation involving w_3

$$\begin{aligned}
&2i \begin{pmatrix} (w_3)_t \\ (\overline{w}_3)_t \end{pmatrix} + \mathcal{A}(v_0^1) \begin{pmatrix} \Delta w_3 \\ \Delta \overline{w}_3 \end{pmatrix} + \left(\mathcal{A}(v_0^1) - \mathcal{A}(v_0^2)\right) \begin{pmatrix} \Delta u_3^2 \\ \Delta \overline{u}_3^2 \end{pmatrix} \\
&+ \sum_{k=1}^{2} \mathcal{E}(v_0^1, v_k^1) \begin{pmatrix} \partial_k w_3 \\ \partial_k \overline{w}_3 \end{pmatrix} + \sum_{k=1}^{2} \left(\mathcal{E}(v_0^1, v_k^1) - \mathcal{E}(v_0^2, v_k^2)\right) \begin{pmatrix} \partial_k u_3^2 \\ \partial_k \overline{u}_3^2 \end{pmatrix} \\
&- \varepsilon^2 \begin{pmatrix} w_5 \\ \overline{w}_5 \end{pmatrix} + \mathcal{I}(V^1, Tv_3^1, \Delta^{-1} v_4) - \mathcal{I}(V^2, Tv_3^2, \Delta^{-1} v_4) = 0
\end{aligned} \tag{54}$$

and we treat it as we have treated (23). We then obtain

$$\sup_{t \in [0,T]} ||w_3||_2 \leq \frac{1}{2} \sup_{t \in [0,T]} \sum_{j=0}^{5} ||v_j^1 - v_j^2||_2 \tag{55}$$

from which follows that for all $j = 0, .., 2$

$$\sup_{t \in [0,T]} ||w_j||_2 \leq \frac{1}{2} \sup_{t \in [0,T]} \sum_{j=0}^{5} ||v_j^1 - v_j^2||_2. \tag{56}$$

Then the mapping $\mathcal{S}$ is a contraction mapping on $X_T(M,r)$ in the norm $||\cdot||_{Y_T}$. The ball $X_T(M,r)$, endowed with the metric of Y_T, is closed. By the contraction mapping principle, there exists a unique solution $U=(u_0,...,u_5)$ of (16), (17), (18), (19) and (20) defined in the interval $[0,T]$ satisfying

$$u_j \in C([0,T];L^2(\mathbb{R}^2)) \cap L^\infty(0,T;H^4(\mathbb{R}^2)), \;\; j=0,...,5. \tag{57}$$

Now, let us set for $j=1,..,2$

$$\widetilde{u}_j = \partial_j u_0$$

and

$$\widetilde{u}_3 = e^{q(|u_0|^2)}\Delta u_0, \;\; \widetilde{u}_4 = \partial_t \Delta u_0 \text{ and } \widetilde{u}_5 = \partial_t^2 \Delta u_0.$$

Then, it is clear that $\widetilde{U}=(u_0,\widetilde{u}_1,...,\widetilde{u}_5)$ satisfies the linear inhomogeneous Cauchy problem (21), (22), (23), (24) and (25) with $V=U$. Hence since $\mathcal{S}$ is a contraction mapping, one has

$$||\widetilde{U}-U||_{Y_T} \leq \frac{1}{2}||U-U||_{Y_T} = 0.$$

As a consequence, we have for $j=1,...,5$, $\widetilde{u}_j = u_j$ and so

$$u_1 = \partial_1 u_0, \;\; u_2 = \partial_2 u_0, \;\; u_3 = e^{q(|u|^2)}\Delta u_0, \;\; u_4 = \partial_t \Delta u_0, \;\; u_5 = \partial_t^2 \Delta u_0. \tag{58}$$

This implies that u_0 is solution of equation (7) with initial condition (5).

Moreover (36), (41), (48) and (49) imply that the unique solution of (7) given by $u=u_0$ satisfies

$$u \in L^\infty(0,T;H^8(\mathbb{R}^2)). \tag{59}$$

Indeed, (36) and (41) show that there is a positive M such that

$$\varepsilon^2 \sup_{t\in[0,T]} ||\partial_t \Delta u||^2_{H^4(\mathbb{R}^2)} \leq M,$$

$$\varepsilon^2 \sup_{t\in[0,T]} ||\partial_t^2 \Delta u||^2_{H^4(\mathbb{R}^2)} \leq M,$$

Thus, since u satisfies (7), there exists a positive constant M independent of ε such that

$$\sup_{t\in[0,T]} ||u||_{H^8(\mathbb{R}^2)} \leq M \tag{60}$$

(to obtain (60), we first apply the operator $(1-\Delta)$ to (7) which allows to include the term $\Delta(\sqrt{1+|u|^2})$ in the term $\mathcal{A}(u)\Delta u$). $\square$

4 Convergence of u_ε towards a solution u of (1)

In this section, we prove Theorem 1.2, namely we prove that the solutions of (4) converge towards the solutions of (1). The estimates obtained on u_ε in Section 3.3 are independent of ε. Equipped with such inequalities, the proof of the convergence of u_ε towards u is very classical. More precisely, let $u_0 \in H^6(\mathbb{R}^2)$ and denote by $u(\cdot,t)$ the solution of (1) with initial condition $u(\cdot,0)=u_0(\cdot)$ which exists at least on a finite interval $[0,T_0]$. According to section 2, we have furthermore

$$u \in L^\infty(0,T_0;H^6(\mathbb{R}^2)) \cap C([0,T_0];H^4(\mathbb{R}^2)).$$

Now let $u_0^\varepsilon \in H^8(\mathbb{R}^2)$ and $u_1^\varepsilon \in H^7(\mathbb{R}^2)$ be two sequences of initial data satisfying

$$u_0^\varepsilon \underset{\varepsilon\to+\infty}{\longrightarrow} u_0 \;\; \text{in} \;\; H^6(\mathbb{R}^2), \;\; u_1^\varepsilon \underset{\varepsilon\to+\infty}{\longrightarrow} 0 \;\; \text{in} \;\; H^7(\mathbb{R}^2). \tag{61}$$

Assume furthermore that u_0^ε and u_1^ε satisfy the hypothesis of Theorem 1.1. Then, there exists a time T_1 independent of ε such that for all $\varepsilon > 0$, there is a unique solution u_ε to the Cauchy problem (4) with initial condition

$$u_\varepsilon(\cdot,0) = u_0^\varepsilon(\cdot), \quad \partial_t u_\varepsilon(\cdot,0) = u_1^\varepsilon(\cdot).$$

Furthermore, we have

$$u_\varepsilon \in L^\infty(0,T_1;H^8(\mathbb{R}^2)) \cap C([0,T_1];H^4(\mathbb{R}^2)),$$

and there exists a constant M independent of ε such that for all $\varepsilon > 0$,

$$\sup_{t\in[0,T_1]} \Big(||u_\varepsilon||^2_{H^8(\mathbb{R}^2)} + \varepsilon^2 ||\partial_t u_\varepsilon||^2_{H^6(\mathbb{R}^2)} + \varepsilon^2 ||\partial_t^2 u_\varepsilon||^2_{H^6(\mathbb{R}^2)} \Big) \le M^2. \tag{62}$$

We denote by $T = \inf(T_0, T_1)$. Our goal is to show that u_ε tends to u as ε goes to zero in the space

$$L^\infty(0,T;H^6(\mathbb{R}^2)).$$

The proof uses an energy estimate of the same type as (46). The equation satisfied by u_ε is

$$\begin{aligned} &-\varepsilon^2 \begin{pmatrix} (u_\varepsilon)_{tt} \\ (\overline{u}_\varepsilon)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (u_\varepsilon)_t \\ (\overline{u}_\varepsilon)_t \end{pmatrix} + \mathcal{A}(u_\varepsilon) \begin{pmatrix} \Delta u_\varepsilon \\ \Delta \overline{u}_\varepsilon \end{pmatrix} \\ &- \begin{pmatrix} \frac{u_\varepsilon}{\gamma_\varepsilon^2} |\nabla u_\varepsilon|^2 - \frac{u_\varepsilon}{4\gamma_\varepsilon^4} |\nabla |u_\varepsilon|^2|^2 + u_\varepsilon g(|u_\varepsilon|^2) \\ \frac{-\overline{u_\varepsilon}}{\gamma_\varepsilon^2} |\nabla u_\varepsilon|^2 + \frac{\overline{u_\varepsilon}}{4\gamma_\varepsilon^4} |\nabla |u_\varepsilon|^2|^2 - \overline{u_\varepsilon} g(|u_\varepsilon|^2) \end{pmatrix} = 0, \end{aligned} \tag{63}$$

while u satisfies

$$2i \begin{pmatrix} (u)_t \\ (\overline{u})_t \end{pmatrix} + \mathcal{A}(u) \begin{pmatrix} \Delta u \\ \Delta \overline{u} \end{pmatrix} - \begin{pmatrix} \frac{u}{\gamma^2} |\nabla u|^2 - \frac{u}{4\gamma^4} |\nabla |u|^2|^2 + u g(|u|^2) \\ \frac{-\overline{u}}{\gamma^2} |\nabla u|^2 + \frac{\overline{u}}{4\gamma^4} |\nabla |u|^2|^2 - \overline{u} g(|u|^2) \end{pmatrix} = 0. \tag{64}$$

As usual, we apply to (63) the operator $(1-\Delta)$ and we set $\phi_\varepsilon^1 = (1-\Delta)u_\varepsilon$ to obtain

$$\begin{aligned} &-\varepsilon^2 \begin{pmatrix} (\phi_\varepsilon^1)_{tt} \\ (\overline{\phi}_\varepsilon^1)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (\phi_\varepsilon^1)_t \\ (\overline{\phi}_\varepsilon^1)_t \end{pmatrix} + \mathcal{A}(u_\varepsilon) \begin{pmatrix} \Delta \phi_\varepsilon^1 \\ \Delta \overline{\phi}_\varepsilon^1 \end{pmatrix} + \sum_{k=1}^2 \mathcal{E}(u_\varepsilon, \partial_k u_\varepsilon) \begin{pmatrix} \partial_k \phi_\varepsilon^1 \\ \partial_k \overline{\phi}_\varepsilon^1 \end{pmatrix} \\ &+ \mathcal{I}_1(D^2 u_\varepsilon) = 0. \end{aligned} \tag{65}$$

where

$$\begin{aligned} \mathcal{I}_1(D^2 u_\varepsilon) = &\sum_{j=1}^2 \partial_j C(u_\varepsilon, \partial_j u_\varepsilon) \begin{pmatrix} \Delta u_\varepsilon \\ \Delta \overline{u}_\varepsilon \end{pmatrix} \\ &+ \sum_{k=1}^2 \partial_j B(u_\varepsilon, \partial_k u_\varepsilon) \begin{pmatrix} \partial_j \partial_k u_\varepsilon \\ \partial_j \partial_k \overline{u}_\varepsilon \end{pmatrix} - \sum_{j=1}^2 \begin{pmatrix} \partial_j F(Du_\varepsilon) \\ -\partial_j \overline{F}(Du_\varepsilon) \end{pmatrix}. \end{aligned}$$

The matrices $\mathcal{E}$, B, C and F are the same as in section 2.2. The only thing to notice is that in F, we formally replace the argument u_j by $\partial_j u_\varepsilon$.

Applying again the operator $(1-\Delta)$ to (65) and setting $\phi_\varepsilon^2 = (1-\Delta)\phi_\varepsilon^1$, we can write

$$\begin{aligned} &-\varepsilon^2 \begin{pmatrix} (\phi_\varepsilon^2)_{tt} \\ (\overline{\phi}_\varepsilon^2)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (\phi_\varepsilon^2)_t \\ (\overline{\phi}_\varepsilon^2)_t \end{pmatrix} + \mathcal{A}(u_\varepsilon) \begin{pmatrix} \Delta \phi_\varepsilon^2 \\ \Delta \overline{\phi}_\varepsilon^2 \end{pmatrix} + \sum_{k=1}^2 \mathcal{H}(u_\varepsilon, \partial_k u_\varepsilon) \begin{pmatrix} \partial_k \phi_\varepsilon^2 \\ \partial_k \overline{\phi}_\varepsilon^2 \end{pmatrix} \\ &+ \mathcal{I}_2(D^4 u_\varepsilon) = 0. \end{aligned} \tag{66}$$

where

$$\mathcal{H}(u_\varepsilon, \partial_k u_\varepsilon) = \mathcal{E}(u_\varepsilon, \partial_k u_\varepsilon) + 2\partial_k \mathcal{A}(u_\varepsilon),$$

$$
\begin{aligned}
\mathcal{I}_2(D^4u_\varepsilon) =& \mathcal{I}_1(D^2u_\varepsilon) - 2\sum_{k=1}^{2}\partial_k\mathcal{A}(u_\varepsilon)\begin{pmatrix}\partial_k\phi_\varepsilon^1\\ \partial_k\overline{\phi}_\varepsilon^1\end{pmatrix}\\
&-\sum_{j=1}^{2}\left(\partial_j^2\mathcal{I}_1(D^2u_\varepsilon)+\partial_j^2\mathcal{A}(u_\varepsilon)\begin{pmatrix}\Delta\phi_\varepsilon^1\\ \Delta\overline{\phi}_\varepsilon^1\end{pmatrix}\right)\\
&-2\sum_{j=1}^{2}\sum_{k=1}^{2}\partial_j\mathcal{E}(u_\varepsilon,\partial_ku_\varepsilon)\begin{pmatrix}\partial_k\partial_j\phi_\varepsilon^1\\ \partial_k\partial_j\overline{\phi}_\varepsilon^1\end{pmatrix}\\
&-\sum_{j=1}^{2}\sum_{k=1}^{2}\partial_j^2\mathcal{E}(u_\varepsilon,\partial_ku_\varepsilon)\begin{pmatrix}\partial_k\phi_\varepsilon^1\\ \partial_k\phi_\varepsilon^1\end{pmatrix}
\end{aligned}
$$

Finally, we can derive the following equation for $\phi_\varepsilon = (1-\Delta)\phi_\varepsilon^2 = (1-\Delta)^3u_\varepsilon$

$$
\begin{aligned}
&-\varepsilon^2\begin{pmatrix}(\phi_\varepsilon)_{tt}\\ (\overline{\phi}_\varepsilon)_{tt}\end{pmatrix}+2i\begin{pmatrix}(\phi_\varepsilon)_t\\ (\overline{\phi}_\varepsilon)_t\end{pmatrix}+\mathcal{A}(u_\varepsilon)\begin{pmatrix}\Delta\phi_\varepsilon\\ \Delta\overline{\phi}_\varepsilon\end{pmatrix}+\sum_{k=1}^{2}\mathcal{W}(u_\varepsilon,\partial_ku_\varepsilon)\begin{pmatrix}\partial_k\phi_\varepsilon\\ \partial_k\overline{\phi}_\varepsilon\end{pmatrix}\\
&+\mathcal{I}_3(D^6u_\varepsilon)=0.
\end{aligned}
\tag{67}
$$

$$
\mathcal{W}(u_\varepsilon,\partial_ku_\varepsilon)=\mathcal{H}(u_\varepsilon,\partial_ku_\varepsilon)+2\partial_k\mathcal{A}(u_\varepsilon),
$$

$$
\begin{aligned}
\mathcal{I}_3(D^6u_\varepsilon) =& \mathcal{I}_2(D^4u_\varepsilon) - 2\sum_{k=1}^{2}\partial_k\mathcal{A}(u_\varepsilon)\begin{pmatrix}\partial_k\phi_\varepsilon^2\\ \partial_k\overline{\phi}_\varepsilon^2\end{pmatrix}\\
&-\sum_{j=1}^{2}\left(\partial_j^2\mathcal{I}_2(D^4u_\varepsilon)+\partial_j^2\mathcal{A}(u_\varepsilon)\begin{pmatrix}\Delta\phi_\varepsilon^2\\ \Delta\overline{\phi}_\varepsilon^2\end{pmatrix}\right)\\
&-2\sum_{j=1}^{2}\sum_{k=1}^{2}\partial_j\mathcal{H}(u_\varepsilon,\partial_ku_\varepsilon)\begin{pmatrix}\partial_j\partial_k\phi_\varepsilon^2\\ \partial_j\partial_k\overline{\phi}_\varepsilon^2\end{pmatrix}\\
&-\sum_{j=1}^{2}\sum_{k=1}^{2}\partial_j^2\mathcal{H}(u_\varepsilon,\partial_ku_\varepsilon)\begin{pmatrix}\partial_k\phi_\varepsilon^2\\ \partial_k\overline{\phi}_\varepsilon^2\end{pmatrix}
\end{aligned}
$$

Keeping the same notations, we can directly write the equation satisfied by $\phi = (1-\Delta)^3u$

$$
\begin{aligned}
&-\varepsilon^2\begin{pmatrix}(\phi)_{tt}\\ (\overline{\phi})_{tt}\end{pmatrix}+2i\begin{pmatrix}(\phi)_t\\ (\overline{\phi})_t\end{pmatrix}+\mathcal{A}(u)\begin{pmatrix}\Delta\phi\\ \Delta\overline{\phi}\end{pmatrix}+\sum_{k=1}^{2}\mathcal{W}(u,\partial_k)\begin{pmatrix}\partial_k\phi\\ \partial_k\overline{\phi}\end{pmatrix}\\
&+\mathcal{I}_3(D^6u)=0.
\end{aligned}
\tag{68}
$$

We are now able to write the equation satisfied by the difference $\chi_\varepsilon = \phi_\varepsilon - \phi$

$$
\begin{aligned}
&-\varepsilon^2\begin{pmatrix}(\phi_\varepsilon)_{tt}\\ (\overline{\phi}_\varepsilon)_{tt}\end{pmatrix}+2i\begin{pmatrix}(\chi_\varepsilon)_t\\ (\overline{\chi}_\varepsilon)_t\end{pmatrix}+\mathcal{A}(u)\begin{pmatrix}\Delta\chi_\varepsilon\\ \Delta\overline{\chi}_\varepsilon\end{pmatrix}+(\mathcal{A}(u_\varepsilon)-\mathcal{A}(u))\begin{pmatrix}\Delta\phi_\varepsilon\\ \Delta\overline{\phi}_\varepsilon\end{pmatrix}\\
&+\sum_{k=1}^{2}\mathcal{W}(u,\partial_ku)\begin{pmatrix}\partial_k\chi_\varepsilon\\ \partial_k\overline{\chi}_\varepsilon\end{pmatrix}+\sum_{k=1}^{2}(\mathcal{W}(u_\varepsilon,\partial_ku_\varepsilon)-\mathcal{W}(u,\partial_ku))\begin{pmatrix}\partial_k\phi_\varepsilon\\ \partial_k\overline{\phi}_\varepsilon\end{pmatrix}\\
&+\mathcal{I}_3(D^6u_\varepsilon)-\mathcal{I}_3(D^6u)=0.
\end{aligned}
\tag{69}
$$

The last thing to do is to use a gauge transform in order to integrate by parts the terms including the first derivatives of χ_ε. Thus, we set $\psi_\varepsilon = e^{p(u)}\chi_\varepsilon$ and we derive

$$\begin{aligned}
&-\varepsilon^2 e^{p(u)} \begin{pmatrix} (\phi_\varepsilon)_{tt} \\ (\overline{\phi}_\varepsilon)_{tt} \end{pmatrix} + 2i \begin{pmatrix} (\psi_\varepsilon)_t \\ (\overline{\psi}_\varepsilon)_t \end{pmatrix} + \mathcal{A}(u) \begin{pmatrix} \Delta\psi_\varepsilon \\ \Delta\overline{\psi}_\varepsilon \end{pmatrix} + e^{p(u)} \left(\mathcal{A}(u_\varepsilon) - \mathcal{A}(u)\right) \begin{pmatrix} \Delta\phi_\varepsilon \\ \Delta\overline{\phi}_\varepsilon \end{pmatrix} \\
&+\sum_{k=1}^{2} \mathcal{X}(u, \partial_k u) \begin{pmatrix} \partial_k\psi_\varepsilon \\ \partial_k\overline{\psi}_\varepsilon \end{pmatrix} + \sum_{k=1}^{2} e^{p(u)} \left(\mathcal{W}(u_\varepsilon, \partial_k u_\varepsilon) - \mathcal{W}(u, \partial_k u)\right) \begin{pmatrix} \partial_k\phi_\varepsilon \\ \partial_k\overline{\phi}_\varepsilon \end{pmatrix} \\
&+e^{p(u)} \left(\mathcal{I}_3(D^6 u_\varepsilon) - \mathcal{I}_3(D^6 u)\right) + \Big(2i\partial_t(e^{-p(u)}) + \Delta(-e^{p(u)})\Big) \begin{pmatrix} \psi_\varepsilon \\ \overline{\psi}_\varepsilon \end{pmatrix} = 0,
\end{aligned} \tag{70}$$

where

$$\mathcal{X}(u, \partial_k u) = \mathcal{W}(u, \partial_k u) - 2 \begin{pmatrix} \partial_k p(u) & 0 \\ 0 & \partial_k p(u) \end{pmatrix}.$$

Multiplying equation (70) by $\mathcal{A}^{-1}(u)$, multiplying by $\overline{\psi}_\varepsilon$ and integrating over $\mathbb{R}^2$ the first line of the resulting equation, we obtain

$$\begin{aligned}
&-\varepsilon^2 \int_{\mathbb{R}^2} e^{p(u)} \frac{(2+|u|^2)}{2} (\phi_\varepsilon)_{tt} \overline{\psi}_\varepsilon dx + \varepsilon^2 \int_{\mathbb{R}^2} e^{p(u)} \frac{u^2}{2} (\overline{\phi}_\varepsilon)_{tt} \overline{\psi}_\varepsilon dx + i \int_{\mathbb{R}^2} (2+|u|^2)(\psi_\varepsilon)_t \overline{\psi}_\varepsilon dx \\
&-i \int_{\mathbb{R}^2} u^2 (\overline{\psi}_\varepsilon)_t \overline{\psi}_\varepsilon dx + \int_{\mathbb{R}^2} \Delta\psi_\varepsilon \overline{\psi}_\varepsilon dx + \sum_{k=1}^{2} \int_{\mathbb{R}^2} \mathcal{X}^{11}(u, \partial_k u)(\partial_k \psi_\varepsilon) \overline{\psi}_\varepsilon dx \\
&+\sum_{k=1}^{2} \int_{\mathbb{R}^2} \mathcal{X}^{12}(u, \partial_k u)(\partial_k \overline{\psi}_\varepsilon) \overline{\psi}_\varepsilon dx + \int_{\mathbb{R}^2} \mathcal{B}^{11}(u_\varepsilon, u) \Delta\phi_\varepsilon \overline{\psi}_\varepsilon dx + \int_{\mathbb{R}^2} \mathcal{B}^{12}(u_\varepsilon, u) \Delta\overline{\phi}_\varepsilon \overline{\psi}_\varepsilon dx \\
&+\sum_{k=1}^{2} \int_{\mathbb{R}^2} \mathcal{Y}^{11}(Du_\varepsilon, Du) \partial_k \phi_\varepsilon \overline{\psi}_\varepsilon dx + \sum_{k=1}^{2} \int_{\mathbb{R}^2} \mathcal{Y}^{12}(Du_\varepsilon, Du) \partial_k \overline{\phi}_\varepsilon \overline{\psi}_\varepsilon dx \\
&+\int_{\mathbb{R}^2} J(D^6 u_\varepsilon, D^6 u) \overline{\psi}_\varepsilon dx = 0,
\end{aligned} \tag{71}$$

In equation (71) we have setted

$$\mathcal{B}(u_\varepsilon, u) = e^{p(u)} \mathcal{A}^{-1}(u) \left(\mathcal{A}(u_\varepsilon) - \mathcal{A}(u)\right)$$

$$\mathcal{Y}(Du_\varepsilon, Du) = e^{p(u)} \mathcal{A}^{-1}(u) \left(\mathcal{W}(u_\varepsilon, \partial_k u_\varepsilon) - \mathcal{W}(u, \partial_k u)\right)$$

$$J(D^6 u_\varepsilon, D^6 u) = e^{p(u)} \mathcal{A}^{-1}(u) \left(\mathcal{I}_3(D^6 u_\varepsilon) - \mathcal{I}_3(D^6 u)\right),$$

and we still denote by $\mathcal{X}$ the product $\mathcal{A}^{-1}(u)\mathcal{X}$. We then take the imaginary part of (71). Since

$$\varepsilon \sup_{t \in [0,T]} ||\partial_t^2 u_\varepsilon||_{H^6(\mathbb{R}^2)} \le M,$$

we can write thanks to Cauchy-Schwarz inequality

$$\begin{aligned}
\left| \varepsilon^2 \int_{\mathbb{R}^2} \frac{(2+|u|^2)}{2} (\phi_\varepsilon)_{tt} \overline{\psi}_\varepsilon dx \right| &\le \varepsilon\, C(M) \varepsilon ||\partial_t^2 \phi_\varepsilon||_2 ||\psi_\varepsilon||_2 \\
&\le \varepsilon\, C(M) ||\psi_\varepsilon||_2
\end{aligned}$$

and

$$\left| \varepsilon^2 \int_{\mathbb{R}^2} \frac{u^2}{2} (\overline{\phi}_\varepsilon)_{tt} \overline{\psi}_\varepsilon dx \right| \le \varepsilon\, C(M) ||\psi_\varepsilon||_2.$$

Furthermore,

$$
\begin{aligned}
&\mathrm{Im}\left(i\int_{\mathbb{R}^2}(2+|u|^2)(\psi_\varepsilon)_t\overline{\psi}_\varepsilon dx - i\int_{\mathbb{R}^2}u^2(\overline{\psi}_\varepsilon)_t\overline{\psi}_\varepsilon dx\right)\\
&=\int_{\mathbb{R}^2}\left(\frac{2+|u|^2}{2}\right)|\psi_\varepsilon|_t^2 - \int_{\mathbb{R}^2}\left(\frac{u^2}{4}(\overline{\psi}_\varepsilon^2)_t+\frac{\overline{u}^2}{4}(\psi_\varepsilon^2)_t\right)dx\\
&=\frac{d}{dt}\left(\int_{\mathbb{R}^2}\left(\frac{2+|u|^2}{2}\right)|\psi_\varepsilon|^2 - \int_{\mathbb{R}^2}\left(\frac{u^2}{4}(\overline{\psi_\varepsilon}^2)+\frac{\overline{u}^2}{4}(\psi_\varepsilon^2)\right)dx\right)\\
&-\int_{\mathbb{R}^2}\frac{|u|_t^2}{2}|\psi_\varepsilon|^2dx+\int_{\mathbb{R}^2}\left(\frac{(u^2)_t}{4}(\overline{\psi}_\varepsilon^2)+\frac{(\overline{u}^2)_t}{4}(\psi_\varepsilon^2)\right)dx.
\end{aligned}
$$

$$
\mathrm{Im}\left(\int_{\mathbb{R}^2}\Delta\psi_\varepsilon\overline{\psi}_\varepsilon dx\right) = -\mathrm{Im}\left(\int_{\mathbb{R}^2}|\nabla\psi_\varepsilon|^2dx\right)=0.
$$

We choose now p such that $\mathrm{Re}(\mathcal{X}^{11})=0$. With such a p, we get

$$
\begin{aligned}
\mathrm{Im}\left(\sum_{k=1}^2\int_{\mathbb{R}^2}\mathcal{X}^{11}(u,\partial_k u)(\partial_k\psi_\varepsilon)\overline{\psi}_\varepsilon dx\right) &= \sum_{k=1}^2\int_{\mathbb{R}^2}\mathrm{Im}(\mathcal{X}^{11}(u,\partial_k u))\partial_k\frac{|\psi_\varepsilon|^2}{2}dx\\
&= -\sum_{k=1}^2\int_{\mathbb{R}^2}\partial_k(\mathrm{Im}(\mathcal{X}^{11}(u,\partial_k u)))\frac{|\psi_\varepsilon|^2}{2}dx
\end{aligned}
$$

Using the continuous embedding of $H^2(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$, we obtain

$$
\left|\mathrm{Im}\left(\sum_{k=1}^2\int_{\mathbb{R}^2}\mathcal{X}^{11}(u,\partial_k u)(\partial_k\psi_\varepsilon)\overline{\psi}_\varepsilon dx\right)\right| \le C(M)||\psi_\varepsilon||_2^2.
$$

Since $\partial_k\overline{\psi}_\varepsilon\overline{\psi}_\varepsilon = \frac{1}{2}\partial_k(\overline{\psi}_\varepsilon^2)$ an integration by parts gives directly

$$
\left|\mathrm{Im}\left(\sum_{k=1}^2\int_{\mathbb{R}^2}\mathcal{X}^{12}(u,\partial_k u)(\partial_k\overline{\psi}_\varepsilon)\overline{\psi}_\varepsilon dx\right)\right| \le C(M)||\psi_\varepsilon||_2^2.
$$

Since the matrices $\mathcal{A}$, $\mathcal{W}$ and $\mathcal{I}_3$ are of class C^1 with respect to their arguments, there exists a positive $C(M)$ such that for $i,j=1,2$

$$
\begin{aligned}
\left|\left|\mathcal{B}^{ij}(u_\varepsilon,u)\right|\right|_{L^\infty(\mathbb{R}^2)} &\le C(M)||u_\varepsilon-u||_{L^\infty(\mathbb{R}^2)}\\
&\le C(M)||\psi_\varepsilon||_2,
\end{aligned}
$$

$$
\begin{aligned}
\left|\left|\mathcal{Y}^{ij}(Du_\varepsilon,Du)\right|\right|_{L^\infty(\mathbb{R}^2)} &\le C(M)\left(||u_\varepsilon-u||_{L^\infty(\mathbb{R}^2)}+||\nabla(u_\varepsilon-u)||_{L^\infty(\mathbb{R}^2)}\right)\\
&\le C(M)||\psi_\varepsilon||_2,
\end{aligned}
$$

$$
\left|\left|J^i(D^6u_\varepsilon,D^6u)\right|\right|_{L^2(\mathbb{R}^2)} \le C(M)||\psi_\varepsilon||_2.
$$

This implies using the fact that

$$
||u_\varepsilon||_{H^8(\mathbb{R}^2)} \le M
$$

and the Cauchy-Schwarz inequality

$$\left|\int_{\mathbb{R}^2} \mathcal{B}^{11}(u_\varepsilon, u)\Delta\phi_\varepsilon \overline{\psi}_\varepsilon dx\right| \le C(M)||\psi_\varepsilon||_2^2,$$

$$\left|\int_{\mathbb{R}^2} \mathcal{B}^{12}(u_\varepsilon, u)\Delta\overline{\phi}_\varepsilon \overline{\psi}_\varepsilon dx\right| \le C(M)||\psi_\varepsilon||_2^2,$$

$$\left|\sum_{k=1}^{2}\int_{\mathbb{R}^2} \mathcal{Y}^{11}(Du_\varepsilon, Du)\partial_k\phi_\varepsilon \overline{\psi}_\varepsilon dx\right| \le C(M)||\psi_\varepsilon||_2^2,$$

$$\left|\sum_{k=1}^{2}\int_{\mathbb{R}^2} \mathcal{Y}^{12}(Du_\varepsilon, Du)\partial_k\overline{\phi}_\varepsilon \overline{\psi}_\varepsilon dx\right| \le C(M)||\psi_\varepsilon||_2^2,$$

and

$$\left|\int_{\mathbb{R}^2} J(D^6u_\varepsilon, D^6u)\overline{\psi}_\varepsilon dx\right| \le C(M)||\psi_\varepsilon||_2^2.$$

Collecting again all these results, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{\mathbb{R}^2}\left(\frac{2+|u|^2}{2}\right)|\psi_\varepsilon|^2 - \int_{\mathbb{R}^2}\left(\frac{u^2}{4}(\overline{\psi_\varepsilon}^2) + \frac{\overline{u}^2}{4}(\psi_\varepsilon^2)\right)dx\right)\\
&-\int_{\mathbb{R}^2}\frac{|u|_t^2}{2}|\psi_\varepsilon|^2dx + \int_{\mathbb{R}^2}\left(\frac{(u^2)_t}{4}(\overline{\psi}_\varepsilon^2) + \frac{(\overline{u}^2)_t}{4}(\psi_\varepsilon^2)\right)dx\\
&\le \varepsilon\, C(M)||\psi_\varepsilon||_2 + C(M)||\psi_\varepsilon||_2^2.
\end{aligned} \tag{72}$$

Integrating (72) from 0 to t gives, recalling that

$$\sup_{t\in[0,T]}||u_t||_{L^\infty(\mathbb{R}^2)} \le M,$$

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left(\frac{2+|u(t)|^2}{2}\right)|\psi_\varepsilon(t)|^2 - \int_{\mathbb{R}^2}\left(\frac{u^2(t)}{4}(\overline{\psi_\varepsilon(t)}^2) + \frac{\overline{u(t)}^2}{4}(\psi_\varepsilon^2)\right)dx\\
&\le \int_{\mathbb{R}^2}\left(\frac{2+|u(0)|^2}{2}\right)|\psi_\varepsilon(0)|^2 - \int_{\mathbb{R}^2}\left(\frac{u^2(0)}{4}(\overline{\psi}_\varepsilon(0)^2) + \frac{\overline{u}^2}{4}(\psi_\varepsilon^2(0))\right)dx\\
&+C(M)\int_0^t\left(||\psi_\varepsilon(s)||_2^2 + \varepsilon^2\right)ds
\end{aligned} \tag{73}$$

Denoting

$$\ell(0) = \int_{\mathbb{R}^2}\left(\frac{2+|u(0)|^2}{2}\right)|\psi_\varepsilon(0)|^2 - \int_{\mathbb{R}^2}\left(\frac{u^2(0)}{4}(\overline{\psi}_\varepsilon(0)^2) + \frac{\overline{u}^2}{4}(\psi_\varepsilon^2(0))\right)dx,$$

we classically obtain by Gronwall lemma,

$$\sup_{t\in[0,T]} ||\psi_\varepsilon||_2^2 \leq C(T)(\ell(0)+2\varepsilon^2). \tag{74}$$

This implies

$$\sup_{t\in[0,T]} ||\phi_\varepsilon||_2^2 \leq C(M)\left(\ell(0)+2\varepsilon^2\right). \tag{75}$$

Thanks to hypothesis (61), it is easy to see that

$$\ell(0) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Thus, we finally obtain

$$u_\varepsilon - u \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{in } L^\infty(0,T;H^6(\mathbb{R}^2)),$$

which ends the proof of Theorem 1.2 □

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