Solitons in quadratic media

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Abstract.— In this paper, we investigate the properties of solitonic structures arising in quadratic media. First, we recall the derivation of systems governing the interaction process for waves propagating in such media and we check local and global well-posedness of the corresponding Cauchy problem. Then, we look for stationary states in the context of normal or anomalous dispersion regimes, that lead us to either elliptic or nonelliptic systems and we address the problem of orbital stability. Finally, some numerical experiments are made in order to compute localized states for several regimes and to study dynamic stability as well as long-time asymptotics.

1 Introduction

Solitons have been subject to very intensive studies as self-guided finite-energy waves propagating in nonlinear media. Historically, a special attention has been paid to Kerr-solitons occurring in $\chi^{(3)}$ centrosymmetric cubic media that play an essential role in various physical contexts. A typical governing model is the nonlinear Schrödinger equation (NLS)

$$i\frac{\partial u}{\partial z} + \Delta u + |u|^{2\sigma}u = 0 \tag{1.1}$$

where $x \in \mathbb{R}^d$ is the transverse coordinate $(d = 1, 2)$, $z \in \mathbb{R}^+$ is the propagation distance normalized to the diffraction length, $\sigma > 0$ and $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_d^2$ (one can refer to [35] for the derivation of (1.1)). The case $\sigma = 1$ corresponds to Kerr nonlinearity and can be found in several areas in physics such as nonlinear optics, plasma physics as well as propagation in nonlinear fibers. Consider then stationnary waves defined as

$$u(x, z) = e^{i\omega z}\varphi(x)$$

where ω stands for a nonnegative parameter and where $\varphi \in H^1(\mathbb{R}^n)$ is a spatially localized function. This naturally leads us to investigate the nonlinear elliptic problem

$$-\omega\varphi + \Delta\varphi + \varphi^{2\sigma+1} = 0 \tag{1.2}$$

assuming that φ is real valued. In the one-dimensional case, (1.2) reduces to a differential equation that can be explicitely solved with use of adapted boundary conditions. Furthermore, it is well-known that in higher space dimensions, (1.2) admits several solutions. Among all this class, the one that minimizes the L^2 norm is referred as the ground state, as opposed to the other ones known as bound states. Ground states are known to be orbitally stable in the subcritical case $\sigma < 2/d$ and unstable both in the critical case $\sigma = 2/d$ and supercritical case $\sigma > 2/d$. Note that in the two last cases, the instability may result in the formation of a singularity at a finite time (see [8] or the survey of [35] for precise statements of these properties). In the case where the spatial derivative operator is no more elliptic arising when time dispersion is taken into account, such as in the equation

$$i\frac{\partial u}{\partial z} + \Delta_\perp u - \frac{\partial^2 u}{\partial x_{d+1}^2} + |u|^{2\sigma}u = 0 \tag{1.3}$$

where x_{d+1} as to be regarded as the physical time, the situation drastically changes. On one hand, it has been proved in [19] that there does not exist localized solitonic solution of (1.3). On the other one, the question of finite time blow-up remains an open problem for critical or supercritical σ, since virial techniques are no more conclusive in this case.

More recently, the same questions concerning solitonic states have been scrutinized for $\chi^{(2)}$ media, where the nonlinear susceptibility tensor only involves quadratic terms. The corresponding model appears as a system of equations governing the evolution of three interacting waves u_1, u_2 and u_3 with corresponding wavenumbers k_1, k_2 and k_3 under the matching condition $k_3 = k_1 + k_2 + \Delta k$ (u_3 is the fundamental; u_1 and u_2 are the harmonics). This system is referred as second harmonic generation (SHG) system in the literature. It writes

$$\begin{aligned}
2ik_1\frac{\partial u_1}{\partial z} + \Delta_\perp u_1 + \gamma_1 u_3\overline{u}_2\exp(-i\Delta kz) = 0\\
2ik_2\frac{\partial u_2}{\partial z} + \Delta_\perp u_2 + \gamma_2 u_3\overline{u}_1\exp(-i\Delta kz) = 0\\
2ik_3\frac{\partial u_3}{\partial z} + \Delta_\perp u_3 + \gamma_3 u_1 u_2\exp(i\Delta kz) = 0,
\end{aligned} \tag{1.4}$$

where z is the longitudinal variable and where the Laplace operator $\Delta_\perp$ refers to transverse variables. A similar system has been derived in [21] as envelope equations for models based on linearly coupled Zakharov-Kuznetsov and Kadomtsev-Petviashvili equations which describe the interaction between long nonlinear waves in fluid flows. A mathematical study of the Cauchy problem for general systems of Schrödinger equations with triad or cubic interactions, has been made in [18] either in elliptic and nonelliptic cases.

There are relatively few mathematical results on systems of type (4.1). The paper [23] concerns a particular case of (4.1) in higher dimensions with no specific connection with quadratic media.

The aim of this paper is to prove some rigorous mathematical results for the system in space dimension 1, 2 and 3 and to compute localized states for such systems. We will in particular establish that the Cauchy problem is globally well-posed (i.e. no self-focusing phenomena occurs), even when the spatial linear part is not elliptic. In fact, contrarily to equations or systems arising from Kerr media, the system is L^2-subcritical in space dimension $d \leq 3$. Next, we focus on solitary wave solutions of (1.4). For some values of the parameters (including the purely non elliptic case), it is shown that such solutions do not exist. In some of the remaining cases, existence and orbital stability can be proved by classical variational techniques. Moreover stationary states can be calculated with gaussian variational approximation and Newton algorithm.

The paper is organized as follows : in Section 2, special attention is paid to the physical derivation of systems arising in quadratic media, as well as their classification. An extensive study of this matter can be found in the survey article [4] and we recall the main steps of the derivation for the sake of completeness. Section 3 is devoted to the normalization procedure for these systems, in order to deal with a reduced number of significant parameters. In Section 4, the Cauchy problem for the SHG system is studied in the spaces $L^2(\mathbb{R}^{d+1})$ and $H^1(\mathbb{R}^{d+1})$. In Section 5, stationary states of these equations are investigated. Pohozaev identities enable us to show nonexistence properties for various classes of parameters. In particular, we find that if anomalous dispersion is involved for each wave vector (leading to hyperbolic second order operators), no localized states can be obtained, in consistency with the non elliptic NLS equation. Nevertheless, the elliptic-hyperbolic case (involving normal time dispersion in the first equation) is not excluded. We construct (non radial) ground state solitary waves in the elliptic-elliptic case by variational methods and prove the orbital stability of the set of ground states.

In the last two sections, we perform numerical simulations to illustrate the previous results and we conclude by a list of open questions.

2 Derivation of models in $\chi^{(2)}$ media

In order to derive the system of equations governing the interaction process, we start from the Maxwell equation

$$\vec{\mathrm{rot}}\ \vec{\mathrm{rot}}\vec{E} + \frac{1}{c^2}\frac{\partial^2 \vec{D}}{\partial t^2} = 0, \tag{2.1}$$

where $\vec{E} = \vec{E}(t, x, y, z)$ and $\vec{D} = \vec{D}(t, x, y, z)$ respectively stand for the electric field and the electric displacement, c denoting the speed of light in vacuum. These two quantities can be related by means of the polarization $\vec{P}$ with use of the constitutive law $\vec{D} = \vec{E} + 4\pi\vec{P}$. In general, the polarization is given as a nonlocal contribution of the electric field. If we assume that the considered media is quadratic, we only retain the two first terms in the expansion in terms of the electric field and we have

$$\begin{aligned} \vec{P}(t, x, y, z) &= \int_{-\infty}^{t} X^{(1)}(t - t_1)(\vec{E}(t_1, x, y, z))\, dt_1 \\ &+ \int_{-\infty}^{t}\int_{-\infty}^{t} X^{(2)}(t - t_1, t - t_2)(\vec{E}(t_1, x, y, z), \vec{E}(t_2, x, y, z))\, dt_2 dt_1, \end{aligned} \tag{2.2}$$

in which $X^{(1)}$ and $X^{(2)}$ are the susceptibility tensors that do not depend on space, meaning that the polarization at a given point only depends on the electric field at this point. We now first focus on the interaction of 3 elementary quasi-monochromatic stationary waves propagating in this nonlinear media. We then assume that the frequencies of these waves exactly match, that is $\omega_1 + \omega_2 = \omega_3$, and that the wavevectors given by the linear dispersion relations satisfy $|\vec{k}_1(\omega)| + |\vec{k}_2(\omega)| - |\vec{k}_3(\omega)| = \Delta k \ll |\vec{k}_3(\omega)|$, Δk being the phase-mismatch parameter. Last, all the wavevectors are assumed to have the same direction. This implies there is no phase-velocity walk-off effect. We then choose z as the direction of wavevectors and x in the plane containing $\vec{k}_j$ and the energy flows given by the Poynting vectors. We expand $\vec{E}$ as

$$\vec{E}(t, x, y, z) = \sum_{j=1}^{3} E_j(x - \rho_j z, y, z) \exp(i(k_j z - \omega_j t))\vec{v}_j + \text{c.c.},$$

where E_j and ρ_j respectively stand for the complex envelope of the elementary wave with frequency ω_j and the walk-off angle between the wavevector and the Poynting vector, each $\vec{v}_j$ being a given unitary vector. The corresponding expansion for the electronic displacement writes

$$\vec{D}(t, x, y, z) = \sum_{j=1}^{3} D_j(x - \rho_j z, y, z) \exp(i(k_j z - \omega_j t))\vec{v}_j + \text{c.c.}, \tag{2.3}$$

meaning that the directions between each elementary electric field and electronic displacement are the same. If we now transform equation (2.1) using the identity $\vec{\text{rot}}\ \vec{\text{rot}} = -\vec{\Delta} + \vec{\text{grad}}\ \text{div}$ and drop the last term off, which does not affect the leading order of the equation (note that for isotropic linear media, $\vec{D} = \vec{E}$ so that the divergence of $\vec{D}$ vanishes), Maxwell equation then rewrites

$$\Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{D}}{\partial t^2} = 0.$$

We now define the susceptibility coefficients as

$$\chi_j^{(1)}(\omega_j) = \int_{-\infty}^{\infty} \overline{e}_j X^{(1)}(e_j) \exp(i\omega_j t) dt$$

(where $X(t)$ is assumed to vanish if $t < 0$) for the linear polarization and

$$\begin{aligned}
\chi_1^{(2)}(\omega_3, -\omega_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{e}_1 X^{(2)}(t,t')(\overline{e}_2, e_3)\exp(i(\omega_3 t' - \omega_2 t))\,dt'dt \\
\chi_2^{(2)}(\omega_3, -\omega_1) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{e}_2 X^{(2)}(t,t')(\overline{e}_1, e_3)\exp(i(\omega_3 t' - \omega_1 t))\,dt'dt \\
\chi_3^{(2)}(\omega_1, \ \ \omega_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{e}_3 X^{(2)}(t,t')(e_2, e_1)\exp(i(\omega_1 t' + \omega_2 t))\,dt'dt
\end{aligned}$$

for the quadratic polarization. From (2.2), it can be shown that the displacement vectors express as, keeping in the expansion only terms oscillating at carrying frequencies ω_j

$$\begin{aligned}
D_1 &= (1 + 4\pi\chi_1^{(1)})E_1 + 8\pi\chi_1^{(2)} E_3 \overline{E}_2 \exp(-i\Delta k z) \\
D_2 &= (1 + 4\pi\chi_2^{(1)})E_2 + 8\pi\chi_2^{(2)} E_3 \overline{E}_1 \exp(-i\Delta k z) \\
D_3 &= (1 + 4\pi\chi_3^{(1)})E_3 + 8\pi\chi_3^{(2)} E_1 E_2 \exp(i\Delta k z).
\end{aligned}$$

It is possible to get rid of the zero-order terms with use of the linear dispersion relations $k_j^2 = \omega_j^2(1 + 4\pi\chi_j^{(1)})$. Furthermore, since the final system is expected to have soliton-like solutions that can be seen as a balance between diffractive terms and nonlinear terms, special care has to be taken to have an adapted scaling (see for instance [3]). In this case, the second cross derivative in x and z give rise to negligible terms that can be neglected when compared with all the other ones. This finally leads us to the following system :

$$\begin{aligned}
2ik_1\left(\frac{\partial E_1}{\partial z} - \rho_1\frac{\partial E_1}{\partial x}\right) + \Delta_\perp E_1 + \frac{8\pi\omega_1^2}{c^2}\chi_1^{(2)} E_3\overline{E}_2\exp(-i\Delta k z) &= 0 \\
2ik_2\left(\frac{\partial E_2}{\partial z} - \rho_2\frac{\partial E_2}{\partial x}\right) + \Delta_\perp E_2 + \frac{8\pi\omega_2^2}{c^2}\chi_2^{(2)} E_3\overline{E}_1\exp(-i\Delta k z) &= 0 \\
2ik_3\left(\frac{\partial E_3}{\partial z} - \rho_3\frac{\partial E_3}{\partial x}\right) + \Delta_\perp E_3 + \frac{8\pi\omega_3^2}{c^2}\chi_3^{(2)} E_1 E_2\exp(i\Delta k z) &= 0.
\end{aligned} \tag{2.4}$$

Among all the possible cases for exactly matching frequencies $\omega_3 = \omega_1 + \omega_2$, it is possible to investigate the particular case $\omega_2 = 2\omega_1 = 2\omega_2 := 2\omega_0$ and $k_3 = 2k_1 - \Delta k = 2k_2 - \Delta k := 2k_0 - \Delta k$: it means that the two fundamental waves have the same frequency. Nevertheless, two different situations are encountered : in the first one, referred as the SHG of type I, both fundamental waves have ordinary polarization (that implies no walk-off effect) and the second harmonic is of extraordinary polarization. Setting $E_1 = E_2$, system (2.4) simply reduces to

$$\begin{aligned}
2ik_0\frac{\partial E_1}{\partial z} + \Delta_\perp E_1 + \frac{8\pi\omega_0^2}{c^2}\chi_1^{(2)} E_3\overline{E}_1\exp(-i\Delta k z) &= 0 \\
4ik_0\left(\frac{\partial E_3}{\partial z} - \rho_3\frac{\partial E_3}{\partial x}\right) + \Delta_\perp E_3 + \frac{32\pi\omega_0^2}{c^2}\chi_3^{(2)} E_1^2\exp(i\Delta k z) &= 0.
\end{aligned} \tag{2.5}$$

In the so-called SHG of type II, one of the fundamental harmonics has a ordinary polarization and the other one has an extraordinary one. In this case, a walk-off angle will be involved in the corresponding equation. Thus, the two fundamental waves have to be distinguished in the system, that is written as

$$\begin{aligned}
&2ik_0\frac{\partial E_1}{\partial z}+\Delta_\perp E_1+\frac{8\pi\omega_0^2}{c^2}\chi_1^{(2)}E_3\overline{E}_2\exp(-i\Delta kz)=0\\
&2ik_0\left(\frac{\partial E_2}{\partial z}-\rho_2\frac{\partial E_2}{\partial x}\right)+\Delta_\perp E_2+\frac{8\pi\omega_0^2}{c^2}\chi_2^{(2)}E_3\overline{E}_1\exp(-i\Delta kz)=0\\
&4ik_0\left(\frac{\partial E_3}{\partial z}-\rho_3\frac{\partial E_3}{\partial x}\right)+\Delta_\perp E_3+\frac{32\pi\omega_0^2}{c^2}\chi_3^{(2)}E_1E_2\exp(i\Delta kz)=0.
\end{aligned}\qquad(2.6)$$

The existence of soliton-like solution is the result of a mutual trapping between the fundamental and the second harmonic beams and is known in the literature as the $\chi^{(2)}:\chi^{(2)}$ cascade. In some situations, the walk-off resulting from the anisotropy may cause the propagation of these structures along different wave vectors and thus generates so-called “walking” solitons. These structures have been observed in second-harmonic generation experiments.

We are now interested in the pulse propagation in the case where the complex envelopes E_j are no more stationary, in the case of a two-dimensional waveguide. Here, defining the linear permittivity tensor as $\epsilon(\omega)=1+4\pi\int_0^\infty X^{(1)}(t)\exp(i\omega t)dt$, the waves can be confined in the (x,y) direction by a linear refraction index given by $n(\omega_j,x,y)=\sqrt{\epsilon_j(x,y)}$. To be consistent with the (x,y) waveguide, the new term $\omega_j^2(\epsilon_j(x,y)-\epsilon_{j0})E_{j,0}/c^2$ (where $\epsilon_j(x,y)-\epsilon_{j,0}\equiv\varepsilon$) has to be added in the system. In this new context, there is no more spatial walk-off effect and the displacement terms in (2.3) express now

$$\begin{aligned}
D_1&=\epsilon_1E_1+i\frac{\partial\epsilon_1}{\partial\omega_1}\frac{\partial E_1}{\partial t}-\frac{1}{2}\frac{\partial^2\epsilon_1}{\partial\omega_1^2}\frac{\partial^2E_1}{\partial t^2}+8\pi\chi_1^{(2)}E_3\overline{E}_2\exp(-i\Delta kz)\\
D_2&=\epsilon_2E_2+i\frac{\partial\epsilon_2}{\partial\omega_2}\frac{\partial E_2}{\partial t}-\frac{1}{2}\frac{\partial^2\epsilon_2}{\partial\omega_2^2}\frac{\partial^2E_2}{\partial t^2}+8\pi\chi_2^{(2)}E_3\overline{E}_1\exp(-i\Delta kz)\\
D_3&=\epsilon_3E_3+i\frac{\partial\epsilon_3}{\partial\omega_3}\frac{\partial E_3}{\partial t}-\frac{1}{2}\frac{\partial^2\epsilon_3}{\partial\omega_3^2}\frac{\partial^2E_3}{\partial t^2}+8\pi\chi_3^{(2)}E_1E_2\exp(i\Delta kz).
\end{aligned}$$

The method of separation of variables can be used here to simplify the problem. In fact, if we seek a solution of the particular form $E_j(t,x,y,z)=F_j(x,y)E_j(t,z)$, then one has to solve the two-dimensional spectral problem (neglecting the high-order terms)

$$\Delta_\perp F_j+\frac{\omega_j^2}{c^2}(\epsilon_j(x,y)-\epsilon_{j,0})F_j+\eta_j^2F_j=0.\qquad(2.7)$$

Taking then the function $F_{j,0}$ that corresponds to the fundamental mode of (2.7) associated to, say $\eta_{j,0}$, the linear dispersion relation will be now written as $k_j^2=\omega_j^2\epsilon_{j,0}/c^2+\varepsilon\eta_{j,0}^2$. The remaining system can be written in a translating frame along the first wave that allows us to drop the first time derivative off in the first equation. Finally, one obtains the system governing the propagation of waves as

$$\begin{aligned}
&2ik_1\frac{\partial E_1}{\partial z}-k_1k_1''\frac{\partial^2E_1}{\partial t^2}+\chi_1q_{\text{eff}}E_3\overline{E}_2\exp(-i\Delta kz)=0\\
&2ik_2\frac{\partial E_2}{\partial z}-2ik_2(k_1'-k_2')\frac{\partial E_2}{\partial t}-k_2k_2''\frac{\partial^2E_2}{\partial t^2}+\chi_2q_{\text{eff}}E_3\overline{E}_1\exp(-i\Delta kz)=0\\
&2ik_3\frac{\partial E_3}{\partial z}-2ik_3(k_1'-k_3')\frac{\partial E_3}{\partial t}-k_3k_3''\frac{\partial^2E_3}{\partial t^2}+\chi_3q_{\text{eff}}E_1E_2\exp(i\Delta kz)=0.
\end{aligned}\qquad(2.8)$$

Here, we have set $t \equiv t - k_1' z$, $k_j' = \partial k_j/\partial\omega_j$ and $k_j'' = \partial^2 k_j/\partial\omega_j^2$. The coefficient q_{eff} is linked to the fundamental modes of the two-dimensional waveguide.

3 Normalization of the systems in $\chi^{(2)}$ media

As seen in the previous section, all the systems governing two or three wave interactions process involved several physical parameters. In order to study theoretical as well as numerical properties for their solutions, it is convenient to perform scalings and changes of unknown functions in order to decrease the number of significant parameters.

In the SHG type I system in the one dimensional transverse case that writes

$$\begin{aligned} &i\frac{\partial E_1}{\partial z} + \gamma_1 \frac{\partial^2 E_1}{\partial x^2} + \chi_1 E_2 \overline{E}_1 \exp(-i\Delta k z) = 0 \\ &i\frac{\partial E_2}{\partial z} - i\delta_2 \frac{\partial E_2}{\partial x} + \gamma_2 \frac{\partial^2 E_2}{\partial x^2} + \chi_2 E_1^2 \exp(i\Delta k z) = 0, \end{aligned} \tag{3.1}$$

where the coefficient γ_1, γ_2, δ_2, χ_1 and χ_2 can be expressed in terms of the coefficients involved in (2.5), it is possible to use the following scalings. We define r_0 and $z_0 = r_0^2 k_2$ as the input beam size and the diffraction length and we set $x' = x/r_0$, $z' = z/z_0$. Setting now $E_1 = u\sqrt{|\beta\gamma_1\gamma_2|/2\chi_1\chi_2 r_0^4}\exp(i\beta z)$, $E_2 = v/\gamma_1 r_0^2 \exp(i(2\beta + \Delta)z)$ where $\delta = z_d \Delta k$, (3.1) becomes, removing the tilde symbols for the sake of clearness :

$$\begin{aligned} &i\frac{\partial u}{\partial z} + r\frac{\partial^2 u}{\partial x^2} - \beta u + v\overline{u} = 0 \\ &i\sigma\frac{\partial v}{\partial z} - i\delta\frac{\partial v}{\partial x} + s\frac{\partial^2 v}{\partial x^2} - \alpha v + \frac{1}{2}u^2 = 0, \end{aligned} \tag{3.2}$$

where $\delta = \delta_2 r_0/|\gamma_2|$, $\sigma = |\gamma_1/\gamma_2| (\simeq 2)$, $\alpha = \sigma(2\beta + \Delta)$, $r = \operatorname{sgn}\gamma_1$ and $s = \operatorname{sgn}\gamma_2$. In this purely spatial case (where x stands for a spatial transverse variable), the only possible choices for r and s are $r = s = 1$ or $r = s = -1$. In the case where the transverse dimension is 2, all these computations remain valid (it suffices to add the same normalization in y as the one used for x) and the system simply becomes

$$\begin{aligned} &i\frac{\partial u}{\partial z} + r\Delta_\perp u - \beta u + v\overline{u} = 0 \\ &i\sigma\frac{\partial v}{\partial z} - i\delta\frac{\partial v}{\partial x} + \Delta_\perp v - \alpha v + \frac{1}{2}u^2 = 0. \end{aligned} \tag{3.3}$$

We now consider the temporal case, given by the system (2.8). Here, the coefficients in (3.1) are

$$\gamma_1 = -\frac{1}{2}\frac{\partial^2 k_0}{\partial\omega^2}, \quad \gamma_2 = -\frac{1}{2}\frac{\partial^2 k_2}{\partial\omega^2} \quad \text{and} \quad \delta_2 = \frac{\partial k_0}{\partial\omega} - \frac{\partial k_2}{\partial\omega}.$$

We respectively rescale the retarded time and the longitudinal coordinate z in terms of the pulse duration t_0 and the dispersion length $z_0 = 2t_0^2/|\partial^2 k_1/\partial\omega_0^2|$: we set $\tilde{x} = x/t_0$ and $\tilde{z} = z/z_0$. The electric fields are then rescaled as $E_1 =$

$u\sqrt{|\gamma_1\gamma_2|/2\chi_1\chi_2 t_0^4}$ and $E_2 = v|\gamma_1|\exp(i\Delta z)/\chi_1 t_0^2$. Defining $\delta = \delta_2 t_0/|\gamma_2|$, we then find the same system as (3.2). However, in this case, the four combinations $(r, s) = (1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$ are made possible. Furthermore, parameter σ differs from 2 in general, as opposed to the spatial case.

If we now investigate the case of three wave interaction, we start from the following problem without walk-off in the two-dimensional context :

$$\begin{aligned} 2ik_1\frac{\partial E_1}{\partial z} + \Delta_\perp E_1 + \chi_1^{(2)} E_3\overline{E}_2 \exp(-i\Delta kz) = 0 \\ 2ik_2\frac{\partial E_2}{\partial z} + \Delta_\perp E_2 + \chi_2^{(2)} E_3\overline{E}_1 \exp(-i\Delta kz) = 0 \\ 2ik_3\frac{\partial E_3}{\partial z} + \Delta_\perp E_3 + \chi_3^{(2)} E_1 E_2 \exp(i\Delta kz) = 0. \end{aligned} \tag{3.4}$$

We normalized the spatial variables as in the two wave interaction $x' = x/r_0$, $y' = y/r_0$, $z' = z/z_0$ where r_0 is the input beam radius and where $z_0 = r_0^2 k_3$. We define the rescaled fields as $E_1 = u\exp(i\beta_v z)/\sqrt{\chi_2\chi_3}r_0^2$, $E_2 = v\exp(i\beta_u z)/\sqrt{\chi_1\chi_3}r_0^2$, $E_3 = w\exp(i(\beta_u+\beta_v+\Delta)z)/\sqrt{\chi_1\chi_2}r_0^2$. Finally defining $\eta = 2k_1/k_3$, the system to be solved writes

$$\begin{aligned} i\eta\frac{\partial v}{\partial z} + \Delta_\perp v - \eta\beta_v v + w\overline{u} = 0 \\ i(2-\eta)\frac{\partial u}{\partial z} + \Delta_\perp u - (2-\eta)\beta_u u + w\overline{v} = 0 \\ 2i\frac{\partial w}{\partial z} + \Delta_\perp w - 2(\beta_u+\beta_v+\Delta)w + uv = 0. \end{aligned} \tag{3.5}$$

If we now set $w = \eta\beta_v\tilde{w}$, $u = \eta\beta_v\tilde{u}$, $v = \eta\beta_v\tilde{v}$ with the new scaling $x = \tilde{x}/\sqrt{\eta\beta_v}$, $y = \tilde{y}/\sqrt{\eta\beta_v}$, $z = \tilde{z}/\eta\beta_v$, $\alpha = 2(\beta_u+\Delta)/(\eta\beta_v)+2/\eta$, $\gamma = (2-\eta)\beta_u/(\eta\beta_v)$, (3.5) finally reduces to

$$\begin{aligned} i\eta\frac{\partial v}{\partial z} + \Delta_\perp v - v + w\overline{u} = 0 \\ i(2-\eta)\frac{\partial u}{\partial z} + \Delta_\perp u - \gamma u + w\overline{v} = 0 \\ 2i\frac{\partial w}{\partial z} + \Delta_\perp z - \alpha w + uv = 0. \end{aligned} \tag{3.6}$$

In the temporal case, the second order derivatives reduce to time derivative. The same scalings as the one previously given in the context of two wave interaction can be made and leads us to a similar system.

Finally, let us consider the spatiotemporal regime, giving second derivatives both in time and space. In [14], the nonelliptic system

$$\begin{aligned} i\frac{\partial u}{\partial z} + \frac{1}{2}\Delta u + \frac{\gamma_1}{2}\frac{\partial^2 u}{\partial t^2} + \bar{u}v = 0 \\ i\frac{\partial v}{\partial z} + \frac{1}{2\sigma}\Delta v + \frac{\gamma_2}{2}\frac{\partial^2 v}{\partial t^2} + \delta kv + \frac{1}{2}u^2 = 0 \end{aligned} \tag{3.7}$$

has been studied in terms of symmetry breaking instabilities of spatial solitons for which $\partial/\partial t \equiv 0$, both in normal ($\gamma_j < 0$) and anomalous ($\gamma_j > 0$) dispersion regimes. This study leads us to investigate the existence of solitons in these different regimes. Note that in the particular case of solutions that do not depend on the transverse variable, dark solitons (that is that are non-localized with respect to the time coordinate) are obtained for $\gamma_1\gamma_2 < 0$ (see [5], [6], [27]).

4 The Cauchy problem

We consider here the Cauchy problem

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta_\perp u + \gamma_1 \dfrac{\partial^2 u}{\partial z^2} + \bar{u}v = 0 & x \in \mathbb{R}^d,\ z \in \mathbb{R},\quad t > 0 \\[2ex] 2i\dfrac{\partial v}{\partial t} + \Delta_\perp v + \gamma_2 \dfrac{\partial^2 v}{\partial z^2} - \beta v + \dfrac{1}{2}u^2 = 0 & x \in \mathbb{R}^d,\ z \in \mathbb{R},\quad t > 0 \\[2ex] u(0,x,z) = u_0(x,z), \qquad v(0,x,z) = v_0(x,z) & x \in \mathbb{R}^d,\ z \in \mathbb{R} \end{cases} \tag{4.1}$$

where $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ (resp. $\Delta_\perp = \partial^2/\partial x^2$) and where γ_1, γ_2 and β are real constants. Here, the longitudinal variable z has been replaced by the time variable t. In a similar way, we will treat also the three waves interaction system (see [25])

$$\begin{cases} i\left(\dfrac{\partial a_1}{\partial t} + v_1\dfrac{\partial a_1}{\partial z}\right) + \dfrac{v_1}{2k_1}\Delta_\perp a_1 + \omega_1''\dfrac{\partial^2 a_1}{\partial z^2} + V\bar{a}_2 a_3 = 0 & x \in \mathbb{R}^d,\ z \in \mathbb{R},\quad t > 0 \\[2ex] i\left(\dfrac{\partial a_2}{\partial t} + v_2\dfrac{\partial a_2}{\partial z}\right) + \dfrac{v_2}{2k_2}\Delta_\perp a_2 + \omega_2''\dfrac{\partial^2 a_2}{\partial z^2} + V\bar{a}_1 a_3 = 0 & x \in \mathbb{R}^d,\ z \in \mathbb{R},\quad t > 0 \\[2ex] i\left(\dfrac{\partial a_3}{\partial t} + v_3\dfrac{\partial a_3}{\partial z}\right) + \dfrac{v_3}{2k_3}\Delta_\perp a_3 + \omega_3''\dfrac{\partial^2 a_3}{\partial z^2} + V a_1 a_2 = 0 & x \in \mathbb{R}^d,\ z \in \mathbb{R},\quad t > 0 \\[2ex] a_j(0,x,z) = a_{0,j}(x,z) \quad (j = 1,2,3), & x \in \mathbb{R}^d,\ z \in \mathbb{R} \end{cases} \tag{4.2}$$

Here, the coefficients v_j and k_j $(j = 1,2,3)$ are strictly positive and $V \in \mathbb{R}$. Note that system (4.1) (resp. system (4.2)) possesses two formal invariants

$$I(u,v) = \int \left(|u|^2 + 4|v|^2\right) dxdz$$

$$\Big(\text{resp. } I(a_1,a_2,a_3) = \int \left(\frac{1}{2}(|a_1|^2 + |a_2|^2) + |a_3|^2\right) dxdz\Big)$$

and

$$E(u,v) = \int \left(\|\nabla_\perp u\|^2 + \|\nabla_\perp v\|^2 + \gamma_1|\partial_z u|^2 + \gamma_2|\partial_z v|^2 + \beta|v|^2 - \mathrm{Re}(\bar{u}^2 v)\right) dxdz$$

$$\begin{aligned} \Big(\text{resp. } E(a_1,a_2,a_3) &= \frac{1}{2}\int \Big(\frac{v_1}{k_1}\|\nabla_\perp a_1\|^2 + \frac{v_2}{k_2}\|\nabla_\perp a_2\|^2 + \frac{v_3}{k_3}\|\nabla_\perp a_3\|^2 \\ &\quad -\omega_1''|\partial_z a_1|^2 - \omega_2''|\partial_z a_2|^2 - \omega_3''|\partial_z a_3|^2 - 2V\mathrm{Re}(a_1 a_2 \bar{a}_3)\Big) dxdz\Big). \end{aligned}$$

Plainly, the energy E does not control the H^1 norm of the solution in the nonelliptic case, that is when γ_1 and/or γ_2 is negative (resp. when one of the ω_j'' is negative). It can nevertheless be proved that the Cauchy problems (4.1) and (4.2) are globally well-posed, as indicated in the

Theorem 1. *Let $(u_0, v_0) \in (L^2(\mathbb{R}^{d+1}))^2$, $d = 1, 2$ (resp. $(a_{0,1}, a_{0,2}, a_{0,3}) \in (L^2(\mathbb{R}^{d+1}))^3$). Then, there exists a unique solution $(u, v) \in C(\mathbb{R}^+, L^2(\mathbb{R}^{d+1}))^2$ (resp. $(a_1, a_2, a_3) \in C(\mathbb{R}^+, L^2(\mathbb{R}^{d+1}))^3$) of (4.1) (resp. of (4.2)). Moreover, we have*

$$I(u, v) = I(u_0, v_0), \quad \forall\, t \geq 0, \ (resp.\ I(a_1, a_2, a_3) = I(a_{0,1}, a_{0,2}, a_{0,3})).$$

If $(u_0, v_0) \in (H^1(\mathbb{R}^{d+1}))^2$, $d = 1, 2$ (resp. $(a_{0,1}, a_{0,2}, a_{0,3}) \in (H^1(\mathbb{R}^{d+1}))^3$). Then, there exists a unique solution $(u, v) \in C(\mathbb{R}^+, H^1(\mathbb{R}^{d+1}))^2$ (resp. $(a_1, a_2, a_3) \in C(\mathbb{R}^+, H^1(\mathbb{R}^{d+1}))^3$) of (4.1) (resp. of (4.2)) and

$$E(u, v) = E(u_0, v_0), \quad \forall\, t \geq 0, \ (resp.\ E(a_1, a_2, a_3) = E(a_{0,1}, a_{0,2}, a_{0,3})).$$

Proof : it is well-known (see [42]) that the Cauchy problem for the equation

$$i\frac{\partial u}{\partial t} + \Delta u \pm |u|^{2\sigma} u = 0, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}^+ \tag{4.3}$$

is subcritical for $\sigma < 2/d$. In particular, $\sigma = 1/2$ corresponds to a subcritical case for $d \leq 3$. Since the Strichartz estimates used in [42] are the same as those obtained when replacing Δ by any second-order nondegenerate constant coefficient operator (see [19]), both (4.1) and (4.2) are L^2-subcritical in $\mathbb{R}^{d+1}$, $d \leq 2$. This implies local well-posedness in $L^2(\mathbb{R}^{d+1})$. Global well-posedness results from the conservation of I that can be obtained using formal computations for the solutions of (4.1) and (4.2).

If the initial data belong to $H^1(\mathbb{R}^{d+1})$, the classical arguments (see [9], [10]) show that the solution persists in $H^1(\mathbb{R}^{d+1})$ as soon as it persists in $L^2(\mathbb{R}^{d+1})$. The conservation of the energy E is standard (see for instance [20]).

When $\gamma_j > 0$, $j = 1, 2$ (resp. $\omega_j'' > 0$, $j = 1, 2, 3$), that is in the elliptic case, one can easily prove using the Gagliardo-Nirenberg inequality and the conservation of I and E that the solution has a uniformly bounded in time H^1 norm. Proving (or disproving) this property in the other cases (the nonelliptic situation) is an interesting open question.

5 Solitons solutions in quadratic media

We first intend to look for nonexistence of localized solutions, by means of Pohozaev identities. We start from the model problem

$$\begin{aligned} &i\frac{\partial u}{\partial t} + \Delta_\perp u + \gamma_1 \frac{\partial^2 u}{\partial z^2} + \bar{u} v = 0 \\ &2i\frac{\partial v}{\partial t} + \Delta_\perp v + \gamma_2 \frac{\partial^2 v}{\partial z^2} - \beta v + \frac{1}{2} u^2 = 0 \end{aligned} \tag{5.1}$$

where $u = u(t,x,z)$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and where $\Delta_\perp = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_d^2$. Note that here, physical situations are obtained for $d \leq 2$. In all that follows, we will also use the notation $\nabla_\perp = (\partial_{x_1}, \ldots, \partial_{x_d})$. By analogy with the terminology used for Davey-Stewartson systems, the cases $(\gamma_1, \gamma_1) = (1,1)$, $(-1,1)$, $(1,-1)$ and $(-1,-1)$ could be referred as elliptic-elliptic, hyperbolic-elliptic, elliptic-hyperbolic and hyperbolic-hyperbolic. Since u and v respectively stand for fundamental and harmonic waves, we naturally seek for solutions of the form $u(t,x,z) \equiv u(x,z)\exp(i\omega t)$ and $v(t,x,z) \equiv v(x,z)\exp(2i\omega t)$, system (5.1) rewrites

$$\begin{aligned} &-\omega u + \Delta_\perp u + \gamma_1 \frac{\partial^2 u}{\partial z^2} + \bar{u} v = 0 \\ &-(4\omega+\beta)v + \Delta_\perp v + \gamma_2 \frac{\partial^2 v}{\partial z^2} + \frac{1}{2}u^2 = 0. \end{aligned} \tag{5.2}$$

5.1 Nonexistence results

Proposition 1. *Assume that system (5.2) admits localized solutions u and v in $H^1(\mathbb{R}^{d+1})$. Then one has the identities*

$$\left\{ \begin{aligned} \omega \int |u|^2 \quad &+ \frac{d-3}{d+1} \int \Big(|\nabla_\perp u|^2 + \gamma_1 |\partial_z u|^2 \Big) \\ &- \frac{4}{d+1} \int \Big(|\nabla_\perp v|^2 + \gamma_2 |\partial_z v|^2 \Big) = 0 \\ (4\omega+\beta) \int |v|^2 \quad &- \frac{2}{d+1} \int \Big(|\nabla_\perp u|^2 + \gamma_1 |\partial_z u|^2 \Big) \\ &+ \frac{d-1}{d+1} \int \Big(|\nabla_\perp v|^2 + \gamma_2 |\partial_z v|^2 \Big) = 0 \end{aligned} \right. \tag{5.3}$$

As a consequence, (5.2) does not admits localized solutions in the following cases :

- $d \leq 3,\ (\gamma_1, \gamma_2) = (1,1),\ \omega \leq 0,$
- $d = 3,\ (\gamma_1, \gamma_2) = (-1,1),\ \omega \leq 0,$
- $d = 1,\ (\gamma_1, \gamma_2) = (1,-1),\ 4\omega + \beta \leq 0,$
- $d = 1,\ (\gamma_1, \gamma_2) = (1,1),\ 4\omega + \beta \leq 0.$

Proof : we first multiply the two equations by $\bar{u}$ and $\bar{v}$ and integrate over $\mathbb{R}^{d+1}$ (that is with respect of both x and z). An integration by parts of the second-order derivatives gives us the two relations

$$-\omega \int |u|^2\, dxdz - \int \Big(\|\nabla_\perp u\|^2 + \gamma_1 |\partial_z u|^2 \Big)\, dxdz + \mathrm{Re} \int \bar{u}^2 v\, dxdz = 0 \tag{5.4}$$

$$-(4\omega+\beta) \int |v|^2\, dxdz - \int \Big(\|\nabla_\perp v\|^2 + \gamma_2 |\partial_z v|^2 \Big)\, dxdz + \frac{1}{2}\mathrm{Re} \int \bar{u}^2 v\, dxdz. \tag{5.5}$$

We now multiply Equations (5.2) by respectively $< x, \nabla \bar{u} > + z\partial_z \bar{u}$ and $< x, \nabla \bar{v} > + z\partial_z \bar{v}$ and still integrate over $\mathbb{R}^{d+1}$. Note that in this paper, we make formal computations : for a rigorous justification of the following ruminations, we refer to Proposition 2.1 of [11]. Claiming that

$$\mathrm{Re} \int \partial_{x_j}^2 u x_j \partial_{x_j} \bar{u}\, dxdz = -\mathrm{Re} \int \partial_{x_j} u \partial_{x_j} (x_j \partial_{x_j} \bar{u})\, dxdz = -\frac{1}{2} \int |\partial_{x_j} u|^2\, dxdz \quad 1 \le j \le d$$

$$\mathrm{Re} \int \partial_{x_j}^2 u x_k \partial_{x_k} \bar{u}\, dxdz = -\mathrm{Re} \int \partial_{x_j} u \partial_{x_j} (x_k \partial_{x_k} \bar{u})\, dxdz = \frac{1}{2} \int |\partial_{x_j} u|^2\, dxdz \qquad j \ne k,$$

$$\mathrm{Re} \int \partial_z^2 u z \partial_z \bar{u}\, dxdz = -\mathrm{Re} \int \partial_z u \partial_z (z \partial_z \bar{u})\, dxdz = -\frac{1}{2} \int |\partial_z u|^2\, dxdz$$

$$\mathrm{Re} \int \partial_z^2 u x_k \partial_{x_k} \bar{u}\, dxdz = -\mathrm{Re} \int \partial_z u \partial_z (x_k \partial_{x_k} \bar{u})\, dxdz = \frac{1}{2} \int |\partial_z u|^2\, dxdz \quad , 1 \le k \le d$$

summing the $d+1$ resulting equations leads us to succesively

$$\begin{aligned} \omega \frac{d+1}{2} \int |u|^2\, dxdz + \frac{d-1}{2} \int \Big(|\nabla_\perp u|^2 + \gamma_1 |\partial_z u|^2 \Big)\, dxdz \\ + \mathrm{Re} \Big(\int \bar{u} v (< x, \nabla_\perp \bar{u} > + z\partial_z \bar{u})\, dxdz \Big) &= 0 \end{aligned} \tag{5.6}$$

$$\begin{aligned} \frac{d+1}{2} (4\omega + \beta) \int |v|^2\, dxdz + \frac{d-1}{2} \int \Big(|\nabla_\perp v|^2 + \gamma_2 |\partial_z v|^2 \Big)\, dxdz \\ + \frac{1}{2} \mathrm{Re} \Big(\int u^2 (< x, \nabla_\perp \bar{v} > + z\partial_z v)\, dxdz \Big) &= 0. \end{aligned} \tag{5.7}$$

Now, since the identity

$$\begin{aligned} \mathrm{Re} \int \bar{u}^2 v\, dxdz &= -\frac{2}{d+1} \Big(\mathrm{Re} \int \bar{u} v (< x, \nabla_\perp \bar{u} > + z\partial_z \bar{u})\, dxdz \\ &+ \frac{1}{2} \mathrm{Re} \int \bar{u}^2 (< x, \nabla_\perp v > + z\partial_z v)\, dxdz \Big) \end{aligned}$$

holds, combining (5.4), (5.5), (5.6) and (5.7) gives us the desired identities. Note that for all values of the parameters given in proposition 1, each integral has to vanish separately and we find that localized solutions cannot exist. This does not exclude the existence of non-localized states (see [19] for an example concerning the nonelliptic Schrödinger equation).

In the hyperbolic-hyperbolic case, a strong argument for nonexistence can be found, as states the

Proposition 2. *For any ω and β, no localized solutions of (5.2) exist for $(\gamma_1, \gamma_2) = (-1, -1)$.*

Proof : first, we multiply Equations (5.2) by respectively $x_k \partial_{x_k} \bar{u}$ and $x_k \partial_{x_k} \bar{v}$ $(1 \le k \le d)$, take the real part of the resulting equations and combine. We obtain, after integration,

$$\omega \int |u|^2 + (4\omega + \beta) \int |v|^2 - \int |\partial_{x_k} u|^2 + \sum_{j \neq k} \int |\partial_{x_j} u|^2 + \gamma_1 \int |\partial_z u|^2$$
$$- \int |\partial_{x_k} v|^2 + \sum_{j \neq k} \int |\partial_{x_j} v|^2 + \gamma_2 \int |\partial_z v|^2$$
$$= -\mathrm{Re} \int x_k \partial_{x_k} (\bar{u}^2 v) = \mathrm{Re} \int \bar{u}^2 v \,. \tag{5.8}$$

Using now $z\partial_z$ instead of $x_k \partial_{x_k}$, we now get

$$\omega \int |u|^2 + (4\omega + \beta) \int |v|^2 + \int |\partial_{x_k} u|^2 + \sum_{j \neq k} \int |\partial_{x_j} u|^2 - \gamma_1 \int |\partial_z u|^2$$
$$+ \int |\partial_{x_k} v|^2 + \sum_{j \neq k} \int |\partial_{x_j} v|^2 - \gamma_2 \int |\partial_z v|^2$$
$$= -\mathrm{Re} \int z \partial_z (\bar{u}^2 v) = \mathrm{Re} \int \bar{u}^2 v \,. \tag{5.9}$$

Finally, subtracting (5.8) and (5.9) gives for all $1 \le k \le d$

$$\int |\partial_{x_k} u|^2 + \int |\partial_{x_k} v|^2 - \gamma_1 \int |\partial_z u|^2 - \gamma_2 \int |\partial_z v|^2 = 0.$$

Thus, $(\gamma_1, \gamma_2) = (-1, -1)$ implies that each term has to vanish. This achieves the proof of Proposition 2.

In the case $d = 0$ (that is when no transverse diffraction occurs), then the stationary states satisfy, after the use of an appropriate scaling, the ordinary differential system

$$\begin{aligned} -\omega u + \gamma_1 u'' + \bar{u} v &= 0 \\ -(4\omega + \beta) v + \gamma_2 v'' + u^2 &= 0, \end{aligned} \tag{5.10}$$

where $''$ refers as the second space derivative with respect to z. In this context, it is possible in some cases to derive explicit formulas (see [5], [6]). A direct integration enables to find for $\gamma_1 = 1$, $\gamma_2 = 2$, $\omega = 1$ and $\beta = -2$

$$u(z) = \sqrt{2} v(z) = \frac{3}{2} \sqrt{2} \mathrm{sech}^2 (\frac{z}{2}).$$

Furthermore, if the differential terms are neglected in the second equation in (5.1), one has $v \equiv \beta u^2$ and we find the asymptotic Schrödinger model

$$i \frac{\partial u}{\partial t} + \gamma_1 \frac{\partial^2 u}{\partial z^2} + \beta |u|^2 u = 0 \tag{5.11}$$

that admits bright solitons (for $\gamma_1 = 1$) or dark solitons (for $\gamma_1 = -1$). The limit obtained for large β is known as the *cascading* limit. Numerical computations of the solutions of (5.10) using the shooting method have shown that the cascading limit is recovered for large β.

5.2 An existence result

The aim of this section is to construct a ground state solution (minimizer of the action among all non trivial solitary waves) to Equations (5.2) in the elliptic-elliptic case, that is $(\gamma_1,\gamma_2)\in\mathbb{R}_+^*$ and to investigate the stability properties of this particular solution. By a scaling on z, u and v, that is

$$\tilde{u}(x,z)=u(x,\sqrt{\gamma_1}z),\quad \tilde{v}=v(x,\sqrt{\gamma_1}z),$$

it is easy to see, droping the tilde, that Equations (5.2) can be reduced to

$$\Delta_\perp u+\partial_z^2u-\omega u+\overline{u}v=0, \tag{5.12}$$

$$\Delta_\perp v+\gamma\partial_z^2v-(4\omega+\beta)v+\frac{1}{2}u^2=0. \tag{5.13}$$

where $\gamma\in\mathbb{R}_+^*$. Note that a solution to (5.12)-(5.13) is not radial and so classical methods using radial functions do not apply here (see [26, 34] for example).

Remark 1. In [23], the authors prove the existence of ground states in the real case for a similar set of equations (in space dimension $n\le 5$) but in the radial case, that is $\gamma=1$ using variational methods.

Define the action associated to (5.12)-(5.13) on $H^1(\mathbb{R}^{d+1})\times H^1(\mathbb{R}^{d+1})$ by

$$\begin{aligned}S(u,v)=&\int_{R^{d+1}}\Big(|\nabla_\perp u|^2+|\nabla_\perp v|^2+|\partial_z u|^2+\gamma|\partial_z v|^2+\omega|u|^2+(4\omega+\beta)|v|^2\Big)dx\\&-\mathrm{Re}\int_{\mathbb{R}^{d+1}}\overline{u}^2v\,dx.\end{aligned}$$

Notice that $(u,v)\in H^1\times H^1$ is a solution to (5.12)-(5.13) if and only if $S^{'}(u,v)=0$. Introduce

$$\mathcal{A}_{\omega,\beta}=\{(u,v)\in H^1(\mathbb{R}^{d+1})\times H^1(\mathbb{R}^{d+1}):S^{'}(u,v)=0,\ (u,v)\neq 0\},$$

and denote by $\mathcal{G}_{\omega,\beta}$ the set of ground states to System (5.12)-(5.13)

$$\mathcal{G}_{\omega,\beta}=\{(\phi,\psi)\in\mathcal{A}_{\omega,\beta}:S(\phi,\psi)\le S(u,v)\ \text{ for all } (u,v)\in\mathcal{A}_{\omega,\beta}\},$$

We now introduce the Nehari manifold associated to System (5.12)-(5.13). For $(u,v)\in H^1(\mathbb{R}^{d+1})\times H^1(\mathbb{R}^{d+1})$, denote

$$\begin{aligned}K_{\omega,\beta}(u,v)=&\int_{R^{d+1}}\Big(|\nabla_\perp u|^2+|\nabla_\perp v|^2+|\partial_z u|^2+\gamma|\partial_z v|^2+\omega|u|^2+(4\omega+\beta)|v|^2\Big)dx\\&-\frac{3}{2}\mathrm{Re}\int_{\mathbb{R}^{d+1}}\overline{u}^2v\,dx\end{aligned}$$

and

$$\mathcal{N}_{\omega,\beta}=\Big\{u,v)\in H^1(\mathbb{R}^{d+1})\times H^1(\mathbb{R}^{d+1}):K_{\omega,\beta}(u,v)=0\Big\}.$$

Lemma 1. *For all $(u,v)\in\mathcal{A}_{\omega,\beta}$, one has $K_{\omega,\beta}(u,v)=0$, that is $(u,v)\in\mathcal{N}_{\omega,\beta}$.*

Proof. Multiply Equation (5.12) by $\overline{u}$ and Equation (5.13) by $\overline{v}$, integrate over $\mathbb{R}^{d+1}$ and make an integration by parts on the second order derivatives terms. We omit the details since the computations are straightforward. □

Remark 2. An important consequence of Lemma 1 is that for any $(u, v) \in \mathcal{A}_{\omega,\beta}$ with $(u, v) \neq (0, 0)$

$$\int_{R^{d+1}} \overline{u}^2 v dx \in \mathbb{R}_+^*.$$

The main result of this section reads :

Theorem 2. *Assume that $d \leq 4$, $\omega > 0$ and $4\omega + \beta > 0$. Then System* (5.12)-(5.13) *admits one ground state solution $(\phi, \psi) \in H^1(\mathbb{R}^{d+1}) \times H^1(\mathbb{R}^{d+1})$.*

Remark 3. In addition, it is easy to prove that the ground state solution has the usual smoothness and decay properties (see [8, 12]), that is :
- $(\phi, \psi) \in C^2(\mathbb{R}^{d+1})$,
- for all $\alpha \in \mathbb{N}^{d+1}$ with $|\alpha| \leq 2$, there exist $(C_\alpha, \delta_\alpha) \in (\mathbb{R}_+^*)^2$ such that

$$|D^\alpha \phi(x, z)| \leq C_\alpha e^{-\delta_\alpha(|x|+|z|)}, \quad |D^\alpha \psi(x, z)| \leq C_\alpha e^{-\delta_\alpha(|x|+|z|)}$$

Proof. We proceed in two steps.

- *step 1:* Construction of a solution to System (5.12)-(5.13) : $\mathcal{A}_{\omega,\beta} \neq \emptyset$.

Introduce the following functional on $H^1(\mathbb{R}^{d+1}) \times H^1(\mathbb{R}^{d+1})$

$$\mathcal{E}(u, v) = \int_{R^{d+1}} \Big(|\nabla_\perp u|^2 + |\nabla_\perp v|^2 + |\partial_z u|^2 + \gamma|\partial_z v|^2 + \omega|u|^2 + (4\omega + \beta)|v|^2\Big) dx.$$

Let $\mu > 0$ be a fixed real and consider the minimization problem

$$I_\mu := \inf_{(u,v)\in K_\mu} \mathcal{E}(u, v), \tag{5.14}$$

where

$$K_\mu = \left\{(u, v) \in H^1(\mathbb{R}^{d+1}) \times H^1(\mathbb{R}^{d+1}) \ / \ \mathrm{Re} \int_{R^{d+1}} \overline{u}^2 v dx = \mu \right\}.$$

A solution to (5.12)-(5.13) is obtained by solving the minimization problem (5.14) using the concentration-compactness principle of P.L. Lions (see [28]).

Note that for any $\mu > 0$, it is obvious that $K_\mu \neq \emptyset$. We begin with two technical lemmas, the first one prove that $I_\mu > 0$ and the second one is concerned with the sub-additivity of the function I_μ.

Lemma 2. *Assume that $d \leq 4$, $\omega > 0$ and $4\omega + \beta > 0$, then for any $\mu > 0$, one has $I_\mu > 0$.*

Proof. Take $\mu > 0$ and two real functions $(u, v) \in K_\mu$. By the Young and Sobolev inequalities, one can find a constant $C > 0$ depending only on ω and β such that

$$0 < \mu = \mathrm{Re} \int_{R^{d+1}} \overline{u}^2 v dx \leq \Big(\int_{R^{d+1}} |u|^3 dx\Big)^{\frac{2}{3}} \Big(\int_{R^{d+1}} |v|^3 dx\Big)^{\frac{1}{3}} \leq CE(u, v),$$

which ends the proof of Lemma 2. □

Lemma 3. *Assume that $d \leq 4$, $\omega > 0$ and $4\omega + \beta > 0$. For any $\mu > 0$ and $\theta > 1$, we have $I_{\theta\mu} < \theta I_\mu$. As a consequence, for each $\mu > 0$ and $\lambda \in (0, \mu)$, $I_\mu < I_\lambda + I_{\mu-\lambda}$.*

Let $\mu > 0$ and $\theta > 1$ and take $(u_n, v_n) \in K_\mu$ be a minimizing sequence for I_μ. For $\alpha = (\theta)^{\frac{1}{3}}$, one has

$$\mathrm{Re} \int_{R^{d+1}} (\alpha \overline{u_n})^2 (\alpha v_n) = \alpha^3 \mathrm{Re} \int_{R^{d+1}} \overline{u_n}^2 v_n = \theta\mu,$$

which provides $(\alpha u_n, \alpha v_n) \in K_{\theta\mu}$. Then

$$I_{\theta\mu} \leq \mathcal{E}(\alpha u_n, \alpha v_n) = \alpha^2 \mathcal{E}(u_n, v_n) = (\theta)^{\frac{2}{3}} \mathcal{E}(u_n, v_n).$$

Letting n goes to $+\infty$, one obtains, since $\theta > 1$

$$I_{\theta\mu} \leq (\theta)^{\frac{2}{3}} I_\mu < \theta I_\mu, \tag{5.15}$$

which proves the first part of Lemma 3. The second part is a classical consequence of inequality (5.15). $\Box$

We are ready to prove that (5.14) admits a non-zero solution.

Proposition 3. *Assume that $d \leq 4$, $\omega > 0$ and $4\omega + \beta > 0$. For any $\mu > 0$, the minimisation problem* (5.14) *admits a solution* $(\phi, \psi) \neq (0, 0)$.

Proof. Let (u_n, v_n) be a minimizing sequence for (5.14). Then $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are bounded in $H^1(\mathbb{R}^{d+1})$. We apply the concentration-compactness principle (see [28]) to the sequence

$$\rho_n = |u_n|^2 + |v_n|^2.$$

Assume first that vanishing occurs, that is, up to a subsequence, for all $R > 0$,

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^{d+1}} \int_{y+B_R} \rho_n dx = 0.$$

By Lemma I.1 of [29], for all $\alpha \in (2, \frac{2(d+1)}{d-1})$, $u_n \longrightarrow 0$ and $v_n \longrightarrow 0$ in $L^\alpha(\mathbb{R}^{d+1})$ as $n \to +\infty$. Take $\alpha = 3$ (recall that $d \leq 4$) and apply Young inequality to obtain

$$0 < \mu = \mathrm{Re} \int_{R^{d+1}} \overline{u_n}^2 v_n dx \leq \Big(\int_{R^{d+1}} |u_n|^3 dx \Big)^{\frac{2}{3}} \Big(\int_{R^{d+1}} |v_n|^3 dx \Big)^{\frac{1}{3}} \underset{n\to+\infty}{\longrightarrow} 0,$$

a contradiction which rules out vanishing.
Assume now that dichotomy occurs. Then there exists $\lambda > 0$ such that

$$\lambda = \lim_{n\to+\infty} \int_{\mathbb{R}^{d+1}} \rho_n dx.$$

By classical arguments (see [29] part IV.1), for some $\gamma \in (0, \lambda)$, one can built four sequences $(u_{\ell,1})_{\ell\in\mathbb{N}}$, $(u_{\ell,2})_{\ell\in\mathbb{N}}$, $(v_{\ell,1})_{\ell\in\mathbb{N}}$ and $(v_{\ell,2})_{\ell\in\mathbb{N}}$ bounded in $H^1(\mathbb{R}^{d+1})$

(where $(u_{\ell,1})_{\ell\in\mathbb{N}}$, $(v_{\ell,1})_{\ell\in\mathbb{N}}$ and $(u_{\ell,2})_{\ell\in\mathbb{N}}$, $(v_{\ell,2})_{\ell\in\mathbb{N}}$ have compact disjoints supports) such that for some subsequences $(u_{n(\ell)})_{\ell\in\mathbb{N}}$ and $(v_{n(\ell)})_{\ell\in\mathbb{N}}$

$$||u_{n(\ell)} - u_{\ell,1} - u_{\ell,2}||_{L^2} \leq \frac{1}{\ell}, \;\; ||v_{n(\ell)} - v_{\ell,1} - v_{\ell,2}||_{L^2} \leq \frac{1}{\ell}, \tag{5.16}$$

$$\lim_{\ell\to+\infty}\inf \int_{\mathbb{R}^{d+1}} \Big(|\nabla_\perp u_{n(\ell)}|^2 - |\nabla_\perp u_{\ell,1}|^2 - |\nabla_\perp u_{\ell,2}|^2\Big) \geq 0, \tag{5.17}$$

$$\lim_{\ell\to+\infty}\inf \int_{\mathbb{R}^{d+1}} \Big(|\nabla_\perp v_{n(\ell)}|^2 - |\nabla_\perp v_{\ell,1}|^2 - |\nabla_\perp v_{\ell,2}|^2\Big) \geq 0. \tag{5.18}$$

Note that since for all $n \in \mathbb{N}$, $\mathrm{Re} \int_{\mathbb{R}^{d+1}} \overline{u_n}^2 v_n dx = \mu$ and reminding that $(u_{\ell,i})_{\ell\in\mathbb{N}}$ and $(v_{\ell,i})_{\ell\in\mathbb{N}}$ are bounded in $H^1(\mathbb{R}^{d+1})$ for $i = 1,2$, there exists $\alpha \in (0,\mu)$ such that as $\ell \to +\infty$

$$\mathrm{Re} \int_{\mathbb{R}^{d+1}} \overline{u_{\ell,1}}^2 v_{\ell,1} dx \longrightarrow \alpha, \;\; \mathrm{Re} \int_{\mathbb{R}^{d+1}} \overline{u_{\ell,2}}^2 v_{\ell,2} dx \longrightarrow \mu - \alpha.$$

Using (5.16)-(5.18), we deduce that

$$\begin{aligned} I_\mu = \liminf_{\ell\to+\infty} \mathcal{E}(u_{n(\ell)}, v_{n(\ell)}) &\geq \liminf_{\ell\to+\infty} \mathcal{E}(u_{\ell,1}, v_{\ell,1}) + \liminf_{\ell\to+\infty} \mathcal{E}(u_{\ell,2}, v_{\ell,2}), \\ &\geq I_\alpha + I_{\mu-\alpha}, \end{aligned}$$

contradicting Lemma 3. Then dichotomy cannot occurs.
The only remaining possibility is the compactness of the minimizing sequence modulo translations, that is there exists a sequence $(y_n)_{n\in\mathbb{N}} \in \mathbb{R}^{d+1}$ such that

$$\forall \varepsilon > 0, \; \exists R_\varepsilon < +\infty \text{ such that } \forall n \in \mathbb{N}, \int_{|x-y_n|\leq R_\varepsilon} \rho_n dx \geq \lambda - \varepsilon. \tag{5.19}$$

Since $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are bounded in $H^1(\mathbb{R}^{d+1})$, there exists two functions u and v in $H^1(\mathbb{R}^{d+1})$ such that $u_n(\cdot - y_n)$ and $v_n(\cdot - y_n)$ converge weakly in H^1 respectively to u and v. Then (5.19) implies that $u_n(\cdot - y_n)$ and $v_n(\cdot - y_n)$ converge strongly in $L^2(\mathbb{R}^{d+1})$ as $n \to +\infty$. Then, by Cauchy-Schwartz and Sobolev inequalities,

$$\begin{aligned} \Big| \int_{\mathbb{R}^{d+1}} \overline{u_n}^2 v_n dx - \int_{\mathbb{R}^{d+1}} \overline{u}^2 v dx \Big| &= \Big| \int_{\mathbb{R}^{d+1}} (\overline{u_n}^2 - \overline{u}^2) v_n dx + \int_{\mathbb{R}^{d+1}} \overline{u}^2 (v_n - v) dx \Big| \\ &\leq ||u_n - u||_{L^2} ||u_n + u||_{L^4} ||v||_{L^4} + ||u||_{L^4} ||v_n - v||_{L^2}, \\ &\leq C\Big(||u_n - u||_{L^2} + ||v_n - v||_{L^2}\Big) \underset{n\to+\infty}{\longrightarrow} 0. \end{aligned}$$

This implies that

$$\mathrm{Re} \int_{\mathbb{R}^{d+1}} \overline{u}^2 v dx = \mu,$$

hence $(u,v) \in K_\mu$ and $u \neq 0$, $v \neq 0$. Finally, from (5.17)-(5.18) we derive

$$\lim_{n\to+\infty}\inf \mathcal{E}(u_n, v_n) = I_\mu \geq \mathcal{E}(u,v),$$

which concludes the proof of Proposition 3. □

By Ekeland principle, there exists a constant $\nu \in \mathbb{R}$ such that

$$\mathcal{E}^{'}(u,v) = \nu F^{'}(u,v), \tag{5.20}$$

where

$$F(u,v) = \mathrm{Re} \int_{\mathbb{R}^{d+1}} \bar{u}^2 v \, dx.$$

We claim that $\nu > 0$. Indeed, applying (5.20) at the point (u,v), one obtains

$$\begin{aligned}
&\int_{R^{d+1}} \Big(|\nabla_\perp u|^2 + |\nabla_\perp v|^2 + |\partial_z u|^2 + \gamma |\partial_z v|^2 + \omega |u|^2 + (4\omega + \beta)|v|^2 \Big) dx \\
&= \frac{3}{2} \nu \mathrm{Re} \int_{R^{d+1}} \overline{u}^2 v dx \\
&= \frac{3}{2} \nu \mu,
\end{aligned}$$

since $(u,v) \in K_\mu$ with $u \neq 0$ and $v \neq 0$. This proves the claim. Introducing

$$\phi = \nu u, \quad \psi = \nu v,$$

a direct computation shows that ϕ, ψ satisfies System (5.12)-(5.13), that is $(\phi,\psi) \in \mathcal{A}_{\omega,\beta}$, and concludes *step 1*.

- *step 2 :* Construction of a ground state $(\phi,\psi) \in \mathcal{G}_{\omega,\beta}$.

Take $\mu > 0$ be fixed and introduce (u,v) solution to the minimization problem (5.14) and ν the associated Lagrange multiplier. As shown in *step 1*, denoting $\phi = \nu u$ and $\psi = \nu v$, then (ϕ,ψ) solve Equations (5.12)-(5.13). We claim that (ϕ,ψ) solve the minimization problem (5.14) with the constraint $\widetilde{\mu} = \nu^3 \mu$. Indeed, take any (a,b) in $H^1 \times H^1$ such that

$$\mathrm{Re} \int_{\mathbb{R}^{d+1}} \overline{a}^2 b dx = \widetilde{\mu}.$$

Then

$$\mathrm{Re} \int_{\mathbb{R}^{d+1}} (\frac{\overline{a}}{\nu})^2 (\frac{b}{\nu}) dx = \mu,$$

which provides, by the characterization of (u,v),

$$\mathcal{E}(\phi,\psi) = \nu^2 \mathcal{E}(u,v) \leq \nu^2 \mathcal{E}(\frac{a}{\nu}, \frac{b}{\nu}) = \mathcal{E}(a,b)$$

and proves the claim.

We now claim that $(\phi,\psi) \in \mathcal{G}_{\omega,\beta}$. Indeed, take any $(a,b) \in \mathcal{A}_{\omega,\beta}$ with $(a,b) \neq (0,0)$ such that $S(a,b) \leq S(\phi,\psi)$. By Remark 2,

$$\mathrm{Re} \int_{R^{d+1}} \overline{a}^2 b dx \in \mathbb{R}_+^*, \ \mathrm{Re} \int_{R^{d+1}} \overline{\phi}^2 \psi dx \in \mathbb{R}_+^*.$$

Take

$$\alpha = \Big(\frac{\mathrm{Re} \int_{R^{d+1}} \overline{\phi}^2 \psi dx}{\mathrm{Re} \int_{R^{d+1}} \overline{a}^2 b dx} \Big)^{\frac{1}{2}}.$$

A direct computation shows that,

$$\mathrm{Re}\int_{R^{d+1}}(\alpha\overline{a})^2 b dx = \mathrm{Re}\int_{R^{d+1}}\overline{\phi}^2\psi dx.$$

Furthermore, by Lemma 1, we have

$$S(a,b)=\frac{1}{2}\mathrm{Re}\int_{R^{d+1}}\overline{a}^2 b dx \leq S(\phi,\psi)=\frac{1}{2}\mathrm{Re}\int_{R^{d+1}}\overline{\phi}^2\psi dx,$$

which provides $\alpha \geq 1$. By *step 1*, we deduce, since $\alpha \geq 1$ and (ϕ,ψ) solves the minimization problem (5.14), and using Lemma 1 that

$$\begin{aligned}
\mathcal{E}(\phi,\psi) &\leq \mathcal{E}(\alpha a, b)\\
&= \alpha^2\int_{R^{d+1}}\Big(|\nabla_\perp a|^2+|\partial_z a|^2+\omega|a|^2\Big)dx+\int_{R^{d+1}}\Big(|\nabla_\perp b|^2+\gamma|\partial_z b|^2+(4\omega+\beta)|b|^2\Big)dx,\\
&\leq \alpha^2\mathcal{E}(a,b),\\
&= \frac{3}{2}\alpha^2\mathrm{Re}\int_{R^{d+1}}\overline{a}^2 b dx,\\
&= \frac{3}{2}\mathrm{Re}\int_{R^{d+1}}\overline{\phi}^2\psi dx,\\
&= \mathcal{E}(\phi,\psi).
\end{aligned}$$

Then all the inequalities are equalities : in particular we have $\alpha = 1$ and $\mathcal{E}(a,b) = \mathcal{E}(\phi,\psi)$, from which we deduce $S(a,b) = S(\phi,\psi)$. This ends the proof of Proposition 2. □

Remark 4. In view of Proposition 1 and Theorem 2, it remains to study the existence of solitary waves in the following cases :

- $d \leq 3,\ (\gamma_1,\gamma_2) = (1,1),\ \omega \geq 0,\ 4\omega+\beta < 0,$
- $d = 1,\ (\gamma_1,\gamma_2) = (1,1),\ \omega < 0,\ 4\omega+\beta > 0,$
- $d = 3,\ (\gamma_1,\gamma_2) = (-1,1),\ \omega > 0,$
- $d \leq 2,\ (\gamma_1,\gamma_2) = (-1,1),$
- $2 \leq d \leq 3,\ (\gamma_1,\gamma_2) = (1,-1),$
- $d = 1, (\gamma_1,\gamma_2) = (1,-1),\ \omega < 0,\ 4\omega+\beta > 0.$

5.3 A stability result.

Owning the notations of Section 5.2, in this section, we prove the orbital stability of the ground states in dimension 3, that is $d = 2$, in the following sense.

Definition 1. Let $(\phi,\psi) \in \mathcal{G}_{\omega,\beta}$. Then we say that $(e^{i\omega t}\phi, e^{2i\omega t}\psi)$ is orbitally stable in the set $\mathcal{G}_{\omega,\beta}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial data $(u_0,v_0) \in H^1 \times H^1$, such that $||(u_0-\phi, v_0-\phi)||_{H^1} \leq \delta$, then the solution $(u(t),v(t))$ of (5.1) with initial condition (u_0,v_0) verifies

$$\sup_{t\in\mathbb{R}_+}\inf_{(a,b)\in\mathcal{G}_{\omega,\beta}}||(u(t)-a, v(t)-b)||_{H^1} < \varepsilon.$$

We first introduce a preliminary result.

Lemma 4. *There exists a constant C_0 such that*

$$i)\ \forall u \in H^1(\mathbb{R}^3),\ \int_{\mathbb{R}^3} |u|^3 dx \le C_0 ||u||_{L^2}^{\frac{4}{3}} \Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx \Big) + C_0 ||u||_{L^2}^2 .$$

ii) $\forall \varepsilon > 0$, there exists a nonnegative function $u_\varepsilon \in H^1(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} |u_\varepsilon|^3 dx \ge (C_0 - \varepsilon) ||u_\varepsilon||_{L^2}^{\frac{4}{3}} \Big(\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx \Big) + (C_0 - \varepsilon) ||u_\varepsilon||_{L^2}^2 . \tag{5.21}$$

Proof. By Gagliardo-Nirenberg and Hölder inequality, one has $\forall u \in H^1(\mathbb{R}^3)$,

$$\begin{aligned}
\int_{\mathbb{R}^3} |u|^3 dx &\le C \Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx \Big)^{\frac{3}{4}} \Big(\int_{\mathbb{R}^3} |u|^2 dx \Big)^{\frac{3}{4}} \\
&\le C \Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx \Big) \Big(\int_{\mathbb{R}^3} |u|^2 dx \Big)^{\frac{2}{3}} + C \int_{\mathbb{R}^3} |u|^2 dx,
\end{aligned}$$

from which the proof of Lemma 4 follows by choosing C_0 as the best constant in the last inequality. In *ii)* we can choose a function $u_\varepsilon > 0$ thanks to the relation $|\nabla u| \ge |\nabla |u||$ a.e. □

The stability result reads :

Proposition 4. *Assume that $d = 2$, $\omega > 0$ and assume that $\beta = 0$. Let $(\phi, \psi) \in \mathcal{G}_{\omega,\beta}$ and define $\nu = I(\phi, \psi)$. If ν satisfies $1 + \max(1, \gamma) - C_0 (\frac{\nu}{5})^{\frac{2}{3}} < 0$ (where C_0 is the constant of Lemma 4), then the solution $(e^{i\omega t}\phi, e^{2i\omega t}\psi)$ is orbitally stable in the set $\mathcal{G}_{\omega,\beta}$.*

Remark 5. To prove Proposition 4, we need $d \le 2$. Indeed, for stability results on equations with quadratic nonlinearities, dimension 3 is critical.

Remark 6. The assumption $\beta = 0$ is only needed to solve the Euler-Lagrange problem associated to (5.22) (see Lemma 8). Since all the other preliminary results holds for $\beta \neq 0$, we conjecture that Proposition 4 is also valid for $\beta \neq 0$.

The proof of Proposition 4 is based on the idea of Cazenave-Lions (see [7] for example). We then introduce the following minimization problem

$$J_\nu := \inf_{I(u,v)=\nu} E(u, v), \tag{5.22}$$

where we recall that

$$E(u, v) = \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} \Big(|\nabla_\perp v|^2 dx + \gamma |\partial_z v|^2 \Big) dx + \beta \int_{\mathbb{R}^3} |v|^2 dx - \mathrm{Re} \int_{\mathbb{R}^3} \overline{u}^2 v dx,$$

$$I(u, v) = \int_{\mathbb{R}^3} \Big(|u|^2 + 4|v|^2 \Big) dx.$$

For $\nu > 0$, denote

$$\mathcal{M}_\nu = \Big\{ (u, v) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3),\ \ I(u, v) = \nu \Big\} .$$

Before proving Proposition 4, we need several lemmas.

Lemma 5. *Let C_0 be the constant of Lemma 4. Then :*

i) there exists $\nu_0 > 0$ such that for all $0 \leq \nu \leq \nu_0$, $J_\nu = 0$.

ii) for all ν such that $1 + \max(1, \gamma) - C_0(\frac{\nu}{5})^{\frac{2}{3}} < 0$ and $\beta < C_0$, one has $J_\nu < 0$.

Proof. For all $(u, v) \in \mathcal{M}_\nu$, one has by Hölder and Cagliardo-Nirenberg inequalites and the continuous embedding $L^3 \hookrightarrow H^1$,

$$\begin{aligned}
\Big| \int_{\mathbb{R}^3} \overline{u}^2 v dx \Big| &\leq ||u||_{L^3}^{\frac{2}{3}} ||v||_{L^3}^{\frac{1}{3}} \\
&\leq C \Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^3} |u|^2 dx \Big)^{\frac{1}{2}} ||v||_{H^1} \\
&\leq C\nu \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\min(1, \gamma, \beta)}{2} ||v||_{H^1}^2,
\end{aligned}$$

from which follows

$$E(u, v) \geq (1 - C\nu) \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\min(1, \gamma, \beta)}{2} ||v||_{H^1}^2 \geq 0,$$

if ν is chosen sufficiently small. Moreover, take any $(u, v) \in \mathcal{M}_\nu$ with $v = 0$ and define $u_\alpha(x) = \alpha u(\alpha^{\frac{2}{3}} x)$. Then it is obvious that $u_\alpha \in \mathcal{M}_\nu$ and

$$E(u_\alpha, 0) = \alpha^{\frac{4}{3}} \int_{\mathbb{R}^3} |\nabla u|^2 dx \longrightarrow 0, \quad \text{as } \alpha \to 0,$$

which proves i). Now take ν such that

$$1 + \max(1, \gamma) - C_0(\frac{\nu}{5})^{\frac{2}{3}} < 0,$$

$(u, v) \in \mathcal{M}_\nu$. Again by Hölder, Cagliardo-Nirenberg and Young inequalites, one has

$$\begin{aligned}
\Big| \int_{\mathbb{R}^3} \overline{u}^2 v dx \Big| &\leq ||u||_{L^4}^2 ||v||_{L^2} \\
&\leq C \Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx \Big)^{\frac{3}{4}} \Big(\int_{\mathbb{R}^3} |u|^2 dx \Big)^{\frac{1}{4}} ||v||_{L^2} \\
&\leq C\nu^{\frac{3}{4}} \Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx \Big)^{\frac{3}{4}} \\
&\leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + C\nu^3,
\end{aligned}$$

from which follows that

$$E(u, v) \geq \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - C\nu^3 \geq -C\nu^3.$$

As a consequence, one has $J_\nu > -\infty$, and remark that this inequality is valid for all $\nu \geq 0$. Furthermore, take $\varepsilon > 0$ such that $\beta - C_0 + \varepsilon < 0$ and

$$1 + \max(1, \gamma) - (C_0 - \varepsilon)(\frac{\nu}{5})^{\frac{2}{3}} < 0,$$

u_ε given by Lemma 4 and define

$$w_\varepsilon^\alpha(x) = u_\varepsilon(\alpha^{\frac{2}{3}}x).$$

Choosing α such that $I(w_\varepsilon^\alpha, w_\varepsilon^\alpha) = \nu$, a direct computation shows that

$$||w_\varepsilon||_{L^3}^3 = \alpha^{-2}||u_\varepsilon||_{L^3}^3, \quad ||w_\varepsilon^\alpha||_{L^2}^2 = \alpha^{-2}||u_\varepsilon||_{L^2}^2,$$

$$||w_\varepsilon^\alpha||_{L^2}^{\frac{4}{3}} = \alpha^{-\frac{4}{3}}||u_\varepsilon||_{L^2}^{\frac{4}{3}}, \quad \int_{\mathbb{R}^3} |\nabla w_\varepsilon^\alpha|^2 dx = \alpha^{-\frac{2}{3}} \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx,$$

from which follows that w_ε^α satisfies (5.21). We deduce

$$\begin{aligned} E(w_\varepsilon^\alpha, w_\varepsilon^\alpha) &\leq \left(1 + \max(1,\gamma) - (C_0 - \varepsilon)(\frac{\nu}{5})^{\frac{2}{3}}\right) \int_{\mathbb{R}^3} |\nabla w_\varepsilon^\alpha|^2 dx \\ &+ (\beta - C_0 + \varepsilon)||w_\varepsilon^\alpha||_{L^2}^2 \\ &< 0. \end{aligned}$$

This ends the proof of the lemma □

We assume now that $1 + \max(1,\gamma) - C_0(\frac{\nu}{5})^{\frac{2}{3}} < 0$ and $\beta < C_0$.

Lemma 6. *Assume that $\nu > 0$ and take $\theta > 1$. Then $J_{\theta\nu} < \theta J_\nu$. As a consequence, for each $\nu > 0$ and $\lambda \in (0,\nu)$, one has $J_\nu < J_\lambda + J_{\nu-\lambda}$.*

Proof. Take $(u_n, v_n) \in \mathcal{M}_\nu$ a minimizing sequence for J_ν. Let $\varepsilon > 0$ and define

$$\alpha = \sqrt{\theta - \varepsilon}, \quad \beta = \Big(\frac{\theta}{\theta - \varepsilon}\Big)^{\frac{1}{3}},$$

$$w_n(x) = \alpha u_n(\frac{x}{\beta}), \quad r_n(x) = \alpha v_n(\frac{x}{\beta}).$$

Then

$$I(w_n, r_n) = \alpha^2\beta^3 I(u_n, v_n) = \theta I(u_n, v_n),$$

and

$$\begin{aligned} J_{\theta\nu} \leq E(w_n, r_n) &= \theta^{\frac{1}{3}}(\theta - \varepsilon)^{\frac{2}{3}}\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} \Big(|\nabla_\perp v_n|^2 dx + \gamma|\partial_z v_n|^2\Big) dx\Big) \\ &+ \beta\theta \int_{\mathbb{R}^3} |v_n|^2 dx - \theta(\theta - \varepsilon)^{\frac{1}{2}} \mathrm{Re} \int_{\mathbb{R}^3} \overline{u_n}^2 v_n dx, \\ &\leq (\theta^{\frac{1}{3}}(\theta - \varepsilon)^{\frac{2}{3}} - \theta)\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} \Big(|\nabla_\perp v_n|^2 dx + \gamma|\partial_z v_n|^2\Big) dx\Big) \\ &+ \theta E(u_n, v_n) \end{aligned}$$

since $\theta(\theta - \varepsilon)^{\frac{1}{2}} \geq \theta$. Note that

$$\liminf \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} \Big(|\nabla_\perp v_n|^2 dx + \gamma|\partial_z v_n|^2\Big) dx = 0$$

implies by Gagliardo-Nirenberg inequality that $\liminf \int_{\mathbb{R}^3} \overline{u_n}^2 v_n dx = 0$ contradicting $J_\nu < 0$. Then passing to the limit $n \to +\infty$, one obtains

$$J_{\theta\nu} < \theta J_\nu,$$

and the lemma is proved. □

Lemma 7. *Let ν such that $1+\max(1,\gamma) - C_0(\frac{\nu}{5})^{\frac{2}{3}} < 0$ and $\beta < C_0$, then every minimizing sequence of the minimizing problem* (5.22) *converges strongly in $H^1 \times H^1$ toward a solution* (5.22).

Let (u_n, v_n) be a minimizing sequence for (5.22). Then $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are bounded in $H^1(\mathbb{R}^{d+1})$. We apply the concentration-compactness principle (see [28]) to the sequence

$$\rho_n = |u_n|^2 + |v_n|^2.$$

Since $J_\nu < 0$, it is classical to prove that vanishing cannot occur whereas Lemma 6 rules out dichotomy (see the proof of Proposition 3 for more details). We then obtain the compactness of the minimizing sequences : in particular, the sequences $(u_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ converge respectively weakly in H^1 and strongly in L^2 toward u and v. This implies that $(u,v) \in \mathcal{M}_\nu$, $(u,v) \neq (0,0)$ and

$$E(u,v) \leq \liminf E(u_n, v_n) = J_\nu,$$

which ends the proof of the lemma. □

Now assume $\beta = 0$, let ϕ, ψ be a ground state of Equation (5.12)-(5.13) given by Theorem 2, define $\nu = I(\phi,\psi)$. We suppose that $1+\max(1,\gamma) - C_0(\frac{\nu}{5})^{\frac{2}{3}} < 0$ and we take (u,v) solution to (5.22). Then, there exists $\lambda \in \mathbb{R}$ such that $E^{'}(u,v) = \lambda\omega I^{'}(u,v)$, that is

$$\begin{aligned} &-\lambda\omega u + \Delta_\perp u + \partial_z^2 u + \bar{u}v = 0 \\ &-4\lambda\omega v + \Delta_\perp v + \gamma\partial_z^2 v + \frac{1}{2}u^2 = 0. \end{aligned} \tag{5.23}$$

Lemma 8. *With the notations above, one has $\lambda = 1$. Moreover, (u,v) is a ground state of Equations* (5.12)-(5.13).

Proof. Observe first that since $\nu = I(u,v) = I(\phi,\psi)$, one as $E(u,v) \leq E(\phi,\psi)$. Applying Proposition 1 to Equation (5.23) and Equations (5.12)-(5.13) with $d=2$ and summing the resulting equations, one has

$$\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} (|\nabla_\perp v|^2 + \gamma|\partial_z v|^2) = \lambda\omega\nu. \tag{5.24}$$

$$\int_{\mathbb{R}^3} |\nabla \phi|^2 + \int_{\mathbb{R}^3} (|\nabla_\perp \psi|^2 + \gamma|\partial_z \psi|^2) = \omega\nu. \tag{5.25}$$

By Lemma 1 we deduce

$$E(u,v) = \lambda\omega\nu - \mathrm{Re}\int_{\mathbb{R}^3} \bar{u}^2 v = -\lambda\omega\nu + \frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^3} \bar{u}^2 v,$$

from which we deduce that

$$\lambda\omega\nu = \frac{3}{4}\mathrm{Re}\int_{\mathbb{R}^3} \bar{u}^2 v, \quad E(u,v) = -\frac{1}{3}\lambda\omega\nu. \tag{5.26}$$

The same procedure on Equation (5.12)-(5.13) furnishes

$$\omega\nu = \frac{3}{4}\mathrm{Re}\int_{\mathbb{R}^3} \bar{\phi}^2 \psi, \quad E(\phi,\psi) = -\frac{1}{3}\omega\nu. \tag{5.27}$$

Then $E(u,v) \le E(\phi,\psi)$, implies directly that

$$\lambda \ge 1.$$

Define

$$\tilde{u} = \frac{1}{\lambda} u(\frac{x}{\sqrt{\lambda}}), \quad \tilde{v} = \frac{1}{\lambda} v(\frac{x}{\sqrt{\lambda}}).$$

A direct computation shows that $(\tilde{u},\tilde{v})$ solves Equations (5.12)-(5.13). As a consequence, using (5.24), (5.25), (5.26) and (5.27), one can write

$$\begin{aligned} S(\phi,\psi) &= \frac{2}{3}\omega\nu \\ &\le S(\tilde{u},\tilde{v}) = \frac{1}{\lambda^{\frac{3}{2}}}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} (|\nabla_\perp v|^2 + \gamma|\partial_z v|^2\Big) + \frac{1}{\lambda^{\frac{1}{2}}}\omega\nu - \frac{1}{\lambda^{\frac{3}{2}}}\mathrm{Re}\int_{\mathbb{R}^3} \bar{u}^2 v \\ &= \frac{2}{3\lambda^{\frac{1}{2}}}\omega\nu, \end{aligned}$$

from which we deduce that $\lambda \le 1$, that is $\lambda = 1$. As a consequence, (u,v) solves Equations (5.12)-(5.13). By (5.24)-(5.27), one has

$$S(\phi,\psi) = \frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^3} \bar{\phi}^2\psi = \frac{2}{3}\omega\nu = \frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^3} \bar{u}^2 v = S(u,v),$$

which ends the proof of the lemma. □

Remark 7. From the proof of Lemma 8, we deduce also that (ϕ,ψ) is a solution to (5.22).

Proof of Proposition 4. We argue by contradiction. Assume that there exists ε_0 and sequences $t^n \in \mathbb{R}_+^*$, $(u_0^n, v_0^n) \in H^1 \times H^1$ such that

$$||(u_0^n - \phi, v_0^n - \psi)||_{H^1} \longrightarrow +\infty \text{ as } n \to 0 \tag{5.28}$$

$$\inf_{(a,b)\in\mathcal{G}_{\omega,\beta}} ||(u^n(t^n) - a, v^n(t^n) - b)||_{H^1} \ge \varepsilon_0, \tag{5.29}$$

where (u^n, v^n) is the solution to (5.1) with initial condition (u_0^n, v_0^n) and $\beta = 0$. By Sobolev and Cauchy-Schwarz inequalities, one has

$$\Big|\int_{R^3} (\overline{u_0^n}^2 v_0^n - \overline{\phi}^2\psi)dx\Big| \longrightarrow 0,$$

as $n \to +\infty$, which provides by (5.28) and the conservation of energy and mass, that $E(u^n(t^n), v^n(t^n)) = E(u_n^0, v_n^0) \longrightarrow E(\phi,\psi)$ and $I(u^n(t^n), v^n(t^n)) = I(u_n^0, v_n^0) \longrightarrow I(\phi,\psi)$ as $n \to +\infty$. By Remark 7, we deduce that $(u^n(t^n), v^n(t^n))$ is a solution to (5.22), then by Lemma 7 and Lemma 8, it converges to a ground state (u,v) of (5.12)-(5.13), contradicting (5.29). □

Remark 8. Note that we are not able to prove the orbital stability of the ground state given by Theorem 2. One of the reason is that we do not solve the problem of unicity of these particular solitary waves.

6 Solitons computation

We now intend to compute solitons arising in χ_2 media in two-dimensional geometry (that is $d = 1$). In the literature, intensive efforts have been made in order to calculate solitons for NLS equations, in the case of radial geometry, for which the initial equation is turned into a differential problem. The major difficulty is then to prescribe the right value at $r = 0$, that will ensure the numerical solution to decrease at infinity. This technique is known as the *shooting* method. Since this procedure fails when seeking nonradial solutions, we have decided to use the coupling of two different methods.

6.1 Description of the numerical method

Starting with the system that takes the form

$$\begin{aligned} -\omega_1 u + \frac{\partial^2 u}{\partial x^2} + \gamma_1 \frac{\partial^2 u}{\partial z^2} + \bar{u} v = 0 \\ -\omega_2 v + \frac{\partial^2 v}{\partial x^2} + \gamma_2 \frac{\partial^2 v}{\partial z^2} + \frac{1}{2} u^2 = 0, \end{aligned} \tag{6.1}$$

we first decide to compute a *Gaussian* ansatz of the solution, meaning that we look for explicit functions

$$u(x, y) = A e^{-(\rho_1 x^2 + \rho_2 z^2)} \quad \text{and} \quad v(x, y) = B e^{-(\delta_1 x^2 + \delta_2 z^2)}, \tag{6.2}$$

where the coefficients are calculated in order that u and v are critical point of the Lagrangian of (6.1). This method has been previously used in [31], [41], [37] for the case of quadratic media. Setting

$$\mathcal{L} := \frac{1}{2} \int_{\mathbb{R}^2} \Big\{ \Big| \frac{\partial u}{\partial x} \Big|^2 + \gamma_1 \Big| \frac{\partial u}{\partial z} \Big|^2 + \Big| \frac{\partial v}{\partial x} \Big|^2 + \gamma_2 \Big| \frac{\partial v}{\partial z} \Big|^2 + \omega_1 |u|^2 + \omega_2 |v|^2 - u^2 v \Big\} \, dx dz,$$

we compute $\mathcal{L}$ with the choice of the trial functions given by (6.2) and choose the coefficients such that $\partial\mathcal{L}/\partial A = \partial\mathcal{L}/\partial B = \partial\mathcal{L}/\partial\rho_j = \partial\mathcal{L}/\partial\gamma_j = 0$ $(j = 1, 2)$. This explicitly gives us the amplitudes A and B as

$$A^2 = \frac{1}{2}(2\rho_1 + \delta_1)(2\rho_2 + \delta_2)(\rho_1 + \gamma_1\rho_2 + \omega_1)(\delta_1 + \delta_2\gamma_2 + \omega_2)\frac{1}{\sqrt{\rho_1\rho_2\delta_1\delta_2}} \tag{6.3}$$

and

$$B = \frac{\sqrt{2\rho_1 + \delta_1}\sqrt{2\rho_2 + \delta_2}}{\sqrt{\rho_1\rho_2}}(\rho_1 + \gamma_1\rho_2 + \omega_1), \tag{6.4}$$

where the four coefficients $(\rho_1, \rho_2, \delta_1, \delta_2)$ solve the nonlinar algebraic system

$$\begin{aligned} 2\rho_1(\rho_1 + \gamma_1\rho_2 + \omega_1) &= (2\rho_1 + \delta_1)(-\rho_1 + \gamma_1\rho_2 + \omega_1) \\ 2\rho_2(\rho_1 + \gamma_1\rho_2 + \omega_1) &= (2\rho_2 + \delta_2)(\rho_1 - \gamma_1\rho_2 + \omega_1) \\ 2\delta_1(\delta_1 + \delta_2\gamma_2 + \omega_2) &= (2\rho_1 + \delta_1)(-\delta_1 + \delta_2\gamma_2 + \omega_2) \\ 2\delta_2(\delta_1 + \delta_2\gamma_2 + \omega_2) &= (2\rho_2 + \delta_2)(\delta_1 - \delta_2\gamma_2 + \omega_2). \end{aligned}$$

The system parameters δ_1, δ_2, ω_1 and ω_2 being prescribed, we solve this system using a Newton algorithm and we deduce A and B with (6.3) and (6.4). Of course, these computations only give an approximation of the localized states. We then decide to make use of a Newton algorithm in order to improve the Gaussian shape of u and v. For this aim, we compute the approximate solution of (6.1) at discrete spatial points $(x_j, z_k) = (j\Delta x, k\Delta z)$, where Δx, Δz are given spatial steps. For any smooth function w depending both on x and y, we set

$$(L_x w)_{j,k} = \frac{1}{\Delta x^2}\Big(w_{j+1,k} - 2w_{j,k} + w_{j-1,k}\Big) = \Big(\frac{\partial^2 w}{\partial x^2}\Big)_{j,k} + \mathcal{O}(\Delta x^2)$$

and

$$(L_z w)_{j,k} = \frac{1}{\Delta z^2}\Big(w_{j,k+1} - 2w_{j,k} + w_{j,k-1}\Big) = \Big(\frac{\partial^2 w}{\partial z^2}\Big)_{j,k} + \mathcal{O}(\Delta z^2),$$

We then replace the initial partial differential equation with a finite differences nonlinear system. We write each equation for $j \in \{-J, \dots, J\}$, $k \in \{-K, \dots, K\}$ and define $(U, V) \in \mathbb{R}^{(2J+1)(2K+1)} \times \mathbb{R}^{(2J+1)(2K+1)}$ as the approximate value of u and v at all the gridpoints. Dropping off the Taylor remainders, the system to solve expresses as

$$F(U, V) = 0 \quad \text{and} \quad G(U, V) = 0, \tag{6.5}$$

where F and G are defined by

$$F(U, V)_{j,k} = (L_x U)_{j,k} + \delta_1 (L_z U)_{j,k} - \omega_1 U_{j,k} + U_{j,k} V_{j,k}$$

and

$$G(U, V)_{j,k} = (L_x V)_{j,k} + \delta_2 (L_z V)_{j,k} - \omega_2 V_{j,k} + \frac{1}{2} U_{j,k}^2,$$

for each index j and k. We compute two sequences $(U_l)_{l \ge 0}$ and $(V_l)_{l \ge 0}$ using the Newton method: we solve for each $l \ge 0$ the system

$$\begin{cases} D_F(\delta U_l, \delta V_l) = -F(U_l, V_l), \\ \\ D_G(\delta U_l, \delta V_l) = -G(U_l, V_l). \end{cases} \tag{6.6}$$

where $\delta U_l = U_{l+1} - U_l$, $\delta V_l = V_{l+1} - V_l$ and where the derivatives $D_F = \nabla F(U, V)$ and $D_G = \nabla G(U, V)$ expressed at some $(H, R) \in \mathbb{R}^{(2J+1)(2K+1)} \times \mathbb{R}^{(2J+1)(2K+1)}$ write

$$(D_F)(H, R)_{j,k} = (L_x H)_{j,k} + \delta_1 (L_z H)_{j,k} - \omega_1 H_{j,k} + U_{j,k} R_{j,k} + H_{j,k} V_{j,k},$$

$$(D_G)(H, R)_{j,k} = (L_x R)_{j,k} + \delta_2 (L_z R)_{j,k} - \omega_2 R_{j,k} + 2U_{j,k} H_{j,k}.$$

The successive linear resolutions are stopped as soon as the relative error between two consecutive iterations becomes small enough. We inject as initial guess the Gaussian variational approximation found above.

6.2 Numerical results

We first present a few experiments concerning the Gaussian variational approximation. In Figures 1 and 2, we plot the values of the coefficients A, B, ρ_1, ρ_2, δ_1 and δ_2 found for $\gamma_1 = 3$, $\gamma_2 = 0.5$, $\omega_1 = 1$ and $\omega_2 \in (0, 1)$. We notice that the case $\omega_2 = 0$ seems to be a limiting case, since the variational approximation gives scaling parameters that seem to vanish, which means that the corresponding solution converges to a constant solution. Note that the same conclusion holds in the case $\omega_1 \in (0, 1)$ and $\omega_2 = 1$ for which both solutions tend to zero whereas in the case $\gamma_1 \in (0, 1)$, $\gamma_2 = 3$ and $\omega_1 = \omega_2 = 1$, we notice that when δ_1 tends to zero, B and ρ seem to diverge, which suggesting that $\gamma_1 = 0$ could be another limiting case. These observations are consistent with the search of constant solutions of (6.1) that lead us to $u_c = \pm\sqrt{2\omega_1\omega_2}$ and $v_c = \omega_1$.

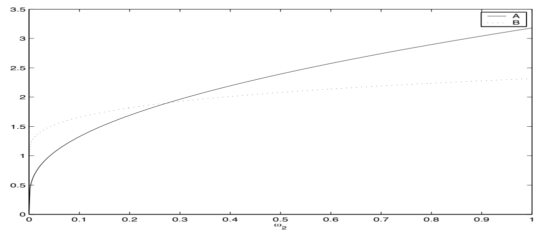

Figure 1: Coefficients A and B, $\gamma_1 = 3$, $\gamma_2 = 0.5$, $\omega_1 = 1$ and $\omega_2 \in (0, 1)$ (elliptic-elliptic nonradial case).

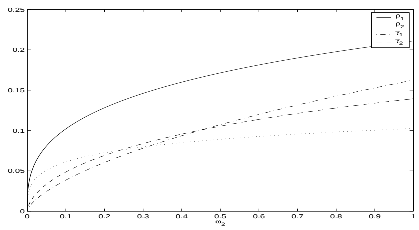

Figure 2: Coefficients ρ_1, ρ_2, δ_1 and δ_2, $\gamma_1 = 3$, $\gamma_2 = 0.5$, $\omega_1 = 1$ and $\omega_2 \in (0, 1)$ (elliptic-elliptic nonradial case).

We then compute the coefficients when $\gamma_1 = 3$, $\gamma_2 \in (-0.212, 1)$, and $\omega_1 = \omega_2 = 1$ (see figures 3 and 4). Surprisingly, it is still possible to calculate the Gaussian solutions when γ_2 changes its sign, which may suggest that solitons can exist in the case of normal dispersion for the harmonic wave (note that this case had not been excluded in the last Section). Nevertheless, this simulation points out that the coefficients ρ_2 and δ_2 may diverge when γ_2 is too small, that leads to a shrinking of the fields across the z axis.

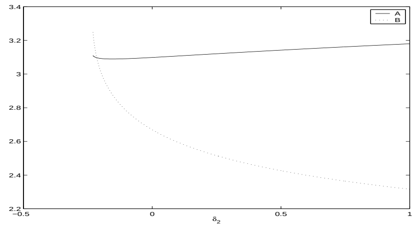

Figure 3: Coefficients A and B, $\gamma_1 = 3$, $\gamma_2 \in (-0.212, 1)$, $\omega_1 = \omega_2 = 1$.

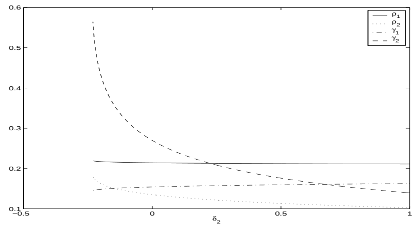

Figure 4: Coefficients ρ_1, ρ_2, δ_1 and δ_2, $\gamma_1 = 3$, $\gamma_2 \in (-0.212, 1)$, $\omega_1 = \omega_2 = 1$.

We then perform computation of stationary states with use of the Newton algorithm. First, we use the same set of parameter than the ones tested for the Gaussian ansatz: $\gamma_1 = 3$, $\gamma_2 = 0.5$ and $\omega_1 = \omega_2 = 1$. The system (6.1) has been solved on the space domain $[-10, 10] \times [-10, 10]$ with the space step $\Delta x = \Delta z = 0.25$. In figures 5 and 6 are plotted the trace of both Newton solution and Gaussian approximation at $z = 0$. It can be noticed a slight difference concerning the asymptotic decay of the soliton: the decreasing does not seem Gaussian.

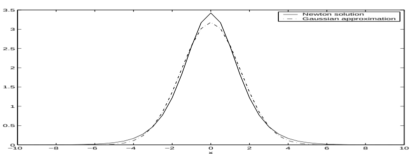

Figure 5: Comparison of the solution computed with the Newton algorithm and Gaussian approximation at $z = 0$: plot of u, $\gamma_1 = 3$, $\gamma_2 = 0.5$, $\omega_1 = \omega_2 = 1$.

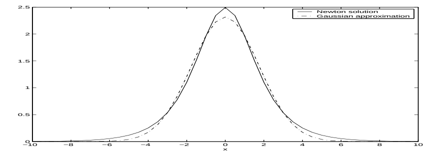

Figure 6: Comparison of the solution computed with the Newton algorithm and Gaussian approximation at $z = 0$: plot of v, $\gamma_1 = 3$, $\gamma_2 = 0.5$, $\omega_1 = \omega_2 = 1$.

The same test has been performed in the elliptic-hyperbolic case $\gamma_2 = -0.05$, where the Newton algorithm still converges to a stationary state (see figures 7 and 8). But a plot of the trace of the different solutions at $z = 0$ (figures 9 and 10) indicates that the two kinds of methods exhibit a quite different behavior for large $|x|$. We have also noticed that for smaller values of γ_2, the Newton algorithm converges to oscillatory solutions.

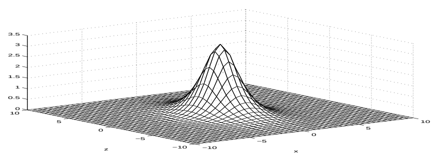

Figure 7: (x, z) plot of u, $\gamma_1 = 3$, $\gamma_2 = -0.05$, $\omega_1 = \omega_2 = 1$.

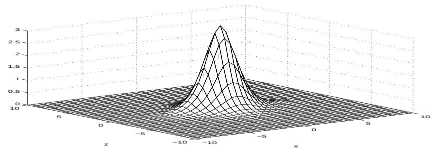

Figure 8: (x, z) plot of v, $\gamma_1 = 3$, $\gamma_2 = -0.05$, $\omega_1 = \omega_2 = 1$.

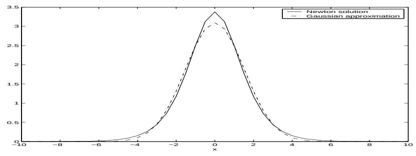

Figure 9: Comparison of the solution computed with the Newton algorithm and Gaussian approximation at $z = 0$: plot of u, $\gamma_1 = 3$, $\gamma_2 = -0.05$, $\omega_1 = \omega_2 = 1$.

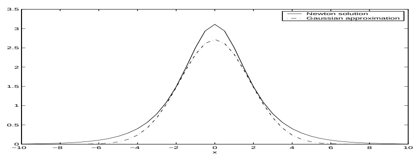

Figure 10: Comparison of the solution computed with the Newton algorithm and Gaussian approximation at $z = 0$: plot of v, $\gamma_1 = 3$, $\gamma_2 = -0.05$, $\omega_1 = \omega_2 = 1$.

7 Dynamics of solitons

In this last Section, we wish to investigate how the solitons that have been previously computed resist to longitudinal displacement. For this purpose, we inject these solitons as initial data and use a finite differences code in both time and space. We briefly show the discretization that is used in the two-dimensional case $d = 1$, that is for the system

$$\begin{aligned} &i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \gamma_1 \frac{\partial^2 u}{\partial z^2} - \omega_1 u + \bar{u} v = 0 \\ &2i\frac{\partial u}{\partial t} + \frac{\partial^2 v}{\partial x^2} + \gamma_2 \frac{\partial^2 v}{\partial z^2} - \omega_2 v + \frac{1}{2} u^2 = 0, \end{aligned} \tag{7.1}$$

where x and z respectively stand for the transverse and longitudinal spatial variables (we restrict ourselves to $d = 1$ for the sake of clearness, since the scheme is similar for higher dimensions or even for the system of 3 waves). Space and time steps Δx, Δz and Δt being given, all the computations will be made in a rectangular grid spatial points $(x_j, z_k) = (j\Delta x, k\Delta z)$ and at discrete times $t_n = n\Delta t$. Let $u^n_{j,k}$ and $v^n_{j,k}$ respectively denote the approximate values of u and v at (t_n, x_j, z_k). We then consider the symmetric Crank-Nicolson discretization of (4.1) : setting $u^{n+1/2} = (u^n + u^{n+1})/2$, the system writes

$$\frac{i}{\Delta t}(u^{n+1}_{j,k} - u^n_{j,k}) + \Big((L_x + \gamma_1 L_z)u^{n+1/2}\Big)_{j,k} - \omega_1 u^{n+1/2}_{j,k} + \bar{u}^{n+1/2}_{j,k} v^{n+1/2}_{j,k} = 0$$

$$\frac{2i}{\Delta t}(v^{n+1}_{j,k} - v^n_{j,k}) + \Big((L_x + \gamma_2 L_z)v^{n+1/2}\Big)_{j,k} - \omega_2 v^{n+1/2}_{j,k} + \frac{1}{2}(u^{n+1/2}_{j,k})^2 = 0,$$

for $n \geq 0$ and $|j| \leq J$, $|k| \leq K$ with J and K given. This is a nonlinear algebraic discrete system that is solved at each time step using a fixed point method. This way of discretizing the system enables us to have the conservation of the discrete mass in the numerical domain

$$I_n = \sum_{j=-J}^{J} \sum_{k=-K}^{K} \Big(\frac{1}{2}|u^n_{j,k}|^2 + |v^n_{j,k}|^2\Big) = I_0, \quad \forall\, n \geq 0$$

(see [16]). The computations will be made in a numerical domain that is chosen in such a way that the boundary will not generate artificial reflections of the numerical solution. In all that follows, we have decided to deal with square domains, setting $\Delta x = \Delta z$ and $J = K$.

We first inject the stationary states found with the Newton method as initial data and solve the time-dependent problem (7.1). Seeking for stationary states with carying frequency ω, we are led to system (6.1) with $\tilde{\omega}_1 = \omega_1 + \omega$ and $\tilde{\omega}_2 = \omega_2 + 4\omega$. We first set $\gamma_1 = 3$, $\gamma_2 = 0.5$, $\omega_1 = \omega_2 = 1$ and $\omega = 0$ (leading us to $\tilde{\omega}_1 = \omega_1$ and $\tilde{\omega}_2 = \omega_2$). We solved (7.1) on the square $[-20, 20] \times [-20, 20]$ with $\Delta x = \Delta z = 0.5$ and $\Delta t = 0.01$ until $T = 4$. We respectively plot in Figures 11 and 12 the profile of u obtained at $z = 0$ compared with the initial trace and the evolution of the maximal amplitudes of u and v. It is observed a persistence of the localized profile through time. The same conclusion also holds for the case $\gamma_2 = -0.05$ (see figures 13 and 14 where the solution has been computed in the case $\omega = 0$). This could indicate that as soon as the Newton method enables to compute solitons, dynamic stability seem to occur.

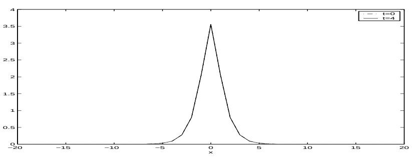

Figure 11: Initial data and amplitude of u at time $T = 4$ and at $z = 0$, $\omega = 0$, $\gamma_2 = 0.5$.

We then study the asymptotic behavior of the solution when we choose as initial data perturbations of stationary states u_s and v_s

$$u_0(x, y) = (1 + \mu_u)u_s(x, y) \;\; \text{and} \;\; v_0(x, y) = (1 + \mu_v)v_s(x, y)$$

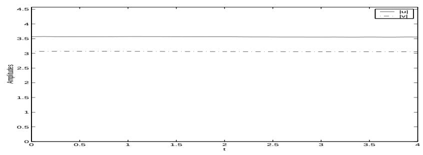

Figure 12: Evolution of $\|u(t,.)\|_{L^\infty}$ and $\|v(t,.)\|_{L^\infty}$ with respect to time, $\omega = 0$, $\gamma_2 = 0.5$.

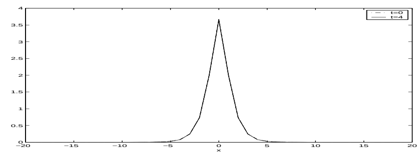

Figure 13: Initial data and amplitude of u at time $T = 4$ and at $z = 0$, $\omega = 0$, $\gamma_2 = -0.05$.

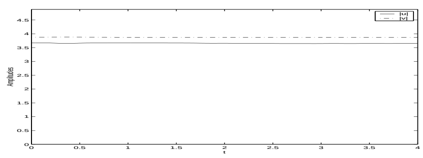

Figure 14: Evolution of $\|u(t,.)\|_{L^\infty}$ and $\|v(t,.)\|_{L^\infty}$ with respect to time, $\omega = 0.1$, $\gamma_2 = -0.05$.

for various values of amplitudes μ_u and μ_v. It can be observed that for negative values of (μ_u, μ_v), the amount of L^2 norm is not sufficient to evolve as the solitonic profile (see figure 15 where tests have been made for $(\mu_u, \mu_v) = (-0.2, -0.3)$) and the dispersion removes the initial structures. For large values of (q_u, q_v), the total mass seems too large to generate an attractive effect of the soliton at moderate times (see figure 16). For the intermediate value $(\mu_u, \mu_v) = (0.01, -0.01)$ close to the ones given by the Gaussian approximate, the time evolution of the profile suggests stability of the soliton (see figure 17). We also plot figure 18 the profiles of the Cauchy data v_0, the solution at final time $t = 4$ and the stationary profile v. This illustrates the way the remaining part of the mass is expunged at infinity through radiation.

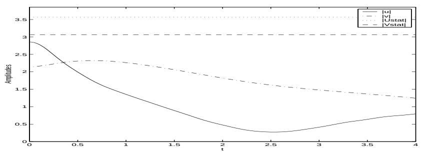

Figure 15: Evolution of $\|u(t,.)\|_{L^\infty}$ and $\|v(t,.)\|_{L^\infty}$ with respect to time, $(\mu_u, \mu_v) = (-0.2, -0.3)$, $\gamma_2 = 0.5$.

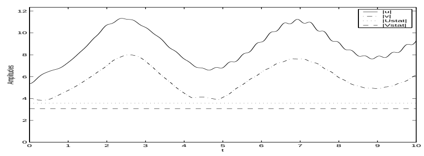

Figure 16: Evolution of $\|u(t,.)\|_{L^\infty}$ and $\|v(t,.)\|_{L^\infty}$ with respect to time, $(\mu_u, \mu_v) = (0.5, 0.3)$, $\gamma_2 = 0.5$.

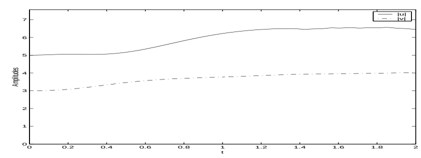

Figure 17: Evolution of $\|u(t,.)\|_{L^\infty}$ and $\|v(t,.)\|_{L^\infty}$ with respect to time, $(\mu_u, \mu_v) = (0.01, -0.01)$, $\gamma_2 = 0.5$.

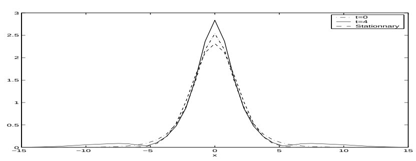

Figure 18: Profile of the trace at $z = 0$ of initial data, computed solution at final time $T = 4$ and stationary state u, $\gamma_2 = 0.5$.

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