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Nonlinear Analysis 56 (2004) 213 – 226
Nonlinear Analysis
www.elsevier.com/locate/na

Solutions for a quasilinear Schrödinger equation: a dual approach

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Received 12 June 2003; accepted 15 September 2003

Abstract

We consider quasilinear stationary Schrödinger equations of the form

$$-\Delta u-\Delta(u^2)u=g(x,u),\qquad x\in\mathbb{R}^N. \tag{1}$$

Introducing a change of unknown, we transform the search of solutions $u(x)$ of (1) into the search of solutions $v(x)$ of the semilinear equation

$$-\Delta v=\frac{1}{\sqrt{1+2f^2(v)}}\,g(x,f(v)),\qquad x\in\mathbb{R}^N, \tag{2}$$

where f is suitably chosen. If v is a classical solution of (2) then $u=f(v)$ is a classical solution of (1). Variational methods are then used to obtain various existence results.

MSC: Primary: 35J60; Secondary: 58E05

Keywords: Quasilinear Schrödinger equations; Minimax methods

1. Introduction

In this paper we deal with equations of the form

$$-\Delta u-\Delta(u^2)u=g(x,u),\qquad u\in H^1(\mathbb{R}^N). \tag{1.1}$$

These equations model several physical phenomena but until recently little had been done to prove rigorously the existence of solutions.

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0362-546X/$ - see front matter
doi:10.1016/j.na.2003.09.008

A major difficulty associated with (1.1) is the following; one may seek to obtain solutions by looking for critical points of the associated "natural" functional, $J: H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \,dx + \int_{\mathbb{R}^N} |\nabla u|^2 u^2 \,dx - \int_{\mathbb{R}^N} G(x,u)\,dx,$$

where $G(x,s) = \int_0^s g(x,t)\,dt$. However except when $N=1$ this functional is not defined on all $H^1(\mathbb{R}^N)$.

The first existence results for equations of the form of (1.1) are, up to our knowledge, due to [12,7]; papers to which we refer for a presentation of the physical motivations of studying (1.1). In [12,7], however, the main existence results are obtained, through a constrained minimization argument, only up to an unknown Lagrange multiplier.

Subsequently a general existence result for (1.1) was derived in [8]. To overcome the undefiniteness of J the idea in [8] is to introduce a change of variable and to rewrite the functional J with this new variable. Then critical points are search in an associated Orlicz space (see [8] for details).

The aim of the present paper is to give a simple and shorter proof of the results of [8], which do not use Orlicz spaces, but rather is developed in the usual $H^1(\mathbb{R}^N)$ space. The fact that we work in $H^1(\mathbb{R}^N)$ also permit to cover a different class of nonlinearities. In particular we give full treatment of the autonomous case and for nonautonomous problems we do not assume that,

$$s \to \frac{g(x,s)}{s}: \,]0,\infty[\, \to \mathbb{R} \text{ is nondecreasing in } s.$$

Following the strategy developed in [4] on a related problem, we also make use of a change of unknown $v = f^{-1}(u)$ and define an associated equation that we shall call dual. If $v \in H^1(\mathbb{R}^N)$ is classical solution of

$$-\Delta v = \frac{1}{\sqrt{1+2f^2(v)}}\, g(x, f(v)), \tag{1.2}$$

$u = f(v)$ is a classical solution of (1.1).

Equations of form (1.2) are of semilinear elliptic type and one can try to solve them by a variational approach. In particular we shall see that, under very general conditions on g, the "natural" functional associated to (1.2), $I: H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \,dx - \int_{\mathbb{R}^N} G(x, f(v))\,dx$$

is well defined and of class C^1 on $H^1(\mathbb{R}^N)$.

The dual approach is introduced in Section 2. In Section 3, we deal with autonomous problems, when (1.1) is of the form

$$-\Delta u - \Delta(u^2)u = g(u), \quad u \in H^1(\mathbb{R}^N). \tag{1.3}$$

Autonomous problems seems to play an important role in physical phenomena (see [3] for example) and we obtain here an existence result under assumptions we believe to be nearly optimal. We assume that the nonlinear term g satisfies:

(g0) $g(s)$ is locally Hölder continuous on $[0,\infty[$.
(g1) $-\infty < \liminf_{s\to 0} g(s)/s \leqslant \limsup_{s\to 0} g(s)/s = -v < 0$ for $N \geqslant 3$,
$\lim_{s\to 0} g(s)/s = -v \in (-\infty, 0)$ for $N = 1,2$.
(g2) When $N \geqslant 3$, $\lim_{s\to\infty} |g(s)|/s^{(3N+2)/(N-2)} = 0$.
When $N = 2$, for any $\alpha > 0$ there exists $C_\alpha > 0$ such that

$$|g(s)| \leqslant C_\alpha e^{\alpha s^2} \quad \text{for all } s \geqslant 0.$$

(g3) When $N \geqslant 2$, there exists $\xi_0 > 0$ such that $G(\xi_0) > 0$,
When N = 1, there exists $\xi_0 > 0$ such that

$$G(\xi) < 0 \text{ for all } \xi \in]0, \xi_0[, G(\xi_0) = 0 \quad \text{and} \quad g(\xi_0) > 0.$$

Remark 1.1. An easy calculation shows that (g0)–(g3) are satisfied in the model case $g(s) = |s|^2 s - vs$.

Theorem 1.2. *Assume that* (g0)–(g3) *hold. Then* (1.3) *admits a solution* $u_0 \in H^1(\mathbb{R}^N)$ *having the following properties*:

(i) $u_0 > 0$ *on* $\mathbb{R}^N$.
(ii) u_0 *is spherically symmetric*: $u_0(x) = u_0(r)$ *with* $r = |x|$ *and* u_0 *decreases with respect to* r.
(iii) $u_0 \in C^2(\mathbb{R}^N)$.
(iv) u_0 *together with its derivatives up to order* 2 *have exponential decay at infinity*

$$|D^\alpha u_0(x)| \leqslant Ce^{-\delta|x|}, \quad x \in \mathbb{R}^N$$

for some $C, \delta > 0$ *and for* $|\alpha| \leqslant 2$.

We prove Theorem 1.2 searching for a critical point of the functional I, which is here autonomous. As we shall see the existence of a critical point follows almost directly, from classical results on scalar field equations due to Berestycki–Lions [2] when $N = 1$ or $N \geqslant 3$ and Berestycki–Gallouët–Kavian [1] when $N = 2$.

In Section 4 we assume that (1.1) is of the form,

$$-\Delta u - \Delta(u^2)u + V(x)u = h(u). \tag{1.4}$$

We require $V \in C(\mathbb{R}^N, \mathbb{R})$ and $h \in C(\mathbb{R}^+, \mathbb{R})$, to be Hölder continuous and to satisfy

(V0) There exists $V_0 > 0$ such that $V(x) \geqslant V_0 > 0$ on $\mathbb{R}^N$.
(V1) $\lim_{|x|\to\infty} V(x) = V(\infty)$ and $V(x) \leqslant V(\infty)$ on $\mathbb{R}^N$.
(h0) $\lim_{s\to 0} h(s)/s = 0$.
(h1) There exists $p < \infty$ if $N = 1,2$ and $p < (3N+2)/(N-2)$ if $N \geqslant 3$ such that $|h(s)| \leqslant C(1 + |s|^p)$, $\forall s \in \mathbb{R}$, for a $C > 0$.
(h2) There exists $\mu \geqslant 4$ such that, $\forall s > 0$,

$$0 < \mu H(s) \leqslant h(s)s \text{ with } H(s) = \int_0^s h(t)\,dt.$$

Our main result is the following:

Theorem 1.3. *Assume that* (V0)–(V1) *and* (h0)–(h1) *hold. Then* (1.4) *has a positive nontrivial solution if one of the following conditions hold*:

(1) (h2) *hold with a* $\mu > 4$.
(2) (h2) *hold with* $\mu = 4$ *with* $p \leqslant 5$ *if* $N = 3$ *and* $p < (3N+4)/N$ *if* $N \geqslant 4$ *in* (h1).

The proof of Theorem 1.3 also relies on the study of the functional I. We first show that I possess a mountain pass geometry and denote by $c > 0$ the mountain pass level (see Lemma 4.2). To find a critical point the main difficulties to overcome are the possible unboundedness of the Palais–Smale (or Cerami) sequences and a lack of compactness since (1.4) is set on all $\mathbb{R}^N$.

For the second difficulty we use some recent results presented in [9,10] which imply that, under conditions (V0)–(V1), the mountain pass level $c > 0$ is below (if $V \not\equiv V(\infty)$) the first level of possible loss of compactness (see Theorem 3.4 and Lemma 4.3).

For the first difficulty we distinguish the cases $\mu > 4$ and $\mu = 4$ in (h2). In the case $\mu > 4$, it is direct to prove that all Cerami sequences of I are bounded. To show it in the case $\mu = 4$ is more involved and for this we make use of an idea introduced in [8].

Notation. Throughout the article the letter C will denote various positive constants whose exact value may change from line to line but are not essential to the analysis of the problem. Also if we take a subsequence of a sequence $\{v_n\}$ we shall denote it again $\{v_n\}$.

2. The dual formulation

We start with some preliminary results. Let f be defined by

$$f'(t) = \frac{1}{\sqrt{1+2f^2(t)}} \quad \text{and} \quad f(0) = 0$$

on $[0, +\infty[$ and by $f(t) = -f(-t)$ on $]-\infty, 0]$.

Lemma 2.1. (1) f *is uniquely defined*, C^∞ *and invertible*.
(2) $|f'(t)| \leqslant 1$, *for all* $t \in \mathbb{R}$.
(3) $\frac{f(t)}{t} \to 1$ *as* $t \to 0$.
(4) $\frac{f(t)}{\sqrt{t}} \to 2^{1/4}$ *as* $t \to +\infty$.

Proof. Points (1)–(3) are immediate. To see (4) we integrate

$$\int_0^t f'(s)\sqrt{1+2f^2(s)}\,\mathrm{d}s = t.$$

Using the changes of variables $x = f(s)$ and $x = \frac{1}{\sqrt{2}} Sh(y)$ we obtain that

$$\frac{1}{2\sqrt{2}}[\sinh^{-1}(\sqrt{2} f(t))] + \frac{1}{4\sqrt{2}} \sinh 2[Sh^{-1}(\sqrt{2} f(t))] = t.$$

Thus, $\sinh 2[Sh^{-1}(\sqrt{2} f(t))] \sim 4\sqrt{2}t$ in the sense that, as $t \to +\infty$,

$$\frac{\sinh 2[\sinh^{-1}(\sqrt{2} f(t))]}{4\sqrt{2}t} \to 1.$$

We set $a(t) = \sinh^{-1}(\sqrt{2} f(t))$. Then $a(t)$ satisfies $\sinh[2a(t)] \sim 4\sqrt{2}t$ and we deduce that

$$a(t) \sim \tfrac{1}{2} \ln(4\sqrt{2}t + \sqrt{32t^2 + 1}).$$

Finally since $2\sinh(t) \sim \mathrm{e}^t$ it follows that

$$2\sqrt{2} f(t) \sim e^{(1/2)\ln(4\sqrt{2}t+\sqrt{32t^2+1})} \sim 2\sqrt{2}\, 2^{1/4}\sqrt{t}$$

and the lemma is proved. □

Lemma 2.2. *For all* $t \in \mathbb{R}$,

$$\frac{1}{2} f(t) \leqslant \frac{t}{\sqrt{1 + 2f^2(t)}} \leqslant f(t).$$

Proof. To establish the first inequality we need to show that, for all $t \geqslant 0$,

$$\sqrt{1 + 2f^2(t)} f(t) \leqslant 2t.$$

In this aim we study the function $g : \mathbb{R}^+ \to \mathbb{R}$, defined by

$$g(t) = 2t - \sqrt{1 + 2f^2(t)} f(t).$$

We have $g(0)=0$ and, since $f'(t)\sqrt{1 + 2f^2(t)}=1, \forall t \in \mathbb{R}$, that $g'(t)=1-2f'^2(t)f^2(t)$. It follows that $g'(t) \geqslant 0$ since $1-2f'^2(t)f^2(t)=f'^2(t)$ and the first inequality is proved. The second one is derived in a similar way. □

We now present our dual approach. For simplicity we set $H = H^1(\mathbb{R}^N)$ and denote by $\|\cdot\|$ its standard norm. We assume that $g(x,s)$ is such that $I : H \to \mathbb{R}$ given by

$$I(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, \mathrm{d}x - \int_{\mathbb{R}^N} G(x, f(v)) \, \mathrm{d}x$$

with $G(x,s) = \int_0^s g(x,t)\, \mathrm{d}t$, is well defined and of class C^1 ($f : \mathbb{R} \to \mathbb{R}$ is the function previously introduced).

Let $v \in H \cap C^2(\mathbb{R}^N)$ be a critical point of I. Since $f'^2(t)(1+2f^2(t)) \equiv 1$, it satisfies

$$-\Delta v = \frac{1}{\sqrt{1 + 2f^2(v)}} g(x, f(v)). \tag{2.1}$$

We set $u = f(v)$ (i.e. $v = f^{-1}(u)$). Clearly $u \in C^2(\mathbb{R}^N)$ and $u \in H$. Indeed $\nabla u = f'(v)\nabla v$ and $|f'(t)| \leqslant 1$, $\forall t \in \mathbb{R}$.

We have $\nabla v = (f^{-1})'(u)\nabla u$ and

$$\Delta v = (f^{-1})''(u)|\nabla u|^2 + (f^{-1})'(u)\Delta u. \tag{2.2}$$

Since $(f^{-1})'(t) = \frac{1}{f'[f^{-1}(t)]}$, it follows that

$$(f^{-1})'(t) = \sqrt{1 + 2f^2(f^{-1}(t))} = \sqrt{1 + 2t^2} \text{ and } (f^{-1})''(t) = \frac{2t}{\sqrt{1 + 2t^2}}.$$

Thus, from (2.2), we deduce that

$$\Delta v = \frac{2u}{\sqrt{1 + 2u^2}}|\nabla u|^2 + \sqrt{1 + 2u^2}\Delta u$$

and consequently, from (2.1), that

$$-\frac{2u}{\sqrt{1 + 2u^2}}|\nabla u|^2 - \sqrt{1 + 2u^2}\Delta u - \frac{1}{\sqrt{1 + 2u^2}}g(x, u) = 0.$$

This can be rewrite as

$$\frac{1}{\sqrt{1 + 2u^2}}[(-1 - 2u^2)\Delta u - 2u|\nabla u|^2 - g(x, u)] = 0.$$

Since $\Delta(u^2)u = 2u|\nabla u|^2 + 2u^2\Delta u$ it shows that $u \in H \cap C^2(\mathbb{R}^N)$ satisfies (1.1).

At this point it is clear that to obtain a classical solution of (1.1) it suffices to obtain a critical point of I of class C^2.

3. Autonomous cases

In this section (1.1) is of the form

$$-\Delta u - \Delta(u^2)u = g(u), \quad u \in H. \tag{3.1}$$

with the nonlinearity g satisfying (g0)–(g3). Because we look for positive solutions we may assume without restriction that $g(s) = 0, \forall s \leqslant 0$. Following our dual approach we shall obtain the existence of solutions for (3.1) studying the associated dual equation

$$-\Delta v = \frac{1}{\sqrt{1 + 2f^2(v)}}g(f(v)), \quad v \in H. \tag{3.2}$$

In this aim, we now recall some classical results due to Berestycki–Lions [2] and Berestycki–Gallouët–Kavian [1] on equations of the form

$$-\Delta v = k(v), \quad v \in H. \tag{3.3}$$

These authors show that the natural functional corresponding to (3.3), $J : H \to \mathbb{R}$ given by

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,dx - \int_{\mathbb{R}^N}K(v)\,dx$$

where $K(s)=\int_0^s k(t)\,dt$ is of class C^1, if k satisfies the conditions:

(k0) $k(s)\in C(\mathbb{R}^+,\mathbb{R})$ (and $k(s)=0, \forall s\leqslant 0$).
(k1) $-\infty<\liminf_{s\to 0}\frac{k(s)}{s}\leqslant \limsup_{s\to 0}\frac{k(s)}{s}=-\nu<0$ for $N\geqslant 3$, $\lim_{s\to 0}\frac{k(s)}{s}=-\nu\in(-\infty,0)$ for $N=1,2$.
(k2) When $N\geqslant 3$, $\lim_{s\to\infty}|k(s)|/s^{(N+2)/(N-2)}=0$.
When $N=2$, for any $\alpha>0$ there exists $C_\alpha>0$ such that

$$|k(s)|\leqslant C_\alpha e^{\alpha s^2} \quad \text{for all } s\geqslant 0.$$

We recall that a solution $v\in H$ of (3.3) is said to be a least energy solution if and only if

$$J(v)=m \text{ where } m=\inf\{J(v), v\in H\setminus\{0\} \text{ is a solution of (3.3)}\}.$$

The following result is given in [2] when $N=1$ or $N\geqslant 3$ and in [1] when $N=2$.

Theorem 3.1. *Assume that* (k0)–(k2) *and* (k3) *hold with*

(k3) *When* $N\geqslant 2$, *there exists* $\xi_0>0$ *such that* $K(\xi_0)>0$.
When $N=1$, *there exists* $\xi_0>0$ *such that*

$$K(\xi)<0 \text{ for all } \xi\in]0,\xi_0[, K(\xi_0)=0 \text{ and } k(\xi_0)>0.$$

Then $m>0$ *and there exists a least energy solution* $\omega(x)$ *of* (3.3) *which satisfies*:

(i) $\omega>0$ *on* $\mathbb{R}^N$.
(ii) ω *is spherically symmetric*: $\omega(x)=\omega(r)$ *with* $r=|x|$ *and* ω *decreases with respect to* r.
(iii) $\omega\in C^2(\mathbb{R}^N)$.
(iv) ω *together with its derivatives up to order* 2 *have exponential decay at infinity*

$$|D^\alpha\omega(x)|\leqslant Ce^{-\delta|x|}, \quad x\in\mathbb{R}^N,$$

for some $C,\delta>0$ *and for* $|\alpha|\leqslant 2$.

Now observe that Eq. (3.2) is of the form $-\Delta v=k(v)$ with

$$k(s)=\frac{1}{\sqrt{1+2f^2(s)}}g(f(s)). \tag{3.4}$$

We claim that if $g(s)$ satisfies (g0)–(g3) then $k(s)$ given by (3.4) satisfies (k0)–(k3). Indeed the fact that (k0) holds is trivial. Conditions (k1), (k2) follow, respectively, from Lemma 2.1(ii) and (iii). To check (k3) when $N\geqslant 2$ it suffices to notice that

$$G(\xi_0)>0 \text{ for a } \xi_0>0 \Leftrightarrow \exists s_0>0 \text{ such that } G(f(s_0))>0.$$

Clearly (k3) also holds when $N=1$. Having proved our claim we directly obtain from Theorem 3.1.

Theorem 3.2. *Assume that* (g0)–(g2) *hold. Then the functional* $I : H \to \mathbb{R}$ *given by*

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \, dx - \int_{\mathbb{R}^N} G(f(v)) \, dx$$

is well defined and of class C^1. *If in addition* g *satisfies* (g3) *then* (3.2) *has a least energy solution* $\omega(x)$ *which possesses the properties* (i)–(iv) *of Theorem* 3.1.

At this point turning back to Eq. (3.1), Theorem 1.2 follows directly from Theorem 3.2 and the properties of f (see Lemma 2.1).

Remark 3.3. In [2] the authors justify the growth restriction (k2) considering the special nonlinearities $k(s) = \lambda|s|^{p-1}s - ms$ where $\lambda, m > 0$. They show that in this case (3.3) has no solution when $p \geqslant (N+2)/(N-2)$. In contrast, Theorem 3.1 says that solutions of (3.1) do exist for all $1 < p < (3N+2)/(N-2)$.

In the next section we shall use the fact that the least energy solution $\omega(x)$ given in Theorem 3.2 has a mountain pass characterization. Indeed, in [9] for $N \geqslant 2$ and in [10] for $N = 1$, Theorem 3.1 is complemented in the following way:

Theorem 3.4. *Assume that* (k0)–(k3) *hold. Then setting*

$$\Gamma = \{\gamma \in C([0,1], H),\ \gamma(0) = 0 \text{ and } J(\gamma(1)) < 0\},$$

we have $\Gamma \neq \emptyset$ *and* $b = m$ *with*

$$b \equiv \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)).$$

Moreover for any least energy solution $\omega(x)$ *as given in Theorem* 3.1, *there exists a path* $\gamma \in \Gamma$ *such that* $\gamma(t)(x) > 0$ *for all* $x \in \mathbb{R}^N$ *and* $t \in (0,1]$ *satisfying* $\omega \in \gamma([0,1])$ *and*

$$\max_{t \in [0,1]} J(\gamma(t)) = b.$$

Remark 3.5. In [9,10] it is also proved that under (k0)–(k2) there exists $\alpha_0 > 0$, $\delta_0 > 0$ *such that*

$$J(v) \geqslant \alpha_0 \|v\|^2 \text{ when } \|v\| \leqslant \delta_0.$$

4. Nonautonomous cases

In this section we assume that (1.1) is of the form

$$-\Delta u - \Delta(u^2)u + V(x)u = h(u), \quad u \in H. \tag{4.1}$$

with the potential $V(x)$ satisfying (V0)–(V1) and the nonlinearity $h(s)$, (h0)–(h2). Here again we use our dual approach and first look to critical points of $I : H \to \mathbb{R}$ given by

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + V(x) f^2(v) \, dx - \int_{\mathbb{R}^N} H(f(v)) \, dx.$$

Namely for solutions $v \in H$ of

$$-\Delta v = \frac{1}{\sqrt{1+2f^2(v)}}[-V(x)f(v) + h(f(v))]. \tag{4.2}$$

From Section 3 we readily deduce that I is well defined and of class C^1 under conditions (V0)–(V1) and (h0)–(h1). Let us show that I has a mountain pass geometry, in the sense that,

$$\Gamma = \{\gamma \in C([0,1], H), \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\} \neq \emptyset,$$

and

$$c \equiv \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0.$$

For this we first mention a direct consequence of (h2).

Remark 4.1. The function $t \to H(st)t^{-4}$ is increasing on $\mathbb{R}^+$, for all $s > 0$. In particular there is $C > 0$ such that $H(s) \geqslant Cs^4$ for $s \geqslant 1$ and $\lim_{s \to +\infty} h(s)s^{-1} = \infty$.

Lemma 4.2. *Under* (V0)–(V1) *and* (h0)–(h2) *I has a mountain pass geometry.*

Proof. From the assumptions (V0)–(V1) we have

$$k_1(s) \leqslant \frac{1}{\sqrt{1+2f^2(v)}}[-V(x)f(v) + h(f(v))] \leqslant k_2(s),$$

where

$$k_1(s) = \frac{1}{\sqrt{1+2f^2(v)}}[-V(\infty)f(v) + h(f(v))]$$

and

$$k_2(s) = \frac{1}{\sqrt{1+2f^2(v)}}[-V_0 f(v) + h(f(v))].$$

The nonlinearities $k_1(s)$ and $k_2(s)$ both satisfy assumptions (k0)–(k3). Thus, from Remark 3.5, we deduce (considering $k_2(s)$) that there exists $\alpha_0 > 0$, $\delta_0 > 0$ such that

$$I(v) \geqslant \alpha_0 \|v\|^2 \text{ when } \|v\| \leqslant \delta_0. \tag{4.3}$$

Namely the origin is a strict local minimum. Also since the functional corresponding to $k_1(s)$ is negative at some point we deduce that $\Gamma \neq \emptyset$. □

Lemma 4.3. *Assume that* (V0)–(V1) *and* (h0)–(h2) *hold. Let* $\{v_n\} \subset H$ *be a bounded Palais–Smale sequence for I at level* $c > 0$. *Then, up to a subsequence,* $v_n \rightharpoonup v \neq 0$ *with* $I'(v) = 0$.

Proof. Since $\{v_n\}$ is bounded, we can assume that, up to a subsequence, $v_n \rightharpoonup v$. Let us prove that $I'(v) = 0$. Noting that $C_0^\infty(\mathbb{R}^N)$ is dense in H, it suffices to check that

$I'(v)\varphi = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. But we readily have, using Lebesgue's Theorem, that

$$I'(v_n)\varphi - I'(v)\varphi = \int_{\mathbb{R}^N} \nabla(v_n - v)\nabla\varphi \, dx + \int_{\mathbb{R}^N} \left(\frac{-f(v_n)}{\sqrt{1+2f^2(v_n)}} + \frac{f(v)}{\sqrt{1+2f^2(v)}} \right) V(x)\varphi \, dx + \int_{\mathbb{R}^N} \left(\frac{h(f(v_n))}{\sqrt{1+2f^2(v_n)}} - \frac{h(f(v))}{\sqrt{1+2f^2(v)}} \right) \varphi \, dx \to 0,$$

since $v_n \rightharpoonup v$ weakly in H and strongly in $L^q_{loc}(\mathbb{R}^N)$ for $q \in [2, 2N/(N-2)[$ if $N \geqslant 3$, $q \geqslant 2$ if $N = 1, 2$. Thus recalling that $I'(v_n) \to 0$ we indeed have $I'(v) = 0$. At this point if $v \neq 0$ the lemma is proved. Thus we assume that $v = 0$. We claim that in this case $\{v_n\}$ is also a Palais–Smale sequence for the functional $\tilde{I} : H \to \mathbb{R}$ defined by

$$\tilde{I}(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(\infty) f^2(v) \, dx - \int_{\mathbb{R}^N} H(f(v)) \, dx$$

at the level $c > 0$. Indeed, as $n \to \infty$,

$$\tilde{I}(v_n) - I(v_n) = \int_{\mathbb{R}^N} [V(\infty) - V(x)] f^2(v_n) \, dx \to 0$$

since $V(x) \to V(\infty)$ as $|x| \to \infty$, $|f(s)| \leqslant |s|, \forall s \in \mathbb{R}$ and $v_n \to 0$ in $L^2_{loc}(\mathbb{R}^N)$. Also, for the same reasons, we have

$$\sup_{\|u\| \leqslant 1} |(\tilde{I}'(v_n) - I'(v_n), u)| = \sup_{\|u\| \leqslant 1} \left| \int_{\mathbb{R}^N} \frac{f(v_n)u}{\sqrt{1+2f^2(v_n)}} [V(\infty) - V(x)] \, dx \right| \to 0.$$

Next we claim that the situation: For all $R > 0$

$$\lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{y+B_R} v_n^2 \, dx = 0$$

which we will refer to as the vanishing case cannot occurs. From (h0)–(h1) and Lemma 2.1, $\forall \varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that

$$h(f(s))f(s) \leqslant \varepsilon s^2 + C_\varepsilon |s|^{(p+1)/2} \quad \text{for all } s \in \mathbb{R}. \tag{4.4}$$

Thus, for any $v \in H$,

$$\int_{\mathbb{R}^N} h(f(v))f(v) \, dx \leqslant \varepsilon \int_{\mathbb{R}^N} v^2 \, dx + C_\varepsilon \int_{\mathbb{R}^N} |v|^{(p+1)/2} \, dx \tag{4.5}$$

and using Lemma 2.2 we see that $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h(f(v_n)) \frac{v_n}{\sqrt{1+2f^2(v_n)}} \, dx \leqslant \lim_{n\to\infty} \int_{\mathbb{R}^N} h(f(v_n))f(v_n) \, dx \leqslant \lim_{n\to\infty} \left[\varepsilon \int_{\mathbb{R}^N} v_n^2 \, dx + C_\varepsilon \int_{\mathbb{R}^N} |v_n|^{(p+1)/2} \, dx \right] \leqslant \varepsilon \lim_{n\to\infty} \int_{\mathbb{R}^N} v_n^2 \, dx$$

because, if $\{v_n\}$ vanish, $v_n \to 0$ strongly in $L^q(\mathbb{R}^N)$ for any $q \in]2, 2N/(N-2)[$ (a proof of this result is given in Lemma 2.18 of [5] and is a special case of Lemma I.1 of [11]). We then deduce that,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h(f(v_n)) \frac{v_n}{\sqrt{1+2f^2(v_n)}} \, dx = 0.$$

This implies, since $I'(v_n)v_n \to 0$, that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 + V(x) f^2(v_n) \, dx \to 0$$

in contradiction with the fact that $I(v_n) \to c > 0$. Thus $\{v_n\}$ does not vanish and there exists $\alpha > 0$, $R < \infty$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{y_n + B_R} v_n^2 \, dx \geqslant \alpha > 0.$$

Let $\tilde{v}_n(x) = v_n(x + y_n)$. Since $\{v_n\}$ is a Palais–Smale sequence for $\tilde{I}$, $\{\tilde{v}_n\}$ also. Arguing as in the case of $\{v_n\}$ we get that $\tilde{v}_n \rightharpoonup \tilde{v}$, up to a subsequence, with $\tilde{I}'(\tilde{v}) = 0$. Since $\{\tilde{v}_n\}$ is nonvanishing we also have that $\tilde{v} \neq 0$.

Now observe that, because of Lemma 2.2, for all $x \in \mathbb{R}^N$, $n \in \mathbb{N}$,

$$f^2(\tilde{v}_n) - \frac{f(\tilde{v}_n)\tilde{v}_n}{\sqrt{1+2f^2(\tilde{v}_n)}} \geqslant 0,$$

also, because of condition (h2), for all $x \in \mathbb{R}^N$, $n \in \mathbb{N}$,

$$\frac{1}{2} \frac{h(f(\tilde{v}_n))\tilde{v}_n}{\sqrt{1+2f^2(\tilde{v}_n)}} - H(f(\tilde{v}_n)) \geqslant 0.$$

Indeed, for all $x \in \mathbb{R}^N$, $n \in \mathbb{N}$,

$$\frac{1}{2} \frac{h(f(\tilde{v}_n))\tilde{v}_n}{\sqrt{1+2f^2(\tilde{v}_n)}} \geqslant \frac{1}{2} h(f(\tilde{v}_n)) f(\tilde{v}_n) \geqslant \frac{\mu}{4} H(f(\tilde{v}_n)) \geqslant H(f(\tilde{v}_n)).$$

Thus, from Fatou's lemma, we get

$$\begin{aligned}
c &= \limsup_{n\to\infty} \left[\tilde{I}(\tilde{v}_n) - \tfrac{1}{2} \tilde{I}'(\tilde{v}_n)\tilde{v}_n \right] \\
&= \limsup_{n\to\infty} \frac{1}{2} \int_{\mathbb{R}^N} \left[f^2(\tilde{v}_n) - \frac{f(\tilde{v}_n)\tilde{v}_n}{\sqrt{1+2f^2(\tilde{v}_n)}} \right] V(\infty) \, dx \\
&\quad + \limsup_{n\to\infty} \int_{\mathbb{R}^N} \left[\frac{1}{2} \frac{h(f(\tilde{v}_n))\tilde{v}_n}{\sqrt{1+2f^2(\tilde{v}_n)}} - H(f(\tilde{v}_n)) \right] dx \\
&\geqslant \frac{1}{2} \int_{\mathbb{R}^N} \left[f^2(\tilde{v}) - \frac{f(\tilde{v})\tilde{v}}{\sqrt{1+2f^2(\tilde{v})}} \right] V(\infty) \, dx + \int_{\mathbb{R}^N} \left[\frac{1}{2} \frac{h(f(\tilde{v}))\tilde{v}}{\sqrt{1+2f^2(\tilde{v})}} - H(f(\tilde{v})) \right] dx \\
&= \tilde{I}(\tilde{v}) - \tfrac{1}{2} \tilde{I}'(\tilde{v})\tilde{v} = \tilde{I}(\tilde{v}).
\end{aligned}$$

Namely $\tilde{v} \neq 0$ is a critical point of $\tilde{I}$ satisfying $\tilde{I}(\tilde{v}) \leqslant c$. We deduce that the least energy level $\tilde{m}$ for $\tilde{I}$ satisfies $\tilde{m} \leqslant c$. We denote by $\tilde{\omega}$ a least energy solution as provided by Theorem 3.1. Now applying Theorem 3.4 to the functional $\tilde{I}$ we can find a path $\gamma(t) \in C([0,1], H)$ such that $\gamma(t)(x) > 0$, $\forall x \in \mathbb{R}^N$, $\forall t \in (0,1]$, $\gamma(0) = 0$, $\tilde{I}(\gamma(1)) < 0$, $\tilde{\omega} \in \gamma([0,1])$ and

$$\max_{t\in[0,1]} \tilde{I}(\gamma(t)) = \tilde{I}(\tilde{\omega}).$$

Without restriction we can assume that $V(x) \leqslant V(\infty)$ but $V \not\equiv V(\infty)$ in (V1) (otherwise there is nothing to prove). Thus

$$I(\gamma(t)) < \tilde{I}(\gamma(t)) \quad \text{for all } t \in (0,1]$$

and it follows that

$$c \leqslant \max_{t\in[0,1]} I(\gamma(t)) < \max_{t\in[0,1]} \tilde{I}(\gamma(t)) \leqslant c.$$

This is a contradiction and the lemma is proved. □

At this point to end the proof of Theorem 1.3 we just need to show that there exists a Palais–Smale sequence for I as in Lemma 4.3. From Lemma 4.2 we know (see [6]) that I possesses a Cerami sequence at the level $c > 0$. Namely a sequence $\{v_n\} \subset H$ such that

$$I(v_n) \to c \text{ and } \|I'(v_n)\|_{H^{-1}}(1 + \|v_n\|) \to 0 \text{ as } n \to \infty.$$

Lemma 4.4. *Assume that* (V0)–(V1) *and* (h0)–(h2) *hold. Then all Cerami sequences for I at the level $c > 0$ are bounded in H.*

Proof. First we observe that if a sequence $\{v_n\} \subset H$ satisfies

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 \,dx + \int_{\mathbb{R}^N} V(x) f^2(v_n) \,dx \quad \text{is bounded} \tag{4.6}$$

then it is bounded in H. To see this we just need to show that $\int_{\mathbb{R}^N} v_n^2 \,dx$ is bounded. We write

$$\int_{\mathbb{R}^N} v_n^2 \,dx = \int_{\{x\,:\,|v_n(x)| \leqslant 1\}} v_n^2 \,dx + \int_{\{x\,:\,|v_n(x)| > 1\}} v_n^2 \,dx.$$

By Remark 4.1, there exists a $C > 0$ such that $H(s) \geqslant Cs^4$ for all $s \geqslant 1$ and thus, because of the behavior of f at infinity, for a $C > 0$, $H(f(s)) \geqslant Cs^2$, for all $s \geqslant 1$. It follows that

$$\int_{\{x\,:\,|v_n(x)| > 1\}} v_n^2 \,dx \leqslant \frac{1}{C} \int_{\{x\,:\,|v_n(x)| > 1\}} H(f(v_n)) \,dx \leqslant \frac{1}{C} \int_{\mathbb{R}^N} H(f(v_n)) \,dx.$$

Also, for a $C > 0$, since $f(s) \geqslant Cs$ for all $s \in [0,1]$, (see Lemma 2.1) we also have

$$\int_{\{x\,:\,|v_n(x)| \leqslant 1\}} v_n^2 \,dx \leqslant \frac{1}{C} \int_{\{x\,:\,|v_n(x)| \leqslant 1\}} f^2(v_n) \,dx \leqslant \frac{1}{C} \int_{\mathbb{R}^N} f^2(v_n) \,dx.$$

At this point the boundedness of $\{v_n\} \subset H$ is clear.

Now let $\{v_n\} \subset H$ be an arbitrary Cerami sequence for I at the level $c > 0$. We have for any $\phi \in H$

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(v_n)\,\mathrm{d}x - \int_{\mathbb{R}^N} H(f(v_n))\,\mathrm{d}x = c + o(1), \tag{4.7}$$

$$I'(v_n)\phi = \int_{\mathbb{R}^N} \nabla v_n \nabla \phi \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)\frac{f(v_n)\phi}{\sqrt{1+2f^2(v_n)}}\,\mathrm{d}x - \int_{\mathbb{R}^N} \frac{h(f(v_n))\phi}{\sqrt{1+2f^2(v_n)}}\,\mathrm{d}x. \tag{4.8}$$

Choosing $\phi = \phi_n = \sqrt{1+2f^2(v_n)}f(v_n)$ we have, from Lemma 2.1, $\|\phi_n\|_2 \leqslant C\|v_n\|_2$ and

$$|\nabla \phi_n| = \left(1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n| \leqslant 2|\nabla v_n|.$$

Thus $\|\phi_n\| \leqslant C\|v_n\|$ and, in particular, recording that $\{v_n\} \subset H$ is a Cerami sequence

$$I'(v_n)\phi_n = \int_{\mathbb{R}^N} \left(1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x) f^2(v_n)\,\mathrm{d}x - \int_{\mathbb{R}^N} h(f(v_n))f(v_n)\,\mathrm{d}x = o(1). \tag{4.9}$$

Now using (h2) it follows computing (4.7) $- 1/\mu$(4.9) that

$$\int_{\mathbb{R}^N} \left(\frac{1}{2} - \frac{1}{\mu}\left(1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right)\right)|\nabla v_n|^2 \,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^N} V(x) f^2(v_n)\,\mathrm{d}x \leqslant c + o(1). \tag{4.10}$$

Since $1 + 2f^2(v_n)/(1+2f^2(v_n)) \leqslant 2$, if $\mu > 4$ we immediately deduce that (4.6) hold and thus $\{v_n\} \subset H$ is bounded. If $\mu = 4$ we obtain from (4.10)

$$\frac{1}{4}\int_{\mathbb{R}^N} \frac{|\nabla v_n|^2}{1+2f^2(v_n)}\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^N} V(x) f^2(v_n)\,\mathrm{d}x \leqslant c + o(1). \tag{4.11}$$

Denoting $u_n = f(v_n)$, we have $|\nabla v_n|^2 = (1 + 2f^2(v_n))|\nabla u_n|^2$ and (4.7), (4.11) give

$$\int_{\mathbb{R}^N} (1 + 2u_n^2)|\nabla u_n|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x) u_n^2 \,\mathrm{d}x - 2\int_{\mathbb{R}^N} H(u_n)\,\mathrm{d}x = 2c + o(1). \tag{4.12}$$

$$\frac{1}{4}\int_{\mathbb{R}^N} |\nabla u_n|^2 \,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^N} V(x) u_n^2 \,\mathrm{d}x \leqslant c + o(1). \tag{4.13}$$

From (4.13) we see that $\{u_n\} \subset H$ is bounded. Thus since, by (h0)–(h1),

$$H(s) \leqslant |s|^2 + C|s|^{p+1} \tag{4.14}$$

we see, from the Sobolev embedding, that if $p \leqslant (N+2)/(N-2)$ then $\int_{\mathbb{R}^N} H(u_n)\,\mathrm{d}x$ is bounded and from (4.12) we get (4.6). When $N = 3$ the condition corresponds to

$p \leqslant 5$. In the case where we assume $p < (3N+4)/N$ let us show that

$$\int_{\mathbb{R}^N} H(u_n)\,\mathrm{d}x = o\left(\int_{\mathbb{R}^N} u_n^2 |\nabla u_n|^2\,\mathrm{d}x\right) \text{ if } \int_{\mathbb{R}^N} u_n^2 |\nabla u_n|^2\,\mathrm{d}x \to \infty. \tag{4.15}$$

Using Holder inequality, we have for $\theta = (N-2)(p-1)/(2N+4)$

$$\int_{\mathbb{R}^N} |u_n|^{p+1}\,\mathrm{d}x \leqslant C\left(\int_{\mathbb{R}^N} |u_n|^2\,\mathrm{d}x\right)^{1-\theta}\left(\int_{\mathbb{R}^N} |u_n|^{4N/(N-2)}\,\mathrm{d}x\right)^{\theta}.$$

Also

$$\left(\int_{\mathbb{R}^N} |u_n^2|^{2N/(N-2)}\,\mathrm{d}x\right)^{\theta} \leqslant C\left(\int_{\mathbb{R}^N} |\nabla(u_n^2)|^2\,\mathrm{d}x\right)^{\theta N/(N-2)}$$
$$= C\left(\int_{\mathbb{R}^N} u_n^2 |\nabla u_n|^2\,\mathrm{d}x\right)^{\theta N/(N-2)},$$

where $\theta N/(N-2) < 1$ since $p < (3N+4)/N$. Recalling (4.14) and the boundedness of $\{u_n\}$ in $L^2(\mathbb{R}^N)$ this proves (4.15). Thus from (4.12) we see that $\int_{R^N} H(u_n)\,\mathrm{d}x$ is bounded and thus (4.6) hold. At this point the lemma is proved. □

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