Cauchy problem and numerical simulation for a quasilinear Zakharov system describing laser-plasma interactions

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Abstract : We present here a type of Zakharov system describing laser-plasma interactions. This system involves a quasilinear part which is not hyperbolic and exhibits some elliptic zones. This difficulty is overcomed using the dispersion. We then give an asymptotic result on a reduced system. Finally, we present some numerical simulations in 1D and 2D.

1. Presentation of the model

The aim of this paper is to study the simulated Raman scattering in a plasma. The starting point is the model introduced for example in [5] that we have modified in [3]. This model describes the coupling effects between the incident laser field, the backscattered Raman component, the electronic-plasma wave and the ionic accoustic wave. For practical reason, this system is written using the vector potential A_C of the incident laser field, the vector potential A_R of the Raman component, the electric field E corresponding to the electronic-plasma wave and p the modulation of density of ions. The system, in 3-D, can be written in a dimensionless form

$$\left(i(\partial_t + v_C\partial_y) + \alpha_1\partial_y^2 + \alpha_2\Delta_\perp\right) A_C = \frac{b^2}{2}pA_C - (\nabla\cdot E)\, A_R e^{-i\theta}, \tag{1.1}$$

$$\left(i(\partial_t + v_R\partial_y) + \beta_1\partial_y^2 + \beta_2\Delta_\perp\right) A_R = \frac{bc}{2}pA_R - (\nabla\cdot E^*)\, A_C e^{i\theta}, \tag{1.2}$$

$$\left(i\partial_t + \gamma\nabla\nabla\cdot - \delta\nabla\times\nabla\times\right) E = \frac{b}{2}pE + \nabla(A_R^*\cdot A_C e^{i\theta}), \tag{1.3}$$

$$(\partial_t^2 - v_s^2\Delta)p = a\Delta\left(|E|^2 + b|A_C|^2 + c|A_R|^2\right). \tag{1.4}$$

The direction of propagation of the laser is y. The tranverse directions are x and z. We denote $\Delta_\perp = \partial_x^2 + \partial_z^2$. With these notations, the constants in system (1.1) – (1.4) are given below. The dispersion coefficients are

$$\alpha_1 = \frac{c_0^2k_0^2\omega_{pe}^2}{2\omega_0^4},\ \ \alpha_2 = \frac{c_0^2k_0^2}{2\omega_0^2},\ \ \beta_1 = \frac{c_0^2k_0^2\omega_{pe}^2}{2\omega_R^3\omega_0},\ \ \beta_2 = \frac{c_0^2k_0^2}{2\omega_R\omega_0},\ \ \gamma = \frac{v_{th}^2k_0^2}{2\omega_{pe}\omega_0},\ \ \delta = \frac{c_0^2k_0^2}{2\omega_{pe}\omega_0},\ \ v_s = \frac{c_sk_0}{\omega_0},$$

where c_s is the sound velocity in the plasma. The group velocity v_C and v_R are given by

$$v_C = \frac{k_0^2c_0^2}{\omega_0^2},\ \ \ v_R = \frac{k_Rk_0c_0^2}{\omega_R\omega_0}.$$

The coefficients of the nonlinearities are

$$a = 4\frac{m_e}{m_i}\frac{\omega_0\omega_R}{\omega_{pe}^2}, \quad b = \frac{\omega_{pe}}{\omega_0}, \quad c = \frac{\omega_{pe}}{\omega_R},$$

where m_e and m_i are respectively the mass of electrons and ions.

The main frequency of the laser is ω_0 and k_0 is the corresponding wave number. They satisfy the dispersion relation for electromagnetic waves in a plasma

$$\omega_0^2 = \omega_{pe}^2 + k_0^2 c_0^2, \tag{1.5}$$

where c_0 is the velocity of light in vaccum and ω_{pe} the electronic-plasma frequency. The main frequency of the Raman component will be denoted by ω_R and k_R will be the wave number. They satisfy the same condition as (1.5)

$$\omega_R^2 = \omega_{pe}^2 + k_R^2 c_0^2. \tag{1.6}$$

Furthermore,

$$\theta = \frac{k_1}{k_0}y - \frac{\omega_1}{\omega_0}t,$$

and k_1, ω_1 satisfy

$$k_0 = k_R + k_1, \tag{1.7}$$

$$\omega_0 = \omega_R + \omega_{pe} + \omega_1. \tag{1.8}$$

This system describes therefore a three-waves interaction. The resonance condition is that the third wave $(\omega_{pe} + \omega_1, k_1)$ satisfies the dispersion relation of the electronic-plasma wave

$$(\omega_{pe} + \omega_1)^2 = \omega_{pe}^2 + v_{th}^2 k_1^2, \tag{1.9}$$

where v_{th} is the thermal velocity of the electrons.

Note that the resonance condition (1.9) can be written

$$2\omega_{pe}\omega_1 + \omega_1^2 = v_{th}^2 k_1^2.$$

Since $\omega_1 \ll \omega_{pe}$, one gets

$$\omega_1 = \frac{v_{th}^2 k_1^2}{2\omega_{pe}},$$

that is

$$\frac{\omega_1}{\omega_0} = \frac{v_{th}^2 k_0^2}{2\omega_{pe}\omega_0}\left(\frac{k_1}{\omega_0}\right)^2 = \gamma\left(\frac{k_1}{\omega_0}\right)^2.$$

It means that in this case, the oscillation $e^{i\theta}$ is resonant for the Schrödinger operator $i\partial_t + \gamma\nabla\nabla\cdot$.

2. The Cauchy problem.

In this section, we recall the result of [3]. Consider the Cauchy problem for system (1.1) − (1.4). The difficulty is that the quasilinear part

$$i\partial_t A_C = -\nabla \cdot E A_R,$$
$$i\partial_t A_R = -\nabla \cdot E^* A_C,$$
$$i\partial_t E = -\nabla(A_R^* A_C),$$

is not hyperbolic. This difficulty is overcomed using the full dispersion. Several changes of unknowns are used in [3] in order to symetrize this quasilinear part. Then Ozawa-Tsutsumi method are coupled to energy estimates to prove the following :

Theorem 2.1. *Let* $(a_C, a_R, e) \in (H^{s+2}(\mathbb{R}^d))^{3d}$*,* $p_0 \in H^{s+1}(\mathbb{R}^d)$ *and* $p_1 \in H^s(\mathbb{R}^d)$ *with* $s > \frac{d}{2}+3$*. There exists* $T^* > 0$ *and a unique maximal solution* (A_C, A_R, E, p) *to* (1.1)−(1.4) *such that*

$$(A_C, A_R, E) \in \mathcal{C}([0, T^*[; H^{s+2})^{3d}, \quad p \in \mathcal{C}([0, T^*[; H^{s+1}) \cap \mathcal{C}^1([0, T^*[; H^s)$$

satisfying

$$(A_C, A_R, E, p)(0) = (a_C, a_R, e, p_0), \quad \partial_t p(0) = p_1.$$

3. Asymptotic analysis of the Raman amplification.

The aim of this section is to provide a simple explanation of the Raman amplification observed in the real life as well as in the numerical simulations of section 4.

We apply the method introduced in [3] to study a reduced problem in the semi-classical scaling, namely

$$\begin{cases} i(\partial_t + v_C\partial_y)A_C + \varepsilon(\alpha_1\partial_y^2 + \alpha_2\Delta_\perp)A_C = -\varepsilon\,(\nabla \cdot E)\,A_R e^{-i\frac{(k_1 y - \omega_1 t)}{\varepsilon}}, \\ \\ i(\partial_t + v_R\partial_y)A_R + \varepsilon(\beta_1\partial_y^2 + \beta_2\Delta_\perp)A_R = -\varepsilon\,(\nabla \cdot E^*)\,A_C e^{i\frac{(k_1 y - \omega_1 t)}{\varepsilon}}, \\ \\ (i\partial_t + \varepsilon\gamma\Delta - \delta\nabla \times \nabla\times)E = \varepsilon\nabla(A_R^* \cdot A_C e^{i\frac{(k_1 y - \omega_1 t)}{\varepsilon}}), \end{cases} \tag{3.1}$$

where ε is a small parameter that will tend to 0. Our main result reads as follows. The asymptotic behaviour of the solution is different wether the resonance condition is satisfied or not. Recall that this condition reads $\omega_1 = \gamma k_1^2$. Denoting

$$E = \mathcal{E}^\varepsilon e^{i\frac{(k_1 y - \omega_1 t)}{\varepsilon}},$$

in the resonant case we introduce the following limit system

$$\begin{cases} (\partial_t + v_C\partial_y)A_C = -(k \cdot \mathcal{E})A_R, \\ (\partial_t + v_R\partial_y)A_R = (k \cdot \mathcal{E}^*)A_C, \\ \partial_t\mathcal{E} + 2\gamma k \cdot \nabla\mathcal{E} = (A_R^* \cdot A_C)k, \end{cases} \tag{3.2}$$

where $k = (0, k_1, 0)$.

In the non-resonant case, the limit system reads, introducing $\mathcal{E}^\varepsilon = F^\varepsilon e^{i(\omega_1-\gamma k_1^2)\frac{t}{\varepsilon}}$,

$$\begin{cases} (\partial_t + v_C\partial_y)A_C = 0, \\ (\partial_t + v_R\partial_y)A_R = 0, \\ (\partial_t + 2\gamma k\cdot\nabla)F = 0. \end{cases} \tag{3.3}$$

The main result of this section reads as follows (see [2] for more details) :

Theorem 3.1. *Take A_C^0, A_R^0, $\mathcal{E}^0$ in $H^s(\mathbb{R}^d)$ for s large enough. There exists a time T independent of ε and a unique solution $(A_C^\varepsilon, A_R^\varepsilon, \mathcal{E}^\varepsilon)$ of (3.1) such that*

$$A_C^\varepsilon(0) = A_C^0,\ \ A_R^\varepsilon(0) = A_R^0,\ \ \mathcal{E}^\varepsilon(0) = \mathcal{E}^0 e^{i\frac{k_1 y}{\varepsilon}}.$$

i) Suppose $\omega_1 = \gamma k_1^2$ and let $(A_C, A_R, \mathcal{E})$ be the solution to (3.2) such that

$$A_C(0) = A_C^0,\ \ A_R(0) = A_R^0,\ \ \mathcal{E}(0) = \mathcal{E}^0,$$

one has

$$\left(A_C^\varepsilon - A_C, A_R^\varepsilon - A_R, Ee^{-i\frac{(k_1 y-\omega_1 t)}{\varepsilon}} - \mathcal{E}\right) \longrightarrow 0$$

as ε goes to 0 in $\left[L^\infty(0,T;H^\sigma(\mathbb{R}^d))\right]^{3d}$ for $\sigma > \frac{d}{2}$.
ii) If $\omega_1 \neq \gamma k_1^2$, let (A_C, A_R, F) be the solution to (3.3) such that

$$A_C(0) = A_C^0,\ \ A_R(0) = A_R^0,\ \ F(0) = \mathcal{E}^0,$$

one has

$$\left(A_C^\varepsilon - A_C, A_R^\varepsilon - A_R, Ee^{-i\frac{(k_1 y-\omega_1 t)}{\varepsilon}} - Fe^{i(\omega_1-\gamma k_1^2)\frac{t}{\varepsilon}}\right) \longrightarrow 0$$

as ε goes to 0 in $\left[L^\infty(0,T;H^\sigma(\mathbb{R}^d))\right]^{3d}$ for $\sigma > \frac{d}{2}$.

Remark 3.1. *In the resonant case, suppose that A_C is fixed to a constant, the last two equations read using $e = \mathcal{E}^*$ as new unknown*

$$\partial_t A_R + v_R\partial_y A_R = (k\cdot e)A_C,$$
$$\partial_t \mathcal{E} + 2\gamma k\cdot\nabla e = (A_R\cdot A_C^*)k,$$

and the amplification rate is $|k\cdot e|$.

Remark 3.2. *In the non resonant case, of course, no amplification occurs.*

4. Numerical simulations

We have used a fractional-step, Crank-Nicolson-type scheme with relaxation directly inspired by that of C. Besse for NLS or Davey-Stewartson system (see [1]). For the accoustic part, we use the scheme introduced by Glassey (see [4]). We will consider a regular mesh in space. We use periodic boundary conditions. We also consider centered discretization for each differential operator.

4.1. The 1-D case

We work on a system in dimensionless form. (The unit of length is $\frac{1}{k_0}$ and the unit of time is $\frac{1}{\omega_0}$). We compute on a space interval $[0, L]$ with $L = 200$ and on a time interval $[0, T]$ with $T = 100$.

We will consider gaussian initial data for A_C of the form

$$A_C(0) = 0.3e^{-0.01(x-40)^2}.$$

Since we deal with simulated Raman effect, we have to begin with a small perturbation on A_R and we take $A_R(0) = 0.01A_C(0)$. Furthermore, E, p and $\partial_t p$ are taken equal to 0 at $t = 0$. Typical number of points discretization in space is $N_y = 500$ and in time $N_t = 200$.

In figure 1, one can find snapshots of the modulus of the fields at nine different times ($t = \frac{n*100}{9}$, $n = 0$ to 8) in the resonant case (i.e. $\omega_1 = \gamma k_1^2$). The continuous line corresponds to A_C, the dashed line to A_R and the semi-doted one to E. Of course A_C and A_R travel in different directions. The growth is rapid. The interaction stops when the supports of A_C and A_R are disjoints.

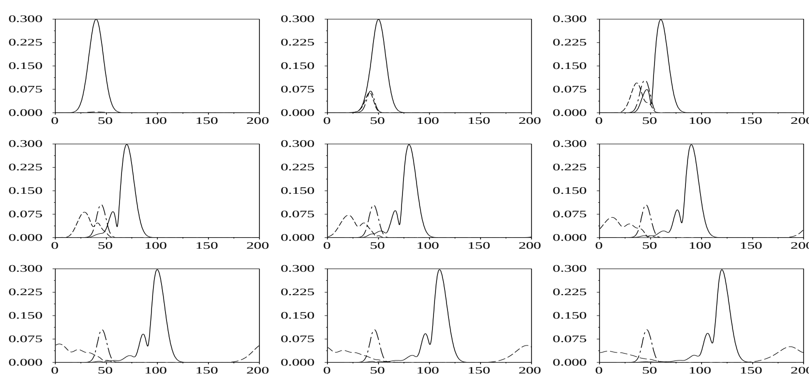

Figure 1. Modulus of the fields at different time steps with $A_C(0) = 0.3e^{-0.01(x-40)^2}$, $\frac{\omega_1}{\omega_0} = 0.01561$

4.2. The 2-D case

Since the system has some transverse dispersion, one expects that the amplitude of the fields will decay more rapidly than in the 1-D case. The spatial domain is $y \in [0, 100]$ and $x \in [0, 60]$ and the time interval is $[0, 50]$. We took $N_y = 100$ points in the y direction, $N_x = 60$ in the x direction and $N_t = 96$ time steps. The initial data for A_C is

$$A_C(0) = 0.3e^{-0.01(y-35)^2}e^{-0.03(x-30)^2}.$$

In Figure 2, the neperian logarithm of the modulus of the fields are drawn. The Raman effect is less efficient than in 1-D as expected. The decrease of A_C in time is more rapid than in 1-D because of the 2-D dispersion. In Figure 3,4, the level curves of respectively A_C and A_R are drawn at time $\frac{n*50}{8}$ for $n = 0$ to 8.

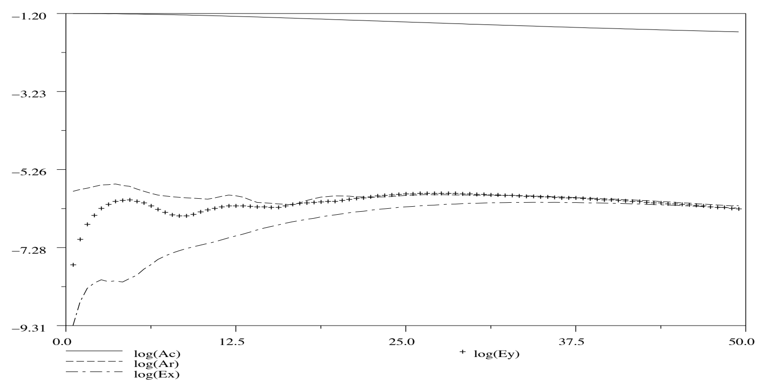

Figure 2. Neperian logarithm of the fiels with $A_C(0) = 0.3e^{-0.01(y-35)^2}e^{-0.03(x-30)^2}$, $\frac{\omega_1}{\omega_0} = 0.01561$

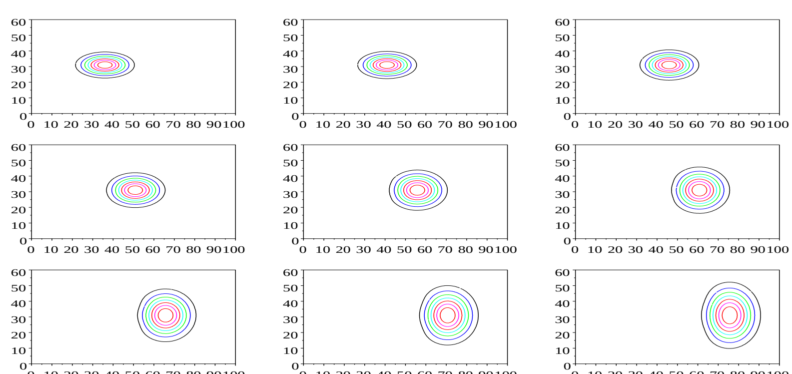

Figure 3. Level curve of A_C with $A_C(0) = 0.3e^{-0.01(y-35)^2}e^{-0.03(x-30)^2}$, $\frac{\omega_1}{\omega_0} = 0.01561$

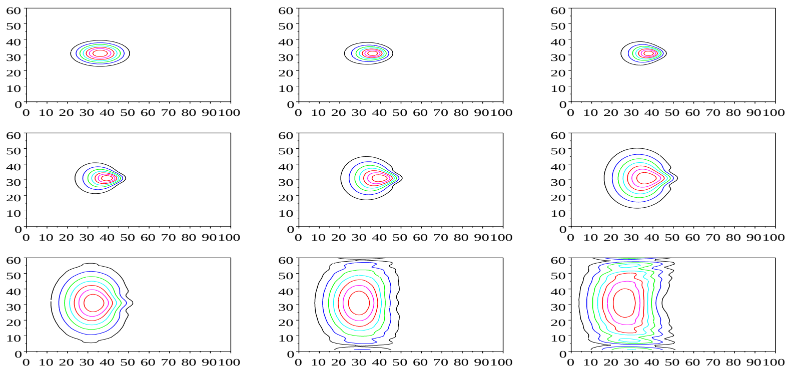

Figure 4. Level curve of A_R with $A_C(0) = 0.3e^{-0.01(y-35)^2}e^{-0.03(x-30)^2}$, $\frac{\omega_1}{\omega_0} = 0.01561$

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