Nonlinear models for laser-plasma interaction.

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Abstract

In this paper, we present a nonlinear model for laser-plasma interaction describing the Raman amplification. This system is a quasilinear coupling of several Zakharov systems. We handle the Cauchy problem and we give some well-posedness and ill-posedness result for some subsystems.

1 Introduction

Powerful laser are used in laboratory in order to simulate nuclear fusion using inertial confinement. During this process, a plasma is created which interacts nonlinearly with the laser. The aim of this paper is to construct and study some nonlinear systems describing the interaction. The more precise models that can be used for describing laser-plasma interaction are probably the kinetic ones. These models involve several distribution functions depending on 7 variables (time, 3 space dimensions and 3 dimensional space for the velocities). The associated computational cost for the application to fusion by inertial confinement are clearly out of reach for the moment. Therefore, one has to used simplified models. The fluid models seems to be more adapted since the physical quantities only depend on 4 variables. However, for practical applications, one has to use very small time and space scales and numerical simulations in 3D on reasonable space domains are impossible. This is why, Zakharov in the 70's has introduced a new type of systems obtained thanks the so-called envelope approximation [27]. Such a typical system reads in dimensionless form:

$$\begin{cases} i\partial_t \nabla\psi + \Delta(\nabla\psi) = \nabla\Delta^{-1}\mathrm{div}(\delta n \nabla\psi), \\ \\ \partial_t^2 \delta n - \Delta\delta n = \Delta(|\nabla\psi|^2). \end{cases} \tag{1.1}$$

This system has been studied in [4, 8, 9]. Of course, variations of this systems exists (see [24] for example). The above system describe the evolution of the electronic plasma waves that are fundamental is the study of nonlinear plasma physics. Note that this system is the equivalent of the Davey-Stewartson system for the study of water-waves see [11]. One of the main instability which has to be undergone is the Raman instability: when the laser pulse enter the plasma, another laser component is created (the Raman component). These two components interact and create some electronic plasma waves. This is a three waves mixing system that is unstable. The high-frequency waves interact nonlinearly and create some ionic acoustic waves. Of course there is a retroaction of the acoustic part to the high frequency parts. Below, we will recall the derivation of such a system initiated in [5].

2 Derivation of the main model

We start from the bifluid Euler-Maxwell systems that reads:

$$\partial_t B + c\nabla \times E = 0, \tag{2.1}$$

$$\partial_t E - c\nabla \times B = 4\pi e \left((n_0 + n_e)v_e - (n_0 + n_i)v_i\right), \tag{2.2}$$

$$(n_0 + n_e)\left(\partial_t v_e + v_e \cdot \nabla v_e\right) = -\frac{\gamma_e T_e}{m_e}\nabla n_e - \frac{e(n_0 + n_e)}{m_e}(E + \frac{1}{c}v_e \times B), \tag{2.3}$$

$$(n_0 + n_i)\left(\partial_t v_i + v_i \cdot \nabla v_i\right) = -\frac{\gamma_i T_i}{m_i}\nabla n_i + \frac{e(n_0 + n_i)}{m_i}(E + \frac{1}{c}v_i \times B), \tag{2.4}$$

$$\partial_t n_e + \nabla \cdot ((n_0 + n_e)v_e) = 0, \tag{2.5}$$

$$\partial_t n_i + \nabla \cdot ((n_0 + n_i)v_i) = 0. \tag{2.6}$$

The unknowns are :

- E and B are respectively the electric and magnetic field.
- v_e and v_i denote respectively the velocity of electrons and ions.
- n_0 is the mean density of the plasma.
- n_e and n_i are the variation of density respectively of electrons and ions with respect to the mean density n_0.

The constants are :

- c is the velocity of light in the vacuum; e is the elementary electric charge.
- m_e and m_i are respectively the electron's and ion's mass.
- T_e and T_i are respectively the electronic and ionic temperature and γ_e and γ_i the thermodynamic coefficients.

For a precise description of this kind of model, see classical textbooks [12]. One of the main points is that the mass of the electrons is very small compared to the mass of the ions : $m_e \ll m_i$. Since the Lorentz force is the same for the ions and the electrons, the velocity of the ions will be neglectable with respect to the velocity of the electrons. The consequence is that we neglect the contribution of the ions in equation (2.2).

The first step is then to study the linearized version around 0 of this system and we write a decomposition of the field as a sum of a longitudinal part and a transverse one: $B = B_\parallel + B_\perp$ with $\nabla \times B_\parallel = 0$ and $\nabla \cdot B_\perp = 0$. Similar decompositions are used for E and v_e. The longitudinal part gives the equations for the electronic plasma waves:

$$\partial_t B_\parallel = 0,\ \partial_t E_\parallel = 4\pi e n_0 v_{e\parallel},$$

$$\partial_t v_{e\parallel} = -\frac{\gamma_e T_e}{m_e n_0}\nabla n_e - \frac{e}{m_e}E_\parallel,\ \partial_t n_e + n_0 \nabla \cdot v_{e\parallel} = 0.$$

Combining these equations leads to

$$\left[\partial_t^2 - v_{th}^2 \Delta + {\omega_{pe}}^2\right] v_{e\parallel} = 0, \tag{2.7}$$

where $\omega_{pe} = \sqrt{\frac{4\pi e^2 n_0}{m_e}}$ is the plasma frequency and $v_{th} = \sqrt{\frac{\gamma_e T_e}{m_e}}$ is the thermal velocity of the plasma. Equation (2.7) gives the following dispersion relation:

$$\omega^2 = \omega_{pe}^2 + k^2 v_{th}^2. \tag{2.8}$$

The same manipulation concerning transverse waves leads to the system for electromagnetic waves:

$$\partial_t B_\perp + c\nabla \times E_\perp = 0$$
$$\partial_t E_\perp - c\nabla \times B_\perp = 4\pi e n_0 v_{e\perp}$$
$$\partial_t v_{e\perp} = -\frac{e}{m_e} E_\perp$$

which reduces to

$$\partial_t^2 E_\perp - c^2 \Delta E_\perp + \omega_{pe}^2 E_\perp = 0. \tag{2.9}$$

The associated dispersion relation is

$$\omega^2 = \omega_{pe}^2 + k^2 c^2. \tag{2.10}$$

For the applications that we have in mind, v_{th} is at least one order of magnitude smaller than c. Therefore, the characteristic variety associated to (2.8) is very flat compared to that of (2.10) and the electromagnetic waves have a different status than that of electronic plasma waves. The electromagnetic waves have to be thought under the form: $e^{i(kx-\omega t)} E_\perp(t,x)$ with $\partial_t E_\perp \ll \omega E_\perp$ and $\partial_x E_\perp \ll k E_\perp$,
whereas the electronic plasma waves have to be search under the form $e^{-i\omega_{pe}t} E_{||}$ with $\partial_t E_{||} \ll \omega_{pe} E_{||}$. In order to obtain a nonlinear model, one needs to perform a weakly nonlinear analysis leading to the equations satisfied by the amplitude during the three waves interaction. The interaction between the 3 waves is effective if the following resonance conditions are satisfied:

$$K_0 = K_R + K_1,\ \omega_0 = \omega_R + \omega_{pe} + \omega_1,$$

where (K_R, ω_R) and (K_0, ω_0) satisfy

$$\omega^2 = \omega_{pe}^2 + K^2 c^2$$

and $(K_1, \omega_{pe} + \omega_1)$ satisfies

$$(\omega_{pe} + \omega_1)^2 = \omega_{pe}^2 + v_{th}^2 K_1^2.$$

Note that (K_0, ω_0) denote the wave vector and the frequency of the laser pulse, (K_R, ω_R) those of the Raman component while $(K_1, \omega_{pe} + \omega_1)$ are those of the electronic plasma waves. The different waves are then written as follows. The incident laser pulse has a vector potential given by $A_L e^{i(k_0 z - \omega_0 t)} + c.c.$, where $c.c.$ stands for the complex conjugate. The Raman component has a vector potential given by$A_R e^{i(k_R z - \omega_R t)} + c.c.$. The electronic plasma wave is described thanks to the electric field: $E_0 e^{-i\omega_{pe} t} + c.c.$

The low-frequency modulation of the density of ions is denoted by p.

The electric field is then recovered by the following formula

$$E = \frac{i\omega_0}{c} A_L(t,x,y,z) e^{i(k_0 z - \omega_0 t)} + \frac{i\omega_R}{c} A_R(t,x,y,z) e^{i(k_R z - \omega_R t)}$$
$$+ E_0(t,x,y,z) e^{-i\omega_{pe} t} + c.c.$$

A WKB-type expansion give the amplitude system that is given at the end of this paper. Here we assume that the vectors K_0, K_R and K_1 are colinear along the direction of the z variable. For some extension, see [7]. The structure of this system is:

$$\begin{aligned} & i\left(\partial_t + \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix} \partial_z \right) \begin{pmatrix} A_L \\ A_R \\ E_0 \end{pmatrix} + \Delta \begin{pmatrix} A_L \\ A_R \\ E_0 \end{pmatrix} \\ & = p \begin{pmatrix} A_L \\ A_R \\ E_0 \end{pmatrix} + \begin{pmatrix} -\nabla \cdot E_0 A_R e^{-i(k_1 z - \omega_1 t)} \\ -\nabla \cdot E_0^* A_L e^{i(k_1 z - \omega_1 t)} \\ \nabla (A_R^* \cdot A_L e^{i(k_1 z - \omega_1 t)}) \end{pmatrix} \\ & \left(\partial_t^2 - \Delta\right) p = \Delta \left(|A_L|^2 + |A_R|^2 + |E_0|^2\right) \end{aligned} \tag{2.11}$$

3 Some result for the Cauchy problem

System (2.11) is an extension of

$$\begin{aligned} & i\partial_t E + \Delta E = pE, \\ & \partial_t^2 p - \Delta p = \Delta |E|^2 \end{aligned} \tag{3.1}$$

which is the original Zakharov system. The Cauchy problem is now well understood see [1, 13, 20, 25, 23] for example, [14, 15] for blowing-up solutions. Basically, the system is well-posed for smooth enough initial data. Note that a huge physical literature exists concerning the computations of the solutions of (3.1) see for example [21, 22] and their references. The asymptotic expansion leading to (3.1) starting from the Euler-Maxwell system has been justified by B. Texier in [26]. One can see [16] for the numerical analysis of this system.

Furthermore, system (2.11) is also an extension of

$$\begin{aligned} & \left(i(\partial_t + \partial_z) + \Delta_\perp\right) A = pA, \\ & \partial_t^2 p - \Delta_\perp p = \Delta_\perp |A|^2. \end{aligned} \tag{3.2}$$

The Cauchy problem for system (3.2) is more subtile. Lineares, Ponce and Saut have proved in [18] that

Theorem 3.1. *System (3.2) is locally well-posed in $H^s(\mathbb{R}^n)$ for s large enough.*

The proof makes use of local and global smoothing effects for the linear Schrödinger operator that correspond to the dispersive properties of the equation in the spirit of [17].

If one consider the case of periodic boundary conditions, the result is drastically different [10]:

Theorem 3.2. *System (3.2) is locally ill-posed in $H^s(\mathbf{T}^n)$ in the sense that for any s, there exist a sequence of times T_k tending to zero and families of solutions $(\underline{A}, 0) + (A_k, p_k)$, in $C^1([0, T_k]; H^s(\mathbf{T}^n))$ such that*

$$\begin{aligned} \|A_k(0),\, p_k(0),\, \partial_t p_k(0)\|_{H^s(\mathbf{T}^n)} &\to 0, \\ \|A_k(T_k),\, p_k(T_k)\|_{L^2(\mathbf{T}^n)} &\to \infty. \end{aligned}$$

where $\mathbf{T}$ denotes the torus of $\mathbb{R}^n$.

This result is typically a result of geometrical optics type.

Now, we look at the complete system (2.11). The difficulties of (3.1) and (3.2) are of course included. Another difficulty comes anyway from the quasilinear terms $\nabla \cdot E_0 A_R$, $\cdot E_0^* A_L$, $\nabla(A_R^* \cdot A_L)$. This quasilinear part is not hyperbolic. Indeed, consider the one-D, real version of the system. Writing $A_L = u_1 + iu_2$, $A_R = u_3 + iu_4$, $E_0 = u_5 + iu_6$, the quasilinear part of (2.11) reads:

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} = M \partial_x \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}$$

with

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & -u_4 & -u_3 \\ 0 & 0 & 0 & 0 & u_3 & -u_4 \\ 0 & 0 & 0 & 0 & -u_2 & u_1 \\ 0 & 0 & 0 & 0 & u_1 & u_2 \\ -u_4 & u_3 & u_2 & -u_1 & 0 & 0 \\ -u_3 & -u_4 & -u_1 & -u_2 & 0 & 0 \end{pmatrix}$$

Clearly, the blocks involving u_1 and u_2 will give complex eigenvalues that lead to an Hadamar instability. We overcome this difficulty using the dispersive terms as follows [5]. We then prove:

Theorem 3.3. *System (2.11) is locally well-posed in $H^s(\mathbb{R}^n)$ or in $H^s(\mathbf{T}^n)$ for s large enough.*

We explain below the key point of our argument. The difficult part of (2.11) lies in the blocks involving u_1 and u_2. In complex form, this reads

$$\begin{aligned}\partial_t A - i\partial_x^2 A = +iP\partial_x E\\ \partial_t E + i\partial_x^2 E = +iP^*\partial_x A\end{aligned}$$

where P is a pump wave considered as being a constant in this computation. The symbol of this system is then $\begin{pmatrix} i\xi^2 & -P\xi \\ -P^*\xi & -i\xi^2 \end{pmatrix}$ and the eigenvalues $i\omega$ satisfy

$$\omega^2 = \xi^4 - |P|^2\xi^2$$

and are real for large ξ. It is a kind of dispersive stabilization and there is an analogy with Kuramoto-Shivashinski equation. Of course, this is not a proof. A detailed proof can be found in [5].

4 A numerical scheme

We perform numerical simulations for system (2.11) by using a numerical scheme inspired from that of [3] for the nonlinear Schrödinger equation. We use this kind of scheme for the following reasons:
i) One can not use splitting schemes because of the quasilinear part that is not hyperbolic (this will lead to an unstable step in the splitting).
ii) The following quantity is a conserved quantity of the continuous system

$$\int 2|A_L|^2 + |A_R|^2 + |E_0|^2(t) = Cte$$

and should be also conserved by the numerical scheme.
iii) One need to handle at the same step dispersion and nonlinearity since the existence proof is done using this method.

Let us recall how Besse's scheme is written on the nonlinear Schrödinger equation:

$$i\partial_t u + \Delta u = |u|^2 u.$$

One introduces the following discretization

$$i\frac{u^{n+1} - u^n}{\delta t} + \Delta\frac{u^{n+1} + u^n}{2} = \varphi^{n+1/2}\frac{u^{n+1} + u^n}{2}$$

where

$$\frac{\varphi^{n+1/2} + \varphi^{n-1/2}}{2} = |u^n|^2.$$

This scheme is at least formally of second order. In order to initialize the scheme, we need a value for $\varphi^{-1/2}$ in order to be able to compute $\varphi^{1/2}$. SInce $\varphi^{n+1/2}$ is some kind of prediction of $|u|^2$ at time $(n+1/2)*dt$, we take $\varphi^{-1/2} = |u^{-1/2}|^2$ where $u^{-1/2}$ is given by an half time-step backward by the explicit Euler scheme.

We now adapt our scheme to our case. We present below the 1-D version, it can be extended to multi-D [6]. We need to introduce two new unknowns, the first one φ corresponding to $\partial_y E$ and ψ corresponding to A_R as follows:

The equation for A_L is discretized as:

$$i\frac{A_L^{n+1} - A_L^n}{\delta t} + \left(iv_1\partial_y + \partial_y^2\right)\frac{A_L^{n+1} + A_L^n}{2}$$

$$= \left(\frac{p^{n+1} + p^n}{2}\right)\frac{A_L^{n+1} + A_L^n}{2} - \frac{1}{2}\varphi^{n+1/2}\frac{A_R^{n+1} + A_R^n}{2}e^{-i\theta^{n+1/2}} - \frac{1}{2}\psi^{n+1/2}\frac{\partial_y E^{n+1} + \partial_y E^n}{2}e^{-i\theta^{n+1/2}},$$

where

$$\frac{\varphi^{n+1/2} + \varphi^{n-1/2}}{2} = \partial_y E^n,$$

and

$$\frac{\psi^{n+1/2} + \psi^{n-1/2}}{2} = A_R^n,$$

The scheme for A_R is:

$$i\frac{A_R^{n+1} - A_R^n}{\delta t} + \left(iv_2\partial_y + \partial_y^2\right)\frac{A_R^{n+1} + A_R^n}{2}$$

$$= \left(\frac{p^{n+1} + p^n}{2}\right)\frac{A_R^{n+1} + A_R^n}{2} - (\varphi^{n+1/2})^*\frac{A_L^{n+1} + A_L^n}{2}e^{i\theta^{n+1/2}}$$

with

$$\frac{\varphi^{n+1/2} + \varphi^{n-1/2}}{2} = \partial_y E^n.$$

The scheme for E is:

$$i\frac{E^{n+1} - E^n}{\delta t} + \partial_y^2\left(\frac{E^{n+1} + E^n}{2}\right)$$

$$= \frac{1}{2}\left(\frac{p^{n+1} + p^n}{2}\right)\left(\frac{E^{n+1} + E^n}{2}\right) + \partial_y\left[(\psi^{n+\frac{1}{2}})^*\left(\frac{A_C^{n+1} + A_C^n}{2}\right)e^{i\theta^{n+\frac{1}{2}}}\right].$$

The discretization for the equation of p is the scheme introduced by Glassey [16] for the Zakharov system:

$$\frac{p^{n+1} - 2p^n + p^{n-1}}{\delta t^2} - \partial_y^2\left(\frac{p^{n+1} + p^{n-1}}{2}\right) = \partial_y^2\left(|E^n|^2 + |A_C^n|^2 + |A_R^n|^2\right).$$

A typical result is give in fig. 1. See [6] for more results and extensions.

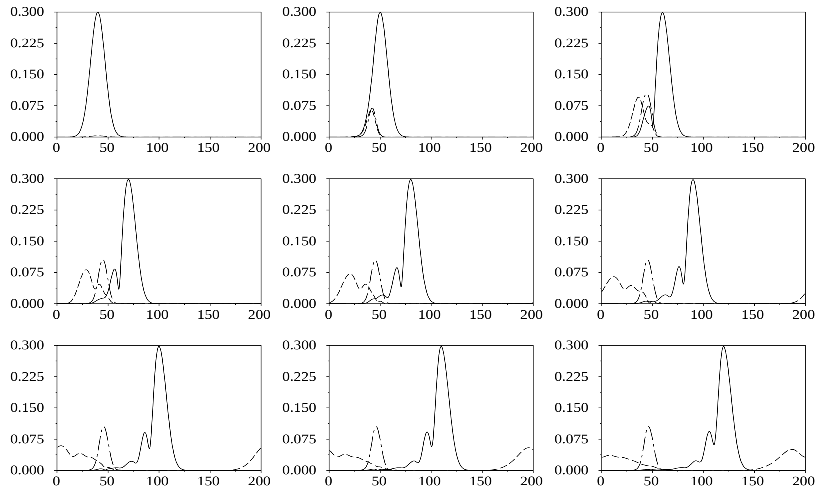

Figure 1: Case 1, 1-D geometry. Modulus of the fields at time $t = n\dfrac{100}{8}$ for $n = 0 \cdots 8$ with $A_C(0) = 0.3e^{-0.01(x-40)^2}$, $\frac{\omega_1}{\omega_0} = 0.01561$. First line, from left to right, $n = 0, 1, 2$, second line, from left to right, $n = 3, 4, 5$, third line, from left to right, $n = 6, 7, 8$. The continuous line corresponds to A_C, the dashed line to A_R and the semi-doted line to E. The value of ω_1 corresponds to the resonant case.

Moreover, the Raman amplification is one of the main cause of the Landau damping phenomena which is a wave-particule interaction. Landau damping is a kinetic phenomena and therefore can not be obtain in our framework starting from the fluid equations. It can be however modelized using a diffusion equation coupled to a Zakharov type system [2].

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