On the existence of ground states for a nonlinear Klein-Gordon- Maxwell type system

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Abstract

In this paper, we study a nonlinear Klein-Gordon equation coupled with a Maxwell equation. Introducing a new constraint minimization problem, we prove the existence of ground states for an associated stationnary elliptic system.

Key words: Klein-Gordon-Maxwell system, standing waves, ground states, constraint minimization method

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1 Introduction

In this paper, we consider the following elliptic system stated in $\mathbb{R}^3$:

$$-\Delta u + (m^2 - \omega^2)u - 2\omega e\phi u - e^2|\phi|^2 u - |u|^{p-2}u = 0, \tag{1.1}$$

$$-\Delta\phi - e\omega|u|^2 - e^2\phi|u|^2 = 0, \tag{1.2}$$

where $m > 0$, $\omega \in \mathbb{R}$, $e \in \mathbb{R}$, $p > 2$, $(u, \phi) \in \mathbb{C} \times \mathbb{C}$. Our aim of this paper is to prove the existence of a ground state solution to the system (1.1)-(1.2) by introducing a new constraint minimization problem.

System (1.4)-(1.5) is closely related to the following nonlinear Klein-Gordon equation coupled with Maxwell equation:

$$\begin{cases} \psi_{tt} - \Delta\psi &= -2ie\phi\psi_t - ie\phi_t\psi + e^2|\phi|^2\psi - 2ie\nabla\psi\cdot\mathbf{A} \\ &\quad -e^2|\mathbf{A}|^2\psi - ie\psi\,\mathrm{div}\mathbf{A} - m^2\psi + |\psi|^{p-2}\psi. \\ \mathbf{A}_{tt} - \Delta\mathbf{A} &= \dfrac{ie}{2}\big(\psi\nabla\bar{\psi} - \bar{\psi}\nabla\psi\big) - e^2|\psi|^2\mathbf{A} + \nabla\phi_t - \nabla\,\mathrm{div}\mathbf{A}. \\ -\Delta\phi &= -\dfrac{ie}{2}\big(\psi\bar{\psi}_t - \bar{\psi}\psi_t\big) - e^2|\psi|^2\phi - \mathrm{div}\mathbf{A}_t. \end{cases} \tag{1.3}$$

where $\psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$, $\mathbf{A} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$, $\phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ and i denotes the unit complex number, that is $i^2 = -1$. In this system, ψ is an electrically charged field and $(\phi, \mathbf{A})$ is a gauge potential which describes an electromagnetic field. System (1.3) describes the interaction of a particle with an electromagnetic field in the following way: the field ψ produces, on one hand, a current which acts as a force for the electromagnetic field and, on the other hand, carries an electric charge which is given by the electromagnetic field. (See [12], Section 3.10 for the detailed derivation.) We also refer to [14], [18] for more physical backgrounds. Note that, to our knowledge, the only results concerning the Cauchy Problem associated with System (1.3) have been established in [9] and [17].

If we look for a standing wave of (1.3) of the type

$$\psi(x,t) = u(x)e^{i\omega t},\ \mathbf{A}(x,t) = \mathbf{0} \text{ and } \phi(x,t) = \phi(x),$$

then we are led to the elliptic system:

$$-\Delta u + (m^2 - \omega^2)u - 2\omega e\phi u - e^2|\phi|^2 u - |u|^{p-2}u = 0, \tag{1.4}$$

$$-\Delta\phi + e\omega|u|^2 + e^2\phi|u|^2 = 0. \tag{1.5}$$

The existence and the non-existence of a solution to system (1.4)-(1.5) have been studied widely (see [1], [3], [6], [10], [11], [13].) Furthermore the existence of a ground state, which is a solution minimizing the action among all non-trivial solutions, has been considered in [2], [19]. More precisely, it was shown that if $4 \le p < 6$, a ground state exists for $|\omega| < m$ ($p = 6$ is the critical Sobolev exponent for the existence in $H^1(\mathbb{R}^3)$) while when $2 < p < 4$, the existence of a ground state has been proved under the condition $\sqrt{g(p)}|\omega| < m$ for some function $g(p) > 1$.

First notice that System (1.4)-(1.5) does not have a good variational structure since the action associated to this set of equations takes the form:

$$\begin{aligned} T_\omega(u,\phi) =& \frac{1}{2}\int_{\mathbb{R}^3} \Big(|\nabla u|^2 - |\nabla\phi|^2 + (m^2-\omega^2)u^2 - 2e\omega\phi u^2 - e^2\phi^2u^2\Big)\,dx \\ &- \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\,dx, \end{aligned}$$

To avoid the indefiniteness of the action T_ω, the authors in [2], [19] used the so-called reduction method. It consists in solving the elliptic equation (1.5) for a fixed function u, which provides a mapping $\phi = \phi(u)$. Then, Equation

(1.4) can be written with the only one variable u associated to a one-variable functional $I_\omega(u) = T_\omega(u, \phi(u))$. In this direction, the proof of the existence of a ground state had been carried out by considering a minimization problem where the constraint is defined by the Nehari manifold.

In this paper, we concentrate on System (1.1)-(1.2) which looks similar to (1.4)-(1.5). Thus we can expect that the study of system (1.1)-(1.2) will give us a better understanding of system (1.4)-(1.5). For that purpose, we introduce the following action $S_\omega : X \to \mathbb{R}$ by

$$\begin{aligned} S_\omega(u, \phi) =& \frac{1}{2} \int_{\mathbb{R}^3} \Big(|\nabla u|^2 + |\nabla \phi|^2 + (m^2 - \omega^2)|u|^2 - 2e\omega\phi|u|^2 - e^2\phi^2|u|^2 \Big)\, dx \\ & - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx, \end{aligned}$$

where the energy space X is given by $X = H^1(\mathbb{R}^3, \mathbb{C}) \times D^{1,2}(\mathbb{R}^3, \mathbb{R})$ and $D^{1,2}(\mathbb{R}^3, \mathbb{R})$ denotes the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm:

$$\|\phi\|^2_{D^{1,2}} := \int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx.$$

We recall that $D^{1,2}(\mathbb{R}^3, \mathbb{R})$ is continuously embedded into $L^6(\mathbb{R}^3)$. At this point, it is important to note that the critical points (u, ϕ) of S_ω are the solutions to (1.1)-(1.2).

We introduce

$$\mathcal{A}_\omega = \Big\{ (u, \phi) \in X \ : \ S'_\omega(u, \phi) = 0, \ (u, \phi) \neq (0, 0) \Big\}$$

and denote by $\mathcal{G}_\omega$ the set of ground states to system (1.1)-(1.2):

$$\mathcal{G}_\omega = \Big\{ (w, \psi) \in \mathcal{A}_\omega \ : \ S_\omega(w, \psi) \le S_\omega(u, \phi) \text{ for all } (u, \phi) \in \mathcal{A}_\omega \Big\}.$$

Then we have the following result.

Theorem 1.1. *Suppose $|\omega| < m$ and $2 < p < 6$. Then system (1.1)-(1.2) has a ground state solution $(w, \psi) \in X$ where w and ψ are real functions.*

We emphasize that our result requires no restriction on p and ω. In our system (1.1)-(1.2), we cannot reduce $S_\omega(u, \phi)$ to a single variable action because we cannot expect that (1.2) is uniquely solvable in general. Moreover one can observe that if we consider the minimization problem on the Nehari manifold (see Lemma 2.2 below), we will face a restriction on p and ω.

To prove the existence of a ground state, we adapt a similar argument as in [8]. Here we briefly explain the strategy. Firstly we introduce a new constraint minimization problem, where the constraint contains a part of the action S_ω. Secondly we prove that after eliminating the Lagrange multiplier, the minimizer gives a solution of (1.1)-(1.2) (see (3.1)). Finally we show that the rescaled minimizer is a ground state of (1.1)-(1.2).

We believe that our result is useful for the study of the stability of standing waves. We refer to [4], [5], [17] for related topics. Especially in [4] and [5], the authors showed that the standing wave, for a similar problem where $m^2u - |u|^{p-2}u$ is replaced by $W'(u)$ with $W(u) \geq 0$ in (1.4)-(1.5), is stable.

This paper is organized as follows. In Section 2, we prepare two identities for solutions of (1.1)-(1.2). In Section 3, we introduce a new constraint minimization problem and prove the existence of a minimizer by applying the concentration compactness principle. Finally we show the existence of a ground state in Section 4.

2 Preliminaries

In this section, we prepare two lemmas. Hereafter in this paper, we write $\gamma = m^2 - \omega^2$ for simplicity.

Lemma 2.1. *Let* (u, ϕ) *be a solution of* (1.1)-(1.2)*. Then,* ϕ *is a real function and one has*

$$\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \gamma |u|^2 - 2e\omega\phi|u|^2 - e^2|\phi|^2|u|^2 - |u|^p \right) dx = 0, \tag{2.1}$$

$$\int_{\mathbb{R}^3} \left(|\nabla \phi|^2 - e\omega\phi|u|^2 - e^2|\phi|^2|u|^2 \right) dx = 0. \tag{2.2}$$

Proof. Multiply (1.1) by $\overline{u}$ and (1.2) by $\overline{\phi}$ respectively, integrate over $\mathbb{R}^3$ and make an integration by parts on the second order derivative terms. We omit the details since all the computations are straightforward. We only write that, from this procedure, we obtain

$$\int_{\mathbb{R}^3} \left(|\nabla \phi|^2 - e\omega\overline{\phi}|u|^2 - e^2|\phi|^2|u|^2 \right) dx = 0$$

from which we deduce that ϕ must be a real function. □

Lemma 2.2. *Let* (u, ϕ) *be a solution of* (1.1)-(1.2). *Then* (u, ϕ) *satisfies the following Pohozaev-type identity:*

$$\int_{\mathbb{R}^3} \Big(|\nabla u|^2 + |\nabla \phi|^2 + 3\gamma |u|^2 - 6e\omega\phi|u|^2 - 3e^2|\phi|^2|u|^2 - \frac{6}{p}|u|^p \Big)\, dx = 0. \quad (2.3)$$

Proof. The proof is also standard, so we omit the details. We refer to [7] and [10] for a proof of Pohozaev type identities. □

3 A new constraint minimization problem

In this section, we introduce a new constraint minimization problem. We show that a solution to equations (1.1)-(1.2) can be obtained as a non-zero solution of a minimizing problem. For that purpose, for a given $\mu > 0$, we put

$$I(u, \phi) = \int_{\mathbb{R}^3} \Big(-\frac{\gamma}{2}|u|^2 + e\omega\phi|u|^2 + \frac{e^2}{2}\phi^2|u|^2 + \frac{1}{p}|u|^p \Big)\, dx$$

and set

$$\mathcal{K}_\mu = \Big\{ (u, \phi) \in X \ : \ I(u, \phi) = \mu \Big\}.$$

We define the functional $E : X \to \mathbb{R}$ by

$$E(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} \Big(|\nabla u|^2 + |\nabla \phi|^2 \Big)\, dx$$

and consider the following minimization problem:

$$J(\mu) := \inf_{(u,\phi)\in\mathcal{K}_\mu} E(u, \phi). \quad (3.1)$$

Note that by a scaling argument, it is obvious that $\mathcal{K}_\mu \neq \emptyset$. Indeed, take any $u \in H^1(\mathbb{R}^3, \mathbb{C})$, $u \neq 0$ and compute, for $\lambda \in \mathbb{R}$

$$I(\lambda u, 0) = \int_{\mathbb{R}^3} \Big(-\lambda^2 \frac{\gamma}{2}|u|^2 + \frac{\lambda^p}{p}|u|^p \Big)\, dx.$$

Since $p > 2$, one has

$$\lim_{\lambda\to+\infty} I(\lambda u, 0) = +\infty, \quad \lim_{\lambda\to 0} I(\lambda u, 0) = 0.$$

A continuity argument shows directly that there exists a $\lambda > 0$ such that $I(\lambda u, 0) = \mu$.

Moreover a direct calculation shows that

$$\begin{aligned}\frac{\partial I}{\partial u}u &= \int_{\mathbb{R}^3}\Big(-\gamma|u|^2+2e\omega\phi|u|^2+e^2|\phi|^2|u|^2+|u|^p\Big)\,dx\\ &= 2I+\left(1-\frac{2}{p}\right)\int_{\mathbb{R}^3}|u|^p\,dx.\end{aligned}$$

This implies that $\frac{\partial I}{\partial u}u \neq 0$ on $\mathcal{K}_\mu$.

Remark 3.1. *For any complex function v, one has $|\nabla u| \geq |\nabla |u|| = |\nabla i|u||$, a.e. in $\mathbb{R}^3$. Consequently, for any $\phi \in D^{1,2}(\mathbb{R}^3,\mathbb{R})$, $E(u,\phi) \geq E(|u|,\phi) = E(i|u|,\phi)$. Moreover, it is obvious that $I(u,\phi) = I(|u|,\phi) = I(i|u|,\phi)$, which means that, in order to solve the minimization problem* (3.1)*, we can restrict ourselves to real or pure imaginary functions u, knowing that ϕ must be a real function by Lemma 2.1.*

Our aim of this section is to show that, for a given $\mu > 0$, the minimization problem (3.1) admits a minimizer $(u_\mu,\phi_\mu) \in X$. According to Remark 3.1, we consider only real functions u. We first give the following lemma which will be useful later on.

Lemma 3.2. *Let (u,ϕ) be a solution to* (1.1)-(1.2)*. Then one has*

$$E(u,\phi) = 3I(u,\phi) \text{ and } S_\omega(u,\phi) = 2I(u,\phi).$$

Proof. Using Lemma 2.2, one has

$$\begin{aligned}E(u,\phi) &= \frac{1}{2}\int_{\mathbb{R}^3}\Big(|\nabla u|^2+|\nabla\phi|^2\Big)\,dx\\ &= \frac{1}{2}\Big(-3\gamma\int_{\mathbb{R}^3}|u|^2\,dx+6e\omega\int_{\mathbb{R}^3}\phi|u|^2\,dx+3e^2\int_{\mathbb{R}^3}\phi^2|u|^2\,dx+\frac{6}{p}\int_{\mathbb{R}^3}|u|^p\,dx\Big)\\ &= 3I(u,\phi)\end{aligned}$$

and

$$\begin{aligned}&S_\omega(u,\phi)\\ &= E(u,\phi)+\frac{\gamma}{2}\int_{\mathbb{R}^3}|u|^2\,dx-e\omega\int_{\mathbb{R}^3}\phi|u|^2\,dx-\frac{e^2}{2}\int_{\mathbb{R}^3}\phi^2|u|^2\,dx-\frac{1}{p}\int_{\mathbb{R}^3}|u|^p\,dx\\ &= 2I(u,\phi).\end{aligned}$$

This completes the proof. □

Next we study the behavior of the function J with respect to μ.

Lemma 3.3. *Assume that $\gamma = m^2 - \omega^2 > 0$. Then for any $\mu > 0$, one has $J(\mu) > 0$.*

Proof. Since it is obvious that $J(\mu) \geq 0$, we only have to prove that $J(\mu) \neq 0$. Let $(u_n, \phi_n) \in X$ be a minimizing sequence for (3.1) and assume that $J(\mu) = 0$, that is,

$$I(u_n, \phi_n) = \mu \text{ and } \int_{\mathbb{R}^3} \Big(|\nabla u_n|^2 + |\nabla \phi_n|^2 \Big)\, dx \longrightarrow 0 \quad \text{as } n \to +\infty. \tag{3.2}$$

By Hölder and Gagliardo-Nirenberg inequalities, we then deduce

$$\begin{aligned}
&\mu + \frac{\gamma}{2} \int_{\mathbb{R}^3} |u_n|^2\, dx \\
&= \int_{\mathbb{R}^3} \Big(e\omega\phi_n |u_n|^2 + \frac{e^2}{2}\phi_n^2 |u_n|^2 + \frac{1}{p}|u_n|^p \Big)\, dx. \\
&\leq |e||\omega| \Big(\int_{\mathbb{R}^3} |\phi_n|^6\, dx \Big)^{\frac{1}{6}} \Big(\int_{\mathbb{R}^3} |u_n|^{\frac{12}{5}}\, dx \Big)^{\frac{5}{6}} + \frac{e^2}{2} \Big(\int_{\mathbb{R}^3} |\phi_n|^6\, dx \Big)^{\frac{1}{3}} \Big(\int_{\mathbb{R}^3} |u_n|^3\, dx \Big)^{\frac{2}{3}} \\
&\quad + C \Big(\int_{\mathbb{R}^3} |\nabla u_n|^2\, dx \Big)^{\frac{3(p-2)}{4}} \Big(\int_{\mathbb{R}^3} |u_n|^2\, dx \Big)^{\frac{6-p}{4}} \\
&\leq C \Big(\|\nabla \phi_n\|_{L^2} \|\nabla u_n\|_{L^2}^{\frac{1}{2}} \|u_n\|_{L^2}^{\frac{3}{2}} + \|\nabla \phi_n\|_{L^2}^2 \|\nabla u_n\|_{L^2} \|u_n\|_{L^2} \\
&\qquad\qquad + \|\nabla u_n\|_{L^2}^{\frac{3(p-2)}{2}} \|u_n\|_{L^2}^{\frac{6-p}{2}} \Big).
\end{aligned}$$

We take $\varepsilon > 0$ such that $\gamma - 6\varepsilon > 0$. By Young inequality, we have

$$\begin{aligned}
&\mu + \frac{\gamma}{2} \int_{\mathbb{R}^3} |u_n|^2\, dx \\
&\leq C(\varepsilon) \Big(\|\nabla \phi_n\|_{L^2}^4 \|\nabla u_n\|_{L^2}^2 + \|\nabla u_n\|_{L^2}^6 \Big) + 3\varepsilon \|u_n\|_{L^2}^2,
\end{aligned}$$

where the constant $C(\varepsilon)$ depends on ε. Then we obtain

$$\mu \leq \mu + \Big(\frac{\gamma}{2} - 3\varepsilon \Big) \|u_n\|_{L^2}^2 \leq C(\varepsilon) \Big(\|\nabla \phi_n\|_{L^2}^4 \|\nabla u_n\|_{L^2}^2 + \|\nabla u_n\|_{L^2}^6 \Big). \tag{3.3}$$

By letting $n \to +\infty$ and using (3.2), we obtain $\mu = 0$, a contradiction. □

Lemma 3.4. *Assume that $\gamma > 0$. Then for any $\mu > 0$ and $\theta > 1$, one has $J(\theta\mu) < \theta J(\mu)$. As a consequence, the function $J(\mu)$ satisfies the sub-additivity condition : $J(\mu) < J(\lambda) + J(\mu - \lambda)$ for all $\mu > 0$ and $\lambda \in (0, \mu)$.*

Proof. Take $\mu > 0$, $\theta > 1$ and let $(u_n, \phi_n)_{n\in\mathbb{N}} \in X$ a minimizing sequence for (3.1), that is, $I(u_n, \phi_n) = \mu$ and $E(u_n, \phi_n) \longrightarrow J(\mu)$ as $n \to +\infty$. We put

$$w_n(x) = u_n\Big(\frac{x}{\theta^{\frac{1}{3}}}\Big) \text{ and } \psi_n(x) = \phi_n\Big(\frac{x}{\theta^{\frac{1}{3}}}\Big).$$

Using the 3-dimensional change of variable $y = \frac{x}{\theta^{\frac{1}{3}}}$, we get

$$\begin{aligned} I(w_n, \psi_n) &= \theta I(u_n, \phi_n) = \theta\mu, \\ E(w_n, \psi_n) &= \frac{1}{2}\int_{\mathbb{R}^3} \Big(|\nabla w_n|^2 + |\nabla \psi_n|^2\Big)\, dx \\ &= \frac{\theta}{2\theta^{\frac{2}{3}}}\int_{\mathbb{R}^3} \Big(|\nabla u_n|^2 + |\nabla \phi_n|^2\Big)\, dy \\ &= \frac{\theta^{\frac{1}{3}}}{2} E(u_n, \phi_n) \\ &= \frac{\theta}{2} E(u_n, \phi_n) + \frac{\theta^{\frac{1}{3}} - \theta}{2} E(u_n, \phi_n). \end{aligned} \tag{3.4}$$

Since $\theta > 1$, one has $\theta^{\frac{1}{3}} - \theta < 0$. Furthermore we have

$$\lim_{n\to+\infty} E(u_n, \phi_n) = J(\mu) > 0,$$

which provide us by passing to the limit in (3.4) that

$$J(\theta\mu) < \theta J(\mu). \tag{3.5}$$

The second part of Lemma 3.4 is a direct consequence of (3.5). □

Lemma 3.5. *Assume that $\gamma > 0$. Then for any $\mu > 0$, the minimization problem* (3.1) *admits a solution $(u_\mu, \phi_\mu) \neq (0, 0)$ where u_μ and ψ_μ are real functions.*

Proof. We argue as in [8]. Let $(u_n, \phi_n)_{n\in\mathbb{N}}$ be a minimizing sequence for (3.1). Then it is clear that $(\phi_n)_{n\in\mathbb{N}}$ is bounded in $D^{1,2}$. Moreover by (3.4), we obtain that $\|u_n\|_{L^2}$ is bounded and hence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$.

Now we apply the concentration-compactness principle (see [15]) to the sequence:

$$\rho_n = |u_n|^2 + |u_n|^p + |\phi_n|^6.$$

If vanishing occurs, there exists a subsequence of $(\rho_n)_{n\in\mathbb{N}}$, still denoted by $(\rho_n)_{n\in\mathbb{N}}$ for simplicity, such that

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{y+B_R} \rho_n\, dx = 0 \text{ for all } R > 0.$$

Here B_R describes a ball of radius R with the center at the origin. Then by Lemma I.1. of [16], it follows that $u_n \longrightarrow 0$ in $L^r(\mathbb{R}^3)$ as $n \to +\infty$ for all $r \in (2,6)$. By Hölder Inequality, one has

$$\begin{aligned}\mu &\le \mu + \frac{\gamma}{2}\int_{\mathbb{R}^3} |u_n|^2\,dx\\ &= \int_{\mathbb{R}^3}\Big(e\omega\phi_n|u_n|^2 + \frac{e^2}{2}\phi_n^2|u_n|^2 + \frac{1}{p}|u_n|^p\Big)\,dx.\\ &\le |e||\omega|\Big(\int_{\mathbb{R}^3}|\phi_n|^6\,dx\Big)^{\frac{1}{6}}\Big(\int_{\mathbb{R}^3}|u_n|^{\frac{12}{5}}\,dx\Big)^{\frac{5}{6}} + \frac{e^2}{2}\Big(\int_{\mathbb{R}^3}|\phi_n|^6\,dx\Big)^{\frac{1}{3}}\Big(\int_{\mathbb{R}^3}|u_n|^3\,dx\Big)^{\frac{2}{3}}\\ &\quad + \frac{1}{p}\int_{\mathbb{R}^3}|u_n|^p\,dx\\ &\quad\longrightarrow 0 \ \text{ as } n \to +\infty,\end{aligned}$$

since $p \in (2,6)$ and $(\phi_n)_{n\in\mathbb{N}}$ is bounded in $L^6(\mathbb{R}^3)$. This is a contradiction, which rules out vanishing.

Assume now that dichotomy occurs, that is, there exists $\lambda > 0$ such that

$$\lambda = \lim_{n\to+\infty}\int_{\mathbb{R}^3}\rho_n\,dx.$$

By classical arguments (see [16] part IV.1), for some $\kappa \in (0,\lambda)$, one can build four sequences $(u_{\ell,1})_{\ell\in\mathbb{N}}$, $(u_{\ell,2})_{\ell\in\mathbb{N}}$ which are bounded in $H^1(\mathbb{R}^3)$ and $(\phi_{\ell,1})_{\ell\in\mathbb{N}}$, $(\phi_{\ell,2})_{\ell\in\mathbb{N}}$ which are bounded in $D^{1,2}(\mathbb{R}^3)$ (where $(u_{\ell,1})_{\ell\in\mathbb{N}}$, $(\phi_{\ell,1})_{\ell\in\mathbb{N}}$ and $(u_{\ell,2})_{\ell\in\mathbb{N}}$, $(\phi_{\ell,2})_{\ell\in\mathbb{N}}$ have disjoint compact supports) such that for some subsequence $(u_{n(\ell)},\phi_{n(\ell)})_{\ell\in\mathbb{N}}$ of (u_n,ϕ_n), it follows that

$$\|u_{n(\ell)}-u_{\ell,1}-u_{\ell,2}\|_{L^2} \le \frac{1}{\ell},\ \|u_{n(\ell)}-u_{\ell,1}-u_{\ell,2}\|_{L^p} \le \frac{1}{\ell},\ \|\phi_{n(\ell)}-\phi_{\ell,1}-\phi_{\ell,2}\|_{L^6} \le \frac{1}{\ell}. \tag{3.6}$$

$$\Big|\int_{\mathbb{R}^3}\Big(|u_{\ell,1}|^2 + |u_{\ell,1}|^p + |\phi_{\ell,1}|^6\Big)\,dx - \kappa\Big| \le \frac{1}{\ell},$$

$$\Big|\int_{\mathbb{R}^3}\Big(|u_{\ell,2}|^2 + |u_{\ell,2}|^p + |\phi_{\ell,2}|^6\Big)\,dx - (\lambda-\kappa)\Big| \le \frac{1}{\ell},$$

$$\liminf_{\ell\to+\infty}\int_{\mathbb{R}^{d+1}}\Big(|\nabla u_{n(\ell)}|^2 - |\nabla u_{\ell,1}|^2 - |\nabla u_{\ell,2}|^2\Big) \ge 0, \tag{3.7}$$

$$\liminf_{\ell\to+\infty}\int_{\mathbb{R}^{d+1}}\Big(|\nabla \phi_{n(\ell)}|^2 - |\nabla \phi_{\ell,1}|^2 - |\nabla \phi_{\ell,2}|^2\Big) \ge 0. \tag{3.8}$$

Now since $(u_{\ell,1})_{\ell\in\mathbb{N}}$, $(\phi_{\ell,1})_{\ell\in\mathbb{N}}$ and $(u_{\ell,2})_{\ell\in\mathbb{N}}$, $(\phi_{\ell,2})_{\ell\in\mathbb{N}}$ have disjoint compact supports, one has $I(u_{n(\ell)},\phi_{n(\ell)}) = I(u_{\ell,1},\phi_{\ell,1}) + I(u_{\ell,2},\phi_{\ell,2})$. Moreover by

(3.6), we can deduce that there exists $\chi \in (0, \mu)$ such that

$$I(u_{\ell,1}, \phi_{\ell,1}) \longrightarrow \chi \text{ and } I(u_{\ell,2}, \phi_{\ell,2}) \longrightarrow \mu - \chi \text{ as } n \to +\infty.$$

Then using (3.7)-(3.8), we deduce that

$$\begin{aligned} J(\mu) = \liminf_{\ell\to+\infty} E(u_\ell, \phi_\ell) &\geq \liminf_{\ell\to+\infty} E(u_{\ell,1}, \phi_{\ell,1}) + E(u_{\ell,2}, \phi_{\ell,2}) \\ &\geq J(\chi) + J(\mu - \chi), \end{aligned}$$

which contradicts to Lemma 3.4.

The only remaining possibility is the compactness of the minimizing sequence modulo translations, that is, there exists a sequence $(y_n)_{n\in\mathbb{N}} \in \mathbb{R}^3$ such that

$$\forall \varepsilon > 0,\ \exists R_\varepsilon < +\infty \text{ such that } \int_{|x-y_n|\leq R_\varepsilon} \rho_n \, dx \geq \lambda - \varepsilon,\ \forall n \in \mathbb{N}. \tag{3.9}$$

Since $(u_n)_{n\in\mathbb{N}}$ and $(\phi_n)_{n\in\mathbb{N}}$ are bounded in $H^1(\mathbb{R}^3)$ and $D^{1,2}(\mathbb{R}^3)$ respectively, there exist two functions $u_\mu \in H^1(\mathbb{R}^3)$ and $\phi_\mu \in D^{1,2}(\mathbb{R}^3)$ such that $u_n(\cdot - y_n)$ converges weakly in $H^1(\mathbb{R}^3)$ to u_μ and $\phi_n(\cdot - y_n)$ converges weakly to ϕ_μ in $D^{1,2}(\mathbb{R}^3)$. Then (3.9) implies that $u_n(\cdot - y_n)$ converge strongly in $L^2(\mathbb{R}^3)$ and in $L^p(\mathbb{R}^3)$ to u_μ, and $\phi_n(\cdot - y_n)$ converge strongly in $L^6(\mathbb{R}^3)$ to ϕ_μ. By Hölder inequality and Sobolev embeddings, one has

$$\begin{aligned} &\Big| \int_{\mathbb{R}^3} \phi_n |u_n|^2 \, dx - \int_{\mathbb{R}^3} \phi_\mu |u_\mu|^2 \, dx \Big| \\ &= \Big| \int_{\mathbb{R}^3} (\phi_n - \phi_\mu) |u_n|^2 \, dx + \int_{\mathbb{R}^3} \phi_\mu (|u_n|^2 - |u_\mu|^2) \, dx \Big| \\ &\leq \|\phi_n - \phi_\mu\|_{L^6} \|u_n\|^2_{L^{\frac{12}{5}}} + \|\phi_\mu\|_{L^6} \|u_n - u_\mu\|_{L^2} \|u_n + u_\mu\|_{L^3} \\ &\leq C\Big(\|\phi_n - \phi_\mu\|_{L^6} + \|u_n - u_\mu\|_{L^2} \Big) \longrightarrow 0 \text{ as } n \to +\infty. \end{aligned}$$

In the same way, one obtains

$$\begin{aligned} &\Big| \int_{\mathbb{R}^3} |\phi_n|^2 |u_n|^2 \, dx - \int_{\mathbb{R}^3} |\phi_\mu|^2 |u_\mu|^2 \, dx \Big| \\ &= \Big| \int_{\mathbb{R}^3} (|\phi_n|^2 - |\phi_\mu|^2) |u_n|^2 \, dx + \int_{\mathbb{R}^3} |\phi_\mu|^2 (|u_n|^2 - |u_\mu|^2) \, dx \Big| \\ &\leq \|\phi_n - \phi_\mu\|_{L^6} \|\phi_n + \phi_\mu\|_{L^6} \|u_n\|^2_{L^3} + \|\phi_\mu\|^2_{L^6} \|u_n - u_\mu\|_{L^2} \|u_n + u_\mu\|_{L^6} \\ &\leq C\Big(\|\phi_n - \phi_\mu\|_{L^6} + \|u_n - u_\mu\|_{L^2} \Big) \longrightarrow 0 \text{ as } n \to +\infty. \end{aligned}$$

As a consequence, one derives

$$\mu = \lim_{n\to\infty} I(u_n, \phi_n) = I(u_\mu, \phi_\mu).$$

This implies that $(u_\mu, \phi_\mu) \neq (0,0)$ and thus

$$J(\mu) = \liminf_{n\to\infty} E(u_n, \phi_n) \geq E(u_\mu, \phi_\mu),$$

that is, (u_μ, ϕ_μ) is a solution to the minimization problem (3.1). Moreover, by Remark 3.1, u_μ is a real function. □

4 Proof of Theorem 1.1.

In this section, we show the existence of a ground state of (1.1)-(1.2) by using the minimizer of (3.1).

Now let $(u, \phi) \in X$ be a minimizer of (3.1), u being a real function (we omit the subscript μ for simplicity). Then by the method of Lagrange multipliers, there exists $\lambda \in \mathbb{R}$ such that $E^{'}(u,\phi) = \lambda I^{'}(u,\phi)$, or equivalently

$$-\Delta u + \lambda\Big(\gamma u - 2\omega e\phi u - e^2|\phi|^2 u - |u|^{p-2}u\Big) = 0, \tag{4.1}$$

$$-\Delta\phi - \lambda\Big(e\omega|u|^2 + e^2\phi|u|^2\Big) = 0. \tag{4.2}$$

Lemma 4.1. *Owning the above notations, one has $\lambda > 0$.*

Proof. First we observe from (3.3) that one has $\lambda \neq 0$. To prove $\lambda > 0$, we start from $I(u,\phi) = \mu$, that is,

$$\mu = -\frac{\gamma}{2}\int_{\mathbb{R}^3}|u|^2\,dx + e\omega\int_{\mathbb{R}^3}\phi|u|^2\,dx + \frac{e^2}{2}\int_{\mathbb{R}^3}|\phi|^2|u|^2\,dx + \frac{1}{p}\int_{\mathbb{R}^3}|u|^p\,dx.$$

This implies that

$$\gamma\int_{\mathbb{R}^3}|u|^2\,dx - 2e\omega\int_{\mathbb{R}^3}\phi|u|^2\,dx - e^2\int_{\mathbb{R}^3}|\phi|^2|u|^2\,dx = -2\mu + \frac{2}{p}\int_{\mathbb{R}^3}|u|^p\,dx. \tag{4.3}$$

On the other hand applying Lemma 2.1 to equations (4.1)-(4.2), we obtain

$$\begin{aligned} 0 = &\int_{\mathbb{R}^3}|\nabla u|^2\,dx \\ &+ \lambda\Big(\gamma\int_{\mathbb{R}^3}|u|^2\,dx - 2e\omega\int_{\mathbb{R}^3}\phi|u|^2\,dx - e^2\int_{\mathbb{R}^3}\phi^2|u|^2\,dx - \int_{\mathbb{R}^3}|u|^p\,dx\Big), \end{aligned} \tag{4.4}$$

$$\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx = \lambda\Big(e\omega\int_{\mathbb{R}^3}\phi|u|^2\,dx + e^2\int_{\mathbb{R}^3}|\phi|^2|u|^2\,dx\Big)\,dx.$$

From (4.3) and (4.4), we have

$$\int_{\mathbb{R}^3} |\nabla u|^2 \, dx = \lambda \left(2\mu + \left(1 - \frac{2}{p} \right) \int_{\mathbb{R}^3} |u|^p \, dx \right).$$

Since $\mu > 0$ and $p > 2$, it follows that $\lambda > 0$. □

We are now able to construct a solution to (1.1)-(1.2) by introducing

$$w(x) = u\Big(\frac{x}{\sqrt{\lambda}}\Big) \text{ and } \psi(x) = \phi\Big(\frac{x}{\sqrt{\lambda}}\Big).$$

Since (u, ϕ) solve (4.1)-(4.2), it is obvious that (w, ψ) solve (1.1)-(1.2). Our next aim is to prove that (w, ψ) is a ground state.

Now a direct computation gives

$$\begin{aligned} I(w, \psi) &= \int_{\mathbb{R}^3} \Big(-\frac{\gamma}{2}|w|^2 + e\omega\psi|w|^2 + \frac{e^2}{2}|\psi|^2|w|^2 + \frac{1}{p}|w|^p \Big) \, dx \\ &= \lambda^{\frac{3}{2}} \int_{\mathbb{R}^3} \Big(-\frac{\gamma}{2}|u|^2 + e\omega\phi|u|^2 + \frac{e^2}{2}|\phi|^2|u|^2 + \frac{1}{p}|u|^p \Big) \, dx \\ &= \lambda^{\frac{3}{2}}\mu. \end{aligned}$$

We claim that the following occurs.

Lemma 4.2. *The couple (w, ψ) satisfies $E(w, \psi) = J(\lambda^{\frac{3}{2}}\mu)$, that is, (w, ψ) is a solution to the minimization problem* (3.1) *where the constraint μ has been replaced by $\tilde{\mu} := \lambda^{\frac{3}{2}}\mu$.*

Proof. First we recall that $I(w, \psi) = \tilde{\mu}$. Let $(\tilde{w}, \tilde{\phi}) \in X$ be such that $I(\tilde{w}, \tilde{\psi}) = \tilde{\mu}$ and define

$$\tilde{u}(x) = \tilde{w}\left(\sqrt{\lambda}x\right) \text{ and } \tilde{\phi}(x) = \tilde{\psi}\left(\sqrt{\lambda}x\right).$$

Then it is obvious that $I(\tilde{u}, \tilde{\phi}) = \mu$. Recalling that (u, ϕ) is a minimizer of (3.1), we obtain

$$\frac{1}{\sqrt{\lambda}} E(w, \psi) = E(u, \phi) \leq E(\tilde{u}, \tilde{\phi}) = \frac{1}{\sqrt{\lambda}} E(\tilde{w}, \tilde{\psi}).$$

This completes the proof. □

Lemma 4.3. *The couple (w, ψ) is a ground state of* (1.1)-(1.2).

Proof. Let $(\hat{u}, \hat{\phi}) \neq (0,0)$ be a solution to (1.1)-(1.2) satisfying

$$S_\omega(\hat{u}, \hat{\phi}) \leq S_\omega(w, \psi). \tag{4.5}$$

We claim that the equality should hold in (4.5).

Now by Lemma 3.2, one has $E(\hat{u}, \hat{\phi}) = 3I(\hat{u}, \hat{\phi})$, which implies that $I(\hat{u}, \hat{\phi}) > 0$. We put

$$\theta := \left(\frac{\lambda^{\frac{3}{2}} \mu}{I(\hat{u}, \hat{\phi})} \right)^{\frac{1}{3}}$$

so that $\hat{w}(x) = \hat{u}\left(\frac{x}{\theta}\right)$ and $\hat{\psi}(x) = \hat{\phi}\left(\frac{x}{\theta}\right)$ satisfy $I(\hat{w}, \hat{\psi}) = \lambda^{\frac{3}{2}} \mu$. Then by using Lemma 3.2, one can write

$$2I(\hat{u}, \hat{\phi}) = S_\omega(\hat{u}, \hat{\phi}) \leq S_\omega(w, \psi) = 2I(w, \psi) = 2\lambda^{\frac{3}{2}} \mu,$$

which implies that $\theta \geq 1$. Moreover, since (w, ψ) solves (3.1) with $\tilde{\mu} = \lambda^{\frac{3}{2}} \mu$ by Lemma 4.2, one derives, using again Lemma 3.2 and $\theta \geq 1$ that

$$\begin{aligned} 3\lambda^{\frac{3}{2}} \mu = 3I(w, \psi) &= E(w, \psi) \\ &\leq E(\hat{w}, \hat{\psi}) = \theta E(\hat{u}, \hat{\phi}) = 3\theta I(\hat{u}, \hat{\phi}) \\ &\leq 3\theta^3 I(\hat{u}, \hat{\phi}) = 3\lambda^{\frac{3}{2}} \mu. \end{aligned}$$

Then all the previous inequalities are equalities. In particular it follows that $\theta = 1$, from which we conclude that

$$S_\omega(\hat{u}, \hat{\phi}) = 2\lambda^{\frac{3}{2}} \mu = S_\omega(w, \psi).$$

Then (w, ψ) is a ground state of (1.1)-(1.2), w and ψ being real functions. This completes the proof of Theorem 1.1. □

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