1 Introduction

1.1 Monodromy and automorphism groups

- $R$ is a strictly henselian DVR of inequal characteristic $(0, p)$.
  $K := \text{Fr}R$; for example $K/\mathbb{Q}_p$ finite.
  $\pi$ a uniformizing parameter.
  $k := R_K/\pi R_K$. 
  $C/K$ smooth projective curve, $g(C) \geq 1$.

- $C$ has potentially good reduction over $K$ if there is $L=K$(finite) such that $C \times_K L$ has a smooth model over $R_L$. Then:

  - There is a minimal extension $L/K$ with this property; it is Galois and called the monodromy extension.
  - $\text{Gal}(L/K)$ is the monodromy group.
  - Its $p$-Sylow subgroup is the wild monodromy group.

- The base change $C \times_K K^{alg}$ induces an homomorphism $\varphi : \text{Gal}(K^{alg}/K) \to \text{Aut}_k C_s$, where $C_s$ is the special fiber of the smooth model over $R_L$ and $L = (K^{alg})^{\text{ker} \varphi}$.

- Let $\ell$ be a prime number, then, $n_\ell := v_\ell(|\text{Gal}(L/K)|) \leq v_\ell(|\text{Aut}_k C_s|)$.

- If $\ell \notin \{2, p\}$, then $\ell^{n_\ell}$ is bounded by the maximal order of an $\ell$-cyclic subgroup of $\text{GL}_{2g}(\mathbb{Z}/\ell\mathbb{Z})$ i.e. $\ell^{n_\ell} \leq O(g)$.

- If $p > 2$, then $n_p \leq \inf_{\ell\neq 2, p} v_p(|\text{GL}_{2g}(\mathbb{Z}/\ell\mathbb{Z})|) = a + [a/p] + ..., \text{ where } a = \lfloor \frac{2g}{p-1} \rfloor$. 

This gives an exponential type bound in $g$ for $|\text{Aut}_k C_s|$. This justifies our interest in looking at Stichtenoth ([St,73]) and Singh ([Si,73]).

**Theorem 1.1.** ([Ra, 90]). Let $Y_K \to X_K$ be a Galois cover with group $G$. Let us assume that:

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*This paper is a report on common work with Claus Lehr. This is a pdf style version of lectures given at Chuo University April 2006. A slide version using beamer is available at http://www.math.u-bordeaux.fr/~matignon/
• $G$ is nilpotent.

• $X_K$ has a smooth model $X$.

• The Zariski closure $B$ of the branch locus $B_K$ in $X$ is étale over $R_K$.

Then, the special fiber of the stable model $Y_K$ is tree-like, i.e. the Jacobian of $Y_K$ has potentially good reduction.

Raynaud’s proof is qualitative and it seems difficult to give a constructive one in the simplest cases.

We have given in [Le-Ma1] such a proof in the case of $p$-cyclic covers of the projective line.

Thanks. The author would like to use this opportunity to thank T. Sekiguchi, N. Suwa and B. Green for the pleasant and working atmosphere during his visit to Tokyo.

2 Automorphism groups of curves in char. $p > 0$

2.1 $p$-cyclic covers of the affine line

$k$ is an algebraically closed of char. $p > 0$.

• $f(X) \in Xk[X]$ monic, deg $f = m > 1$ prime to $p$.

• $C_f : W^p - W = f(X)$. Let $\infty$ be the point of $C_f$ above $X = \infty$ and $z$ a local parameter. Then, $g := g(C_f) = \frac{n-1}{2}(m-1) > 0$.

• $G_{\infty}(f) := \{ \sigma \in \text{Aut}_kC_f \mid \sigma(\infty) = \infty \}$.

• $G_{\infty,1}(f) := \{ \sigma \in \text{Aut}_kC_f \mid v_{\infty}(\sigma(z) - z) \geq 2 \}$, the $p$-Sylow.

• ([St,73]) Let $g(C_f) \geq 2$, then $G_{\infty,1}(f)$ is a $p$-Sylow of $\text{Aut}_kC_f$.

• It is normal except for $f(X) = X^m$ where $m|1 + p$.

2.2 Structure of $G_{\infty,1}(f)$

• Let $\rho(X) = X$, $\rho(W) = W + 1$, then $< \rho >= G_{\infty,2} \subset Z(G_{\infty,1})$

• $0 \rightarrow < \rho \rightarrow G_{\infty,1} \rightarrow V \rightarrow 0$, $V := \{ \tau_y \mid \tau_y(X) = X + y, y \in k \}$.

• $f(X + y) = f(X) + f(y) + (F - \text{Id})(P(X, y))$, $P(X, y) \in Xk[X]$.

• $V \simeq (\mathbb{Z}/p\mathbb{Z})^v$ as a subgroup of $k$.

• Let $\tau_y(W) := W + a_y + P(X, y)$, $a_y \in \mathbb{F}_p$, then $[\tau_y, \tau_z] = \rho^{(y,z)}$, where $\epsilon : V \times V \rightarrow \mathbb{F}_p$ is an alternating form.

• $\epsilon$ is non degenerated iff $< \rho >= Z(G_{\infty,1})$. 


2.3 Bounds for $|G_{\infty,1}(f)|$

Lemma 2.1. If $f(X) = \sum_{1\leq i\leq m} t_i X^i \in k[X]$ is monic, then:

- $\Delta(f)(X,Y) := f(X+Y) - f(X) - f(Y) = R(X,Y) + (F - \text{Id})(P_f(X,Y))$, where $R \in \bigoplus \{ \frac{1}{p} \}^{1 \leq i < m, (i,p)=1} k[Y]X^{fp(i)}$ and $P_f \in Xk[X,Y]$.

- $P_f = (\text{Id} + F + \ldots + F^{n-1}) (\Delta(f)) \mod X^{\lceil \frac{m-1}{p} \rceil + 1}$.

Let us denote by $\text{Ad}_f(Y)$ the content of $R(X,Y) \in k[Y][X]$, then

- $\text{Ad}_f(Y)$ is an additive and separable polynomial.

- $Z(\text{Ad}_f(Y)) \simeq V$.

Let $m - 1 = \ell p^s$ with $(\ell, p) = 1$.

- ([St 73]) $|G_{\infty,1}| = p \deg \text{Ad}_f \leq p(m - 1)^2$, i.e. $\frac{|G_{\infty,1}|}{g} \leq \frac{4p}{(p-1)^2}$.

- ([St 73]) $s = 0$ i.e. $(m - 1, p) = 1$, then $|G_{\infty,1}| = p$.

- If $s > 0$,
  - $\ell > 1, p = 2$, then $\frac{|G_{\infty,1}|}{g} \leq \frac{2}{3}$.
  - $\ell > 1, p > 2$, then $\frac{|G_{\infty,1}|}{g} \leq \frac{p}{p-1}$.

- ([St 73]) $\ell > 1, m = 1 + p^s$, then $\frac{|G_{\infty,1}|}{g} \leq 2p^s \frac{p}{p-1}$ (with equality for $f(X) = X^{1+p^s}$).

2.4 Characterization of $G_{\infty,1}(f)$

- We consider the extensions $0 \rightarrow N \simeq Z/pZ \rightarrow G \rightarrow (Z/pZ)^n \rightarrow 0$ (note that $G_{\infty,1}(f)$ is an extension of this type). Then $G' \subset N \subset Z(G)$.

- If $G' = Z(G)$, $G$ is called extraspecial.
  - Then, $|G| = p^{2s+1}$ and there are 2 isomorphism classes for a given $s$.
  - If $p > 2$, we denote by $E(p^3)$ (resp. $M(p^3)$) the non abelian group of order $p^5$ and exponent $p$ (resp. $p^2$). Then, $G \simeq E(p^3) * E(p^3) * \ldots * E(p^3)$ or $M(p^3) * E(p^3) * \ldots * E(p^3)$, according as the exponent is $p$ or $p^2$.
  - If $p = 2$, then $G \simeq D_8 * D_8 * \ldots * D_8$ or $Q_8 * D_8 * \ldots * D_8$ (in both cases, the exponent is $2^2$).

- If $G' \subset Z(G)$, $G$ is a subgroup of an extraspecial group $E$ with $Z(E) \subset G$.

Theorem 2.2. ([Le-Ma 1]). Let $f(X) = X\Sigma(F)(X) \in Xk[X]$, $\Sigma(F) = \sum_{0 \leq i \leq s} a_i F^i \in k\{F\}$ an additive polynomial with $\deg f = 1 + p^s$. Then,

- $\text{Ad}_f(Y) = F^s(\sum_{0 \leq i \leq s} (a_i F^i + F^{-i} a_i)(Y))$, a palindromic polynomial.

- $G_{\infty,1}(f)$ is an extraspecial group with cardinal $p^{2s+1}$ and exponent $p$ for $p > 2$, and of type $Q_8 * D_8 * \ldots * D_8$ for $p = 2$. 

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Theorem 2.3. ([Le-Ma 1]). If $G$ is an extension of type $0 \to \mathbb{Z}/p\mathbb{Z} \to G \to (\mathbb{Z}/p\mathbb{Z})^n \to 0$, there is $f \in Xk[X]$ with $G \simeq G_{\infty,1}(f)$.

- Sketch proof: Extraspecial groups with exponent $p^2$ are realized by a modification by a Witt cocycle of the polynomial $f$ in the previous theorem.
- We can see $G$ as a subgroup of an extraspecial group $E$, then we realize $E$ with $f_E$ and a suitable modification of $f_E$ will limit $G_{\infty,1}(f_E)$ to $G$.

3 Actions of $p$-groups over a curve $C$ with $g(C') \geq 2$

3.1 Big actions (I)

Theorem 3.1. ([Le-Ma 1]). Let $f(X) \in Xk[X]$ with $(\deg f, p) = 1$. If $\frac{|G_{\infty,1}|}{g} > \frac{p}{p-1}$ ($\frac{p}{2}$ for $p = 2$), then $f(X) = cX + X\Sigma(F)(X) \in k[X]$.

- Sketch proof: One shows that monomials in $f$ with a degree $\not\equiv 1 + p\mathbb{N}$ will limit the degree of $\text{Ad}_f$.
- Let $(C, G)$ with $G \subset \text{Aut}_kC$, a $p$-group. We say that $(C, G)$ is a big action if:
  
  (N) $gC > 0$ and $\frac{|G|}{gC} > \frac{2p}{p-1}$.

  It follows from ([Na 87]) that there is $\infty \in C$, with
  
  - $C \to C/G \simeq \mathbb{P}^1_k - \infty$ is étale and $G = G_{\infty,1}$.
  - $G_{\infty,2} \neq G_{\infty,1}$ and $C/G_{\infty,2} \simeq \mathbb{P}^1_k$
  - Then, $G_{\infty,1}/G_{\infty,2}$ acts as a group of translations of the affine line $C/G_{\infty,2} - \{\infty\}$.

- Transfert of condition (N) to quotients. Let $(C, G)$ a big action, if $H \lhd G$ and if $g(C/H) > 0$, then $(C/H, G/H)$ is a big action.

3.2 Condition (N) and $G_2$

In this section $(C, G)$ is a big action. Let $G_i$ be the lower ramification groups.

- Let $H \lhd G$ and $H$ with index $p$ in $G_2$ ($H$ exists!), then $(C/H, G/H)$ satisfies (N).
- $(G/H)_2 = G_2/H \simeq \mathbb{Z}/p\mathbb{Z}$.
- There is $S(F) \in k\{F\}$, $f_1 = cX + X\Sigma(F)(X) \in k[X]$ with $C/H \simeq C_{f_1}$.
- If $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^t$, then $k(C) = k(X, W_1, ..., W_t)$ and $\varphi(W_1, ..., W_t) = (f_1(X), f_2(X), ..., f_t(X)) \in (k[X])^t$
- $f_1(X), ..., f_t(X)$ are $\bar{F}$-free \ mod $\varphi(k[X])$.
- The group extension $0 \to G_2 \to G_1 \to V = (\mathbb{Z}/p\mathbb{Z})^n \to 0$ induces a representation $\rho : V \to \text{Gl}_t(\bar{F})$.
- dual to the one given by $V$ acting via translation: $(v \in V) \times (f_1(X), f_2(X), ..., f_t(X)) \mod \varphi(k[X])^t \to \to (f_1(X + v), f_2(X + v), ..., f_t(X + v)) \mod \varphi(k[X])^t$.
• Imp is a unipotent subgroup of $\text{GL}_1(\mathbb{F}_p)$ which is the identity iff $G_2 \subset Z(G)$. In this case $f_1(X) = c_1X + X \Sigma_1(F)(X)$ where $\Sigma_1(F) \in k\{F\}$ and $v \in V$ is a common zero to the palyndromic polynomials $\text{Ad}_{f_1} \in k\{F, F^{-1}\}$.

• Let $f_1 := X(\alpha F)(X) = \alpha X^{1+p}$ with $\alpha^p + \alpha = 0$; then $\text{Ad}_{f_1} = Y^{p^2} - Y$.

• Let $f_2 := X^{1+2p} - X^{2+p}$, then $f_2(X + Y) - f_2(X) - f_2(Y) = 2(Y^p - Y)X^{1+p} + (Y - Y^{p^2})X^{2p} + (Y^{2p^2} - Y^2 + 2Y^{1+p} - 2Y^{p+p^2})XP \mod \varphi(k[X, Y])$

• If $y \in Z(\text{Ad}_{f_1}) = \mathbb{F}_{p^2}$ one has $f_2(X + y) = \frac{2(y^p - y)}{\alpha} f_1(X) + f_2(X) + \varphi(P_2)$.

• $y \rightarrow \frac{2(y^p - y)}{\alpha}$ is a non zero linear form over $\mathbb{F}_{p^2}$ with value in $\mathbb{F}_p$.

• $|G| = p^2p^2$ and $g = \frac{p - 1}{2}(p + p(2p))$.

• $\frac{|G|}{g} = \frac{2p}{p - 1 + 2p}$.

• $\frac{|G|}{g^2} = \frac{4p}{(p - 1)^2(1 + 2p)^2}$.

**Theorem 3.2.** ([Le-Ma 4]) Let $(C, G)$ be a big action then $G_2 = G'$.

• Sketch proof: If $G' \neq G_2$, there is $H < G$ with $G' \subset H \subset G_2$ and $[G_2 : H] = p$. $(C/H, G/H)$ satisfies condition (N);

• $C/H : W^p - W = f := X \Sigma_1(F)(X), \deg(f) = 1 + p^a$.

• $(\text{Aut}C/H)_{\infty, 1} := E$, is extraspecial with order $p^{2s+1}$.

• $G/H$ is abelian and normal in $E$.

• ([Hu 67] Satz 13.7 p. 353) $|G/H| \leq p^{s+1}$ and so $|G/H|/g(C/H) \leq \frac{2p^{s+1}}{(p - 1)p^2} = \frac{2p}{p - 1}$, a contradiction.

We deduce the following corollary from ([Su 86] 4.21 p.75).

**Corollary 3.3.** If $|G_2| = p^3$, then $G_2$ is abelian.

### 3.3 Riemann surfaces

• In characteristic 0, an analogue of big actions is given by the actions of a finite group $G$ on a compact Riemann surface $C$ with $g_C \geq 2$ such that $|G| = 84(g_C - 1)$ (we say that $C$ is an Hurwitz curve) ([Co 90]).

• Let us mention Klein’s quartic $(G \simeq PSL_2(\mathbb{F}_7))$ ([El 99]).

• The Fricke-Macbeath quartic curves with genus 7 $(G \simeq PSL_2(\mathbb{F}_8))$ ([Mc].65).

• Let $C$ be an Hurwitz curve with genus $g_C$. Let $n > 1$ and $C_n$ the maximal unramified Galois cover whose group is abelian with exponent $n$. The Galois group of $C_n/C$ is $(\mathbb{Z}/n\mathbb{Z})^{2g_c}$. It follows from the unicity of $C_n$ that the $k$-automorphisms of $C$ have $n^{2p}$ prolongations to $C_n$. Therefore $g_{C_n} - 1 = n^{2g}(g_C - 1)$ and $n^{2g} |\text{Aut}_k C| \leq |\text{Aut}_k C_n|$, where $|\text{Aut}_k C_n| \geq 84(g_{C_n} - 1)$; $C_n$ is an Hurwitz curve ([Mc].61).
3.4 Ray class fields

- If \((C, G)\) is a big action in \(\text{char}, p > 0\), then \(C \to C/G\) is an tame cover of the affine line whose group is a \(p\)-group; it follows that the Hasse-Witt invariant of \(C\) is zero; therefore, in order to adapt the previous proof to char. \(p > 0\), one needs to accept ramification. This is done with the so called ray class fields of function fields over finite fields.

- Let \(K := \mathbb{F}_q(X)\) where \(q = p^e\), \(S\) the set of finite rational places \((X - v), v \in \mathbb{F}_q\) and \(m \in \mathbb{N}\). Let \(K^{alg}\) be an algebraic closure. Let \(K_S^m \subset K^{alg}\) be the biggest abelian extension \(L\) of \(K\) with conductor \(\leq m\infty\) and such that the places in \(S\) are completely decomposed.

3.5 Big actions (II)

From now on, \(k\) is any algebraically closed field and \((C, G)\) is a big action.

- If \(G_2 \simeq \mathbb{Z}/p^n\mathbb{Z}\), then \(n = 1\) ([Le-Ma 4]).

  - Sketch proof: Let \(H = G_2^{p-2}\) then \((C/H, G/H)\) is a big action, it follows that one can assume that \(n = 2\). Then \(C \to C/G_2\) is given by \(\varphi(W_0, W_1) = (f_0, f_1)\) with \(f_0 = X\Sigma(F)(X)\), \(\deg f_0 = 1 + p^s\).

  - Let \(v \in V := Z(\text{Ad}_{f_0})\) and \(P \in k[X]\) with \(f_0(X + v) = f_0(X) + \varphi(P)\) then \(f_1(X + v) - f_1(X) = \ell(v)f_0(X) + \frac{1}{p^e}(f_0(X)^p + P(X)^p - P(X)^p - (f_0(X) + P(X))^p - f_0(X + v)^p + (f_0(X + v) + P(X))^p)\)

    \[= \ell(v)f_0(X) + \sum_{1 \leq i \leq p-1} \frac{(-1)^{i+1}}{i!} v^i X^{p^{i+1}-1} \mod X^{p^{i+1}}\]

    where \(\ell : V \to \mathbb{F}_p\) is a linear form.

  - More generally for \(G_2\) abelian with exponent \(p^e\), \(e \geq 2\), one can expect a lower bound in \(O(\log(g_G))\) for the \(p\)-rank of \(G_2\). This is the case in the preceding situation i.e. \((C, G) = (C_{m_2}, G(m_2))\) ([M. Rocher, thesis in preparation]).
3.6 Maximal curves

Let us assume that \((C, G)\) is a big action.

- Let \(i_0\) with \(G_2 = G_3 = \ldots = G_{i_0} \supseteq G_{i_0+1}\). Then \(g_{(C/G_{i_0+1})} = \frac{1}{2}(|G_2/G_{i_0+1}| - 1)(i_0 - 1)\).
- If \(0 < M \leq \frac{|G|}{g_C}\), then
  \[|G_{i_0+1}| \leq \frac{1}{M} \frac{|G_{i_0+1}|}{g_C} \leq \frac{1}{M} \frac{4|G_{i_0+1}|}{(|G_2/G_{i_0+1}| - 1)^2}.

**Theorem 3.4.** ([Le-Ma 1]) If \(\frac{|G|}{g_C} \geq \frac{4}{(p-1)^2}\), then there is \(\Sigma(F) \in k\{F\}\) and \(f = cX + X\Sigma(F)(X) \in k[X]\) with \(C \cong C_f\).

Moreover there are two possibilities for \(G\):

- \(\frac{|G|}{g_C} = \frac{4p}{(p-1)^2}\) and \(G = G_{\infty, 1}(f)\) or
- \(\frac{|G|}{g_C} = \frac{4}{(p-1)^2}\) and \(G \subset G_{\infty, 1}(f)\) has index \(p\).

- Note that the sequence \(\frac{p^n}{(p^n - 1)^2}\) is decreasing and that \(|G_{i_0+1}| \in p^N\).
- We deduce bounds for \(|G_2/G_{i_0+1}|, |G_{i_0+1}|\) and so for \(|G_2|\).

We still assume that \((C, G)\) is a big action.

- One can push the “classification” of big actions up to the condition \(\frac{|G|}{g_C} \geq \frac{4}{(p^n-1)^2}\).
  Namely

- One first show that \(|G_2|\) divides \(p^n\).
- \(G_2\) is abelian by corollary 7.
- Applying ([Mr 71]) to the case of abelian extensions with group \(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z}\), one shows that \(G_2\) has exponent \(p\) (we have seen in 3.5 that \(G_2\) is cyclic iff \(G_2 = \mathbb{Z}/p\mathbb{Z}\)).

**Theorem 3.5.** ([Le-Ma 4]) For all \(M > 0\), the set \(\frac{|G|}{g_C} > M\), for \((C, G)\) a big action with \(G_2\) abelian with exponent \(p\), is finite.

Sketch proof: We saw that \(|G_2|\) and so \(t\) are bounded above. We use the notations introduce in 3.2. moreover we can choose the \(f_i\) and the \(m_i := \deg f_i\) with \(m_1 \leq m_2 \leq \ldots \leq m_t\) and in such a way that \(\deg(\sum_{1 \leq i \leq t} \lambda_i f_i) \in \{m_i, 1 \leq i \leq t\}\) for \(\lambda_i \in \mathbb{P}_{n-1}(\mathbb{F}_p)\).

We distinguish two cases:

- If \(\text{Im} \rho\) is trivial.
  - Then \(m_i - 1 = p^{\nu_i}\) and \(\nu_1 \leq \ldots \leq \nu_t\)
  - \(|G| = p^t |V| \leq p^{t+2n_1}\)
  - \(g_C = \frac{(p-1)}{2}(\sum_{1 \leq i \leq t} p^{i-1}p^{\nu_i})\)
  - \(M \leq \frac{p^t |V|}{g_C} \leq \frac{4p^t}{(p-1)^2(\sum_{1 \leq i \leq t} p^{i-1}p^{\nu_i})^2}\)
  - \(\nu_i - \nu_1\) is bounded above.
1) Marked stable model
There is

- $\frac{p^{2i}}{|V|} \leq \frac{4^p}{M(p-1)^2(\sum_{i \leq 1} p^{r-1}p^{i-1})}$ and so $\{\frac{p^{2i}}{|V|}\}$ is finite.
- $\{\frac{|G|}{g_C} = \frac{4^p|V|}{(p-1)^2(\sum_{i \leq 1} p^{r-1}p^{i-1})} \}$ is finite.

- If $\text{Im} \rho$ isn’t trivial.
  - There is a smallest $i_0$ such that $f_{i_0+1}(X) \neq cX + X \Sigma(F)(X)$ (exercise).
  - For $v \in V$ $f_{i_0+1}(X + v) = f_{i_0+1}(X) + \sum_{1 \leq i \leq i_0} \ell_i(v)f_i(X) \mod \varphi(k[X])$
  - $\ell_i$ is a non zero linear form on the $\mathbb{F}_p$-space $V$.
  - Let $W := \cap_{1 \leq i \leq i_0} \ker \ell_i$, then $|W| \geq |V|/p^{n_0}$.
  - $g_C = \frac{(p-1)}{2}(\sum_{1 \leq i \leq 1} p^{i-1}(m_i - 1)) \geq \frac{(p-1)}{2}(p^{10}(m_{i_0+1} - 1))$.
  - $g_C \geq \frac{p^{10}p^{i_0}(m_{i_0+1} - 1)}{2p^{1-1}}$
  - $M \leq \frac{p^{i}|V|}{g_C} \leq \frac{4^p|V|}{(p-1)^2|V|}$
  - $|V|$ is bounded above and $g_C^2 \leq \frac{p^i|V|}{M}$ is also bounded above.
  - $\{\frac{|G|}{g_C} = \frac{|G|}{g_C} \}$ is finite. ///

4 Monodromy polynomial

- Let $C \longrightarrow \mathbb{P}^1_k$ birationally given by the equation: $Z_0^p = f(X_0) = \prod_{1 \leq i \leq m}(X_0 - x_i)^{n_i} \in R[X_0]$, $(n_i, p) = 1$ and $(\deg f, p) = 1$, $v(x_i - x_j) = v(x_i) = 0$ for $i \neq j$.

- $f'(Y)/f(Y) = S_1(Y)/S_0(Y), (S_0(Y), S_1(Y)) = 1$; then $\deg(S_1(Y)) = m - 1$ and $\deg(S_0(Y)) = m$.

- $f(X + Y) = f(Y)((1 + a_1(Y)X + \ldots + a_r(Y)X^r) + \sum_{r+1 \leq i \leq n} A_i(Y)X^i)$, where $r + 1 = [n/p], a_i(Y), A_i(Y) \in K(Y)$.

- There is a unique $\alpha$ such that $r < p^\alpha < n < p^{\alpha+1}$

- There is $T(Y) \in R[Y]$ with $A_{\alpha}(Y) = -(\frac{1}{p^{\alpha-1}})^p \frac{S_1(Y)^{p^\alpha + pT(Y)}}{S_0(Y)^{p^\alpha}}$.

- $\mathcal{L}(Y) := S_1(Y)^{p^\alpha} + pT(Y)$. This is a polynomial of degree $p^\alpha(m - 1)$ which is called the monodromy polynomial of $f(Y)$.

4.1 Marked stable model
We mean the $R$-model $\mathcal{C}_R$ defined by $Z_0^p = f(X_0) = \prod_{1 \leq i \leq m}(X_0 - x_i)^{n_i} \in R[X_0]$ (cf. fig 1).

Theorem 4.1. ([Le-Ma 3])

- The components with genus $> 0$ of the marked stable model of $C$ correspond bijectively to the Gauss valuations $v_{x_j}$, with $\rho_jX_j = X_0 - y_j$, where $y_j$ is a zero of the monodromy polynomial $\mathcal{L}(Y)$.

- $\rho_j \in R^\text{alg}$ satisfies $v(\rho_j) = \max\{\frac{1}{x_j} \frac{X^p}{X_{A_i(y_j)}} \}$ for $r + 1 \leq i \leq n$.

- The dual graph of the special fiber of the marked stable model of $C$ is an oriented tree whose ends are in bijection with the components of genus $> 0$. 


4.2 Potentially good reduction

**Theorem 4.2. ([Le-Ma 3])**

- \( p > 2, q = p^n, n \geq 1, K = \mathbb{Q}_p^{ur}(p^{n/(q+1)}) \) and \( C \to \mathbb{P}_K^1 \) is birationally defined by the equation \( Z_0^p = f(X_0) = 1 + p^{n/(q+1)}X_0^q + X_0^{q+1} \).

- Then, \( C \) has potentially good reduction and \( \mathcal{L}(Y) \) is irreducible over \( K \).

- The monodromy \( L/K \) is the extension of the decomposition field of \( \mathcal{L}(Y) \) obtained by adjoining the \( p \)-roots \( f(y)^{1/p} \), for \( y \) describing the zeroes of \( \mathcal{L}(Y) \).

- The monodromy group is the extraspecial group with exponent \( p^2 \) and order \( pq^2 \) (which is maximal for this conductor).

4.3 Genus 2

- Case \( p = 2 \) and \( m = 5 \) (i.e. curves with genus 2 over a 2-adic field \( \subset \mathbb{Q}_2^{tame} \)).

- There are 3 types of degeneration for the marked stable model.

\[ C \to \mathbb{P}_K^1 \] is birationally defined by the equation \( Z_0^p = f(X_0) \) with \( f(X_0) = 1 + b_2X_0^2 + b_3X_0^3 + b_4X_0^4 + X_0^5 \in R[X_0] \).

Now, we see that the monodromy can be maximal for the 3 types of degeneration.

- a) \( f(X_0) = 1 + 2^{3/5}X_0^2 + X_0^3 + 2^{2/5}X_0^4 + X_0^5 \) and \( K = \mathbb{Q}_2^{ur}(2^{1/5}); \)

- \( C \) has a marked stable model of type 1.
The maximal monodromy group is \( Q_8 \times Q_8 \).

b) Let \( K = \mathbb{Q}_2^ur(a) \) with \( a^9 = 2 \) and \( f(X_0) = 1 + a^3X_0^2 + a^6X_0^3 + X_0^5 \).

- \( C \) has a marked stable model of type 2.
- The maximal monodromy group is \( (Q_8 \times Q_8) \rtimes \mathbb{Z}/2\mathbb{Z} \), where \( \mathbb{Z}/2\mathbb{Z} \) exchanges the 2 factors.

c) \( K = \mathbb{Q}_2^ur \) and \( f(X_0) = 1 + X_0^4 + X_0^5 \).

- \( C \) has potentially good reduction (i.e. is of type 3)
- The maximal monodromy group is \( Q_8 * D_8 \).

References


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