Automorphisms and monodromy *

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1 Introduction

1.1 Monodromy and automorphism groups

- R is a strictly henselian DVR of inequal characteristic $(0, p)$.

  $K := \mathrm{Fr}R$; for example $K/\mathbb{Q}_p^{ur}$ finite.

  π a uniformizing parameter.

  $k := R_K/\pi R_K$.

  C/K smooth projective curve, $g(C) \geq 1$.

- C has potentially good reduction over K if there is L/K (finite) such that $C \times_K L$ has a smooth model over R_L. Then:

- There is a minimal extension L/K with this property; it is Galois and called the **monodromy** extension.

- $\mathrm{Gal}(L/K)$ is the **monodromy group**.

- Its p-Sylow subgroup is the **wild monodromy group** .

- The base change $C \times_K K^{alg}$ induces an homomorphism $\varphi : \mathrm{Gal}(K^{alg}/K) \to \mathrm{Aut}_k C_s$, where C_s is the special fiber of the smooth model over R_L and $L = (K^{alg})^{\ker \varphi}$.

- Let ℓ be a prime number, then, $n_\ell := v_\ell(|\mathrm{Gal}(L/K)|) \leq v_\ell(|\mathrm{Aut}_k C_s|)$.

- If $\ell \notin \{2, p\}$, then ℓ^{n_ℓ} is bounded by the maximal order of an ℓ-cyclic subgroup of $\mathrm{GL}_{2g}(\mathbb{Z}/\ell\mathbb{Z})$ i.e. $\ell^{n_\ell} \leq O(g)$.

- If $p > 2$, then $n_p \leq \inf_{\ell \neq 2,p} v_p(|\mathrm{GL}_{2g}(\mathbb{Z}/\ell\mathbb{Z})|) = a + [a/p] + ...$, where $a = [\frac{2g}{p-1}]$.

  This gives an exponential type bound in g for $|\mathrm{Aut}_k C_s|$. This justifies our interest in looking at Stichtenoth ([St,73]) and Singh ([Si,73]).

Theorem 1.1. *([Ra, 90]). Let $Y_K \to X_K$ be a Galois cover with group G. Let us assume that:*

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*This paper is a report on common work with Claus Lehr. This is a pdf style version of lectures given at Chuo University April 2006. A slide version using beamer is available at http://www.math.u-bordeaux.fr/˜matignon/

- *G is nilpotent.*
- *X_K has a smooth model X.*
- *The Zariski closure B of the branch locus B_K in X is étale over R_K.*

Then, the special fiber of the stable model Y_K is tree-like, i.e. the Jacobian of Y_K has potentially good reduction.

Raynaud's proof is qualitative and it seems difficult to give a constructive one in the simplest cases.

We have given in [Le-Ma1] such a proof in the case of p-cyclic covers of the projective line.

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2 Automorphism groups of curves in char.$p > 0$

2.1 p -cyclic covers of the affine line

k is an algebraically closed of char. $p > 0$.

- $f(X) \in Xk[X]$ monic,deg $f = m > 1$ prime to p.
- $C_f : W^p - W = f(X)$. Let ∞ be the point of C_f above $X = \infty$ and z a local parameter. Then, $g := g(C_f) = \frac{p-1}{2}(m-1) > 0$.
- $G_\infty(f) := \{\sigma \in \mathrm{Aut}_k C_f \mid \sigma(\infty) = \infty\}$.
- $G_{\infty,1}(f) := \{\sigma \in \mathrm{Aut}_k C_f \mid v_\infty(\sigma(z) - z) \geq 2\}$, the p-Sylow.
- ([St,73]) Let $g(C_f) \geq 2$, then $G_{\infty,1}(f)$ is a p-Sylow of $\mathrm{Aut}_k C_f$.
- It is normal except for $f(X) = X^m$ where $m|1+p$.

2.2 Structure of $G_{\infty,1}(f)$

- Let $\rho(X) = X$, $\rho(W) = W + 1$, then $< \rho >= G_{\infty,2} \subset Z(G_{\infty,1})$
- $0 \to < \rho > \to G_{\infty,1} \to V \to 0, \quad V := \{\tau_y | \ \tau_y(X) = X + y, \ y \in k\}$.
  $f(X+y) = f(X) + f(y) + (F - \mathrm{Id})(P(X,y))$, $P(X,y) \in Xk[X]$.
  $V \simeq (\mathbb{Z}/p\mathbb{Z})^v$ as a subgroup of k.
- Let $\tau_y(W) := W + a_y + P(X,y)$, $a_y \in \mathbb{F}_p$, then $[\tau_y, \tau_z] = \rho^{\epsilon(y,z)}$, where $\epsilon : V \times V \to \mathbb{F}_p$ is an alternating form.
- ϵ is non degenerated iff $< \rho >= Z(G_{\infty,1})$.

2.3 Bounds for $|G_{\infty,1}(f)|$

Lemma 2.1. *If $f(X)=\sum_{1\leq i\leq m} t_i X^i \in k[X]$ is monic, then:*

- $\Delta(f)(X,Y) := f(X+Y)-f(X)-f(Y) = R(X,Y)+(F-\mathrm{Id})(P_f(X,Y))$, *where* $R \in \bigoplus_{\lfloor \frac{m}{p} \rfloor \leq ip^{n(i)}<m,\ (i,p)=1} k[Y]X^{ip^{n(i)}}$ *and* $P_f \in Xk[X,Y]$.
- $P_f = (\mathrm{Id}+F+...+F^{n-1})(\Delta(f)) \mod X^{[\frac{m-1}{p}]+1}$.

Let us denote by $\mathrm{Ad}_f(Y)$ *the content of* $R(X,Y)\in k[Y][X]$*, then*

- $\mathrm{Ad}_f(Y)$ *is an additive and separable polynomial.*
- $Z(\mathrm{Ad}_f(Y)) \simeq V$.

Let $m-1=\ell p^s$ with $(\ell,p)=1$.

- ([St 73]) $|G_{\infty,1}| = p\deg \mathrm{Ad}_f \leq p(m-1)^2$, i.e. $\frac{|G_{\infty,1}|}{g^2} \leq \frac{4p}{(p-1)^2}$.
- ([St 73]) $s=0$ i.e. $(m-1,p)=1$, then $|G_{\infty,1}|=p$.
- If $s>0$,
  - $\ell>1$, $p=2$, then $\frac{|G_{\infty,1}|}{g} \leq \frac{2}{3}$.
  - $\ell>1$, $p>2$, then $\frac{|G_{\infty,1}|}{g} \leq \frac{p}{p-1}$.
  - ([St 73]) $\ell>1$, $m=1+p^s$, then $\frac{|G_{\infty,1}|}{g} \leq 2p^s\frac{p}{p-1}$ (with equality for $f(X)=X^{1+p^s}$).

2.4 Characterization of $G_{\infty,1}(f)$

- We consider the extensions $0\to N\simeq Z/p\mathbb{Z}\to G\to(\mathbb{Z}/p\mathbb{Z})^n\to 0$ (note that $G_{\infty,1}(f)$ is an extension of this type). Then $G'\subset N\subset Z(G)$.
- If $G'=Z(G)$, G is called extraspecial.
  - Then, $|G|=p^{2s+1}$ and there are 2 isomorphism classes for a given s.
  - If $p>2$, we denote by $E(p^3)$ (resp. $M(p^3)$) the non abelian group of order p^3 and exponent p (resp. p^2). Then, $G\simeq E(p^3)*E(p^3)*...*E(p^3)$ or $M(p^3)*E(p^3)*...*E(p^3)$, according as the exponent is p or p^2.
  - If $p=2$, then $G\simeq D_8*D_8*...*D_8$ or $Q_8*D_8*...*D_8$ (in both cases, the exponent is 2^2).
- If $G'\subset Z(G)$, G is a subgroup of an extraspecial group E with $Z(E)\subset G$.

Theorem 2.2. *([Le-Ma 1]). Let* $f(X)=X\Sigma(F)(X)\in Xk[X]$, $\Sigma(F)=\sum_{0\leq i\leq s} a_iF^i \in k\{F\}$ *an additive polynomial with* $\deg f = 1+p^s$*. Then,*

- $\mathrm{Ad}_f(Y)=F^s(\sum_{0\leq i\leq s}(a_iF^i+F^{-i}a_i)(Y))$*, a palyndromic polynomial.*
- $G_{\infty,1}(f)$ *is an extraspecial group with cardinal* p^{2s+1} *and exponent* p *for* $p>2$*, and of type* $Q_8*D_8*...*D_8$ *for* $p=2$.

Theorem 2.3. *([Le-Ma 1]). If G is an extension of type* $0 \to \mathbb{Z}/p\mathbb{Z} \to G \to (\mathbb{Z}/p\mathbb{Z})^n \to 0$*, there is* $f \in Xk[X]$ *with* $G \simeq G_{\infty,1}(f)$.

- Sketch proof: Extraspecial groups with exponent p^2 are realized by a modification by a Witt cocycle of the polynomial f in the previous theorem.
- We can see G as a subgroup of an extraspecial group E, then we realize E with f_E and a suitable modification of f_E will limit $G_{\infty,1}(f_E)$ to G.

3 Actions of p-groups over a curve C with $g(C) \geq 2$

3.1 Big actions (I)

Theorem 3.1. *([Le-Ma 1]). Let* $f(X) \in Xk[X]$ *with* $(\deg f, p) = 1$*. If* $\frac{|G_{\infty,1}|}{g} > \frac{p}{p-1}$ *(*$\frac{2}{3}$ *for* $p = 2$*), then* $f(X) = cX + X\Sigma(F)(X) \in k[X]$.

- Sketch proof: One shows that monomials in f with a degree $\notin 1 + p^{\mathbb{N}}$ will limit the degree of Ad_f.
- Let (C, G) with $G \subset \mathrm{Aut}_k C$, a p-group. We say that (C, G) is a **big action** if:
  (N) $g_C > 0$ and $\frac{|G|}{g_C} > \frac{2p}{p-1}$.
  It follows from ([Na 87]) that there is $\infty \in C$, with
  - $C \to C/G \simeq \mathbb{P}^1_k - \infty$ is étale and $G = G_{\infty,1}$.
  - $G_{\infty,2} \neq G_{\infty,1}$ and $C/G_{\infty,2} \simeq \mathbb{P}^1_k$
  - Then, $G_{\infty,1}/G_{\infty,2}$ acts as a group of translations of the affine line $C/G_{\infty,2} - \{\infty\}$.
- **Transfert of condition (N) to quotients.** Let (C, G) a big action, if $H \lhd G$ and if $g(C/H) > 0$, then $(C/H, G/H)$ is a big action.

3.2 Condition (N) and G_2

In this section (C, G) **is a big action.** Let G_i be the lower ramification groups.

- Let $H \lhd G$ and H with index p in G_2 (H exists!), then $(C/H, G/H)$ satisfies (N).
- $(G/H)_2 = G_2/H \simeq \mathbb{Z}/p\mathbb{Z}$.
- There is $S(F) \in k\{F\}$, $f_1 = cX + X\Sigma(F)(X) \in k[X]$ with $C/H \simeq C_{f_1}$.
- If $G_2 \simeq (\mathbb{Z}/p\mathbb{Z})^t$, then $k(C) = k(X, W_1, ..., W_t)$ and $\wp(W_1, ..., W_t) = (f_1(X), f_2(X), ..., f_t(X)) \in (k[X])^t$
- $f_1(X), .., f_t(X)$ are $\mathbb{F}_p$-free $\mod \wp(k[X])$.
- The group extension $0 \to G_2 \to G_1 \to V = (\mathbb{Z}/p\mathbb{Z})^v \to 0$ induces a representation $\rho : V \to \mathrm{Gl}_t(\mathbb{F}_p)$
- dual to the one given by V acting via translation: $(v \in V) \times (f_1(X), f_2(X), ..., f_t(X)) \mod \wp(k[X])^t \to \to (f_1(X+v), f_2(X+v), ..., f_t(X+v)) \mod \wp(k[X])^t$

- Imρ is a unipotent subgroup of $\mathrm{Gl}_t(\mathbb{F}_p)$ which is the identity iff $G_2 \subset Z(G)$. In this case $f_i(X) = c_iX + X\Sigma_i(F)(X)$ where $\Sigma_i(F) \in k\{F\}$ and $v \in V$ is a commun zero to the palyndromic polynomials $\mathrm{Ad}_{f_i} \in k\{F, F^{-1}\}$.
- Let $f_1 := X(\alpha F)(X) = \alpha X^{1+p}$ with $\alpha^p + \alpha = 0$; then $\mathrm{Ad}_{f_1} = Y^{p^2} - Y$.
- Let $f_2 := X^{1+2p} - X^{2+p}$, then
- $f_2(X+Y) - f_2(X) - f_2(Y) = 2(Y^p - Y)X^{1+p} + (Y - Y^{p^2})X^{2p} + (Y^{2p^2} - Y^2 + 2Y^{1+p} - 2Y^{p+p^2})X^p \mod \wp(k[X,Y])$
- If $y \in Z(\mathrm{Ad}_{f_1}) = \mathbb{F}_{p^2}$ one has

  $f_2(X+y) = \frac{2(y^p-y)}{\alpha} f_1(X) + f_2(X) + \wp(P_2)$.
- $y \longrightarrow \frac{2(y^p-y)}{\alpha}$ is a non zero linear form over $\mathbb{F}_{p^2}$ with value in $\mathbb{F}_p$.
- $|G| = p^2p^2$ and $g = \frac{p-1}{2}(p + p(2p))$.
- $\frac{|G|}{g} = \frac{2p}{p-1}\frac{p^2}{1+2p}$.
- $\frac{|G|}{g^2} = \frac{4p}{(p-1)^2}\frac{p}{(1+2p)^2}$.

Theorem 3.2. *([Le-Ma 4]) Let (C, G) be a big action then $G_2 = G'$.*

- Sketch proof: If $G' \neq G_2$, there is $H \lhd G$ with $G' \subset H \subset G_2$ and $[G_2 : H] = p$. $(C/H, G/H)$ satisfies condition (N);
- $C/H : W^p - W = f := X\Sigma(F)(X)$, $\deg(f) = 1 + p^s$.
- $(\mathrm{Aut}C/H)_{\infty,1} := E$, is extraspecial with order p^{2s+1}.
- G/H is abelian and normal in E.
- ([Hu 67] Satz 13.7 p. 353) $|G/H| \leq p^{s+1}$ and so $|G/H|/g(C/H) \leq \frac{2p^{s+1}}{(p-1)p^s} = \frac{2p}{p-1}$, a contradiction.

  We deduce the following corollary from ([Su 86] 4.21 p.75).

Corollary 3.3. *If $|G_2| = p^3$, then G_2 is abelian.*

3.3 Riemann surfaces

- In characteristic 0, an analogue of big actions is given by the actions of a finite group G on a compact Riemann surface C with $g_C \geq 2$ such that $|G| = 84(g_C - 1)$ (we say that C is an **Hurwitz curve**) ([Co 90]).
- Let us mention Klein's quartic ($G \simeq PSL_2(\mathbb{F}_7)$) ([El 99]).
- The Fricke-Macbeath curves with genus 7 ($G \simeq PSL_2(\mathbb{F}_8)$) ([Mc],65).
- Let C be an Hurwitz curve with genus g_C. Let $n > 1$ and C_n the maximal unramified Galois cover whose group is abelian with exponent n. The Galois group of C_n/C is $(\mathbb{Z}/n\mathbb{Z})^{2g_C}$. It follows from the unicity of C_n that the k-automorphisms of C have n^{2g} prolongations to C_n. Therefore $g_{C_n} - 1 = n^{2g}(g_C - 1)$ and $n^{2g}|\mathrm{Aut}_kC| \leq |\mathrm{Aut}_kC_n|$, where $|\mathrm{Aut}_kC_n| \geq 84(g_{C_n} - 1)$; C_n is an Hurwitz curve ([Mc],61).

3.4 Ray class fields

- If (C, G) is a big action in char.$p > 0)$, then $C \to C/G$ is an tale cover of the affine line whose group is a p-group; it follows that the Hasse-Witt invariant of C is zero; therefore, in order to adapt the previous proof to char. $p > 0$, one needs to accept ramification. This is done with the so called ray class fields of function fields over finite fields.

- Let $K := \mathbb{F}_q(X)$ where $q = p^e$, S the set of finite rational places $(X - v)$, $v \in \mathbb{F}_q$ and $m \in \mathbb{N}$. Let K^{alg} be an algebraic closure. Let $K_S^m \subset K^{alg}$ be the biggest abelian extension L of K with conductor $\leq m\infty$ and such that the places in S are completely decomposed.

- ([La 99], [Au 00]) The constant field of K_S^m is $\mathbb{F}_q$ and $G_S(m) := \mathrm{Gal}(K_S^m/K) \simeq (1 + T\mathbb{F}_q[[T]])/ < 1 + T^m\mathbb{F}_q[[T]], 1 - vT, v \in \mathbb{F}_q >$, is a p-group.

- ([Ma-Le 4]) Let $C_m/\mathbb{F}_q$ be the smooth projective curve with function field K_S^m. The translations $X \to X + v$, $v \in \mathbb{F}_q$ stabilize S and ∞; they can be extended to $\mathbb{F}_q$-automorphisms of K_S^m. In this way, we get an action of a p-group $G(m)$ on C_m with $0 \to G_S(m) \to G(m) \to \mathbb{F}_q \to 0$

- ([Au 00] If $n_m := |G_S(m)|$, then $g_{C_m} = 1 + n_m(-1 + m/2) - (1/2)\sum_{0\leq j\leq m-1} n_j \leq n_m(-1 + m/2)$

- $\frac{|G(m)|}{g_{C_m}} \geq \frac{n_m q}{n_m(-1+m/2)} = \frac{q}{-1+m/2}$. This is a "big action" as soon as $\frac{q}{-1+m/2} > \frac{2p}{p-1}$ (we have $G_2 = G_S(m)$)

- Let $N_q := |C_m(\mathbb{F}_q)|$. Then, $N_q = 1 + |G(m)|$, and the quotient $\frac{|G(m)|}{g_{C_m}} \sim \frac{N_q}{g_{C_m}}$.

- ([La 99]) If $q = p^e$,$m_2 := p^{\lceil e/2 \rceil + 1} + p + 1$ is the smallest conductor m such that the exponent of G_S^m is $> p$.

- If $e > 2$, $(C_{m_2}, G(m_2))$ is a big action and G_2 is abelian with exponent p^2.

3.5 Big actions (II)

From now on, k is any algebraically closed field and (C, G) is a big action.

- If $G_2 \simeq \mathbb{Z}/p^n\mathbb{Z}$, then $n = 1$ ([Le-Ma 4]).
  - Sketch proof: Let $H = G_2^{p^{n-2}}$ then $(C/H, G/H)$ is a big action, it follows that one can assume that $n = 2$. Then $C \to C/G_2$ is given by $\wp(W_0, W_1) = (f_0, f_1)$ with $f_0 = X\Sigma(F)(X)$, $\deg f_0 = 1 + p^s$.
  - Let $v \in V := Z(\mathrm{Ad}_{f_0})$ and $P \in k[X]$ with $f_0(X + v) = f_0(X) + \wp(P)$ then $f_1(X + v) - f_1(X) = \ell(v)f_0(X) + \frac{1}{p}(f_0(X)^p + P(X)^p - P(X)^{p^2} - (f_0(X) + P(X))^p - f_0(X + v)^p + (f_0(X + v) + P(X)^p)^p)$
  - $= \ell(v)f_0(X) + \sum_{1\leq i\leq p-1} \frac{(-1)^{i-1}}{i} v^i X^{p-i+p^{s+1}} \mod X^{p^{s+1}}$ where $\ell : V \to \mathbb{F}_p$ is a linear form.

- More generally for G_2 abelian with exponent p^e, $e \geq 2$, one can expect a lower bound in $O(log(g_C))$ for the p-rank of G_2. This is the case in the preceding situation i.e. $(C, G) = (C_{m_2}, G(m_2))$ ([M. Rocher, thesis in preparation]).

3.6 Maximal curves

Let us assume that (C,G) is a big action.

- Let i_0 with $G_2 = G_3 = ... = G_{i_0} \supsetneqq G_{i_0+1}$. Then $g_{(C/G_{i_0+1})} = \frac{1}{2}(|G_2/G_{i_0+1}| - 1)(i_0 - 1)$.
- If $0 < M \leq \frac{|G|}{g_C^2}$, then
  $|G_{i_0+1}| \leq \frac{1}{M}\frac{|G/G_{i_0+1}|}{g^2_{C/G_{i_0+1}}} \leq \frac{1}{M}\frac{4|G_2/G_{i_0+1}|}{(|G_2/G_{i_0+1}|-1)^2}$.

Theorem 3.4. *([Le-Ma 1]) If $\frac{|G|}{g_C^2} \geq \frac{4}{(p-1)^2}$, then there is $\Sigma(F) \in k\{F\}$ and $f = cX + X\Sigma(F)(X) \in k[X]$ with $C \simeq C_f$.*

Moreover there are two possibilities for G:

- $\frac{|G|}{g_C^2} = \frac{4p}{(p-1)^2}$ *and* $G = G_{\infty,1}(f)$ *or*
- $\frac{|G|}{g_C^2} = \frac{4}{(p-1)^2}$ *and* $G \subset G_{\infty,1}(f)$ *has index* p.

- Note that the sequence $\frac{p^n}{(p^n-1)^2}$ is decreasing and that $|G_{i_0+1}| \in p^{\mathbb{N}}$.
- We deduce bounds for $|G_2/G_{i_0+1}|$, $|G_{i_0+1}|$ and so for $|G_2|$.

We still assume that (C,G) is a big action.

- One can push the "classification " of big actions up to the condition $\frac{|G|}{g_C^2} \geq \frac{4}{(p^2-1)^2}$. Namely
- One first show that $|G_2|$ divides p^3.
- G_2 is abelian by corollary 7.
- Applying ([Mr 71]) to the case of abelian extensions with group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z}$, one shows that G_2 has exponent p (we have seen in 3.5 that G_2 is cyclic iff $G_2 = \mathbb{Z}/p\mathbb{Z}$).

Theorem 3.5. *([Le-Ma 4]) For all $M > 0$, the set $\frac{|G|}{g_C^2} > M$, for (C,G) a big action with G_2 abelian with exponent p, is finite.*

Sketch proof: We saw that $|G_2|$ and so t are bounded above. We use the notations introduce in 3.2. moreover we can choose the f_i and the $m_i := \deg f_i$ with $m_1 \leq m_2 \leq ... \leq m_t$ and in such a way that $\deg(\sum_{1\leq i\leq t} \lambda_i f_i) \in \{m_i,\ 1 \leq i \leq t\}$ for $[\lambda_i] \in \mathbb{P}^{t-1}(\mathbb{F}_p)$.

We distinguish two cases:

- If $\mathrm{Im}\rho$ is trivial.
  - Then $m_i - 1 = p^{\nu_i}$ and $\nu_1 \leq ... \leq \nu_t$
  - $|G| = p^t|V| \leq p^{t+2\nu_1}$.
  - $g_C = \frac{(p-1)}{2}(\sum_{1\leq i\leq t} p^{i-1}p^{\nu_i})$
  - $M \leq \frac{p^t|V|}{g^2} \leq \frac{4p^t}{(p-1)^2(\sum_{1\leq i\leq t} p^{i-1}p^{\nu_i-\nu_1})^2}$
  - $\nu_i - \nu_1$ is bounded above.

- $\frac{p^{2\nu_1}}{|V|} \leq \frac{4p^t}{M(p-1)^2(\sum_{1\leq i\leq t} p^{i-1}p^{\nu_i-\nu_1})^2}$ and so $\{\frac{p^{2\nu_1}}{|V|}\}$ is finite.
- $\{\frac{|G|}{g_C^2} = \frac{4p^t|V|p^{-2\nu_1}}{(p-1)^2(\sum_{1\leq i\leq t} p^{i-1}p^{\nu_i-\nu_1})^2}\}$ is finite.

- If Imρ isn't trivial.
  - There is a smallest i_0 such that $f_{i_0+1}(X) \neq cX + X\Sigma(F)(X)$ (exercise).
  - For $v \in V$ $f_{i_0+1}(X+v) = f_{i_0+1}(X) + \sum_{1\leq i\leq i_0} \ell_i(v)f_i(X) \mod \wp(k[X])$
  - ℓ_i is a non zero linear form on the $\mathbb{F}_p$-space V.
  - Let $W := \cap_{1\leq i\leq i_0} \ker \ell_i$, then $|W| \geq \frac{|V|}{p^{i_0}}$.
  - $g_C = \frac{(p-1)}{2}(\sum_{1\leq i\leq t} p^{i-1}(m_i-1)) \geq \frac{(p-1)}{2}(p^{i_0}(m_{i_0+1}-1))$.
  - $\frac{2p|W|}{(p-1)(m_{i_0+1}-1)} \leq \frac{2p}{p-1}$
  - $g_C \geq \frac{p-1}{2}p^{i_0}(m_{i_0+1}-1) \geq \frac{p-1}{2}|V|$
  - $M \leq \frac{p^t|V|}{g^2} \leq \frac{4p^t|V|}{(p-1)^2|V|^2}$
  - $|V|$ is bounded above and $g_C^2 \leq \frac{p^t|V|}{M}$ is also bounded above .
  - $\{\frac{|G|}{g_C^2} = \frac{|G_2||V|}{g_C^2}\}$ is finite. ///

4 Monodromy polynomial

- Let $C \longrightarrow \mathbb{P}^1_K$ birationally given by the equation: $Z_0^p = f(X_0) = \prod_{1\leq i\leq m}(X_0 - x_i)^{n_i} \in R[X_0]$, $(n_i, p) = 1$ and $(\deg f, p) = 1$, $v(x_i - x_j) = v(x_i) = 0$ for $i \neq j$.

- $f'(Y)/f(Y) = S_1(Y)/S_0(Y)$, $(S_0(Y), S_1(Y)) = 1$; then $\deg(S_1(Y)) = m-1$ and $\deg(S_0(Y)) = m$.

- $f(X+Y) = f(Y)((1 + a_1(Y)X + ... + a_r(Y)X^r)^p - \sum_{r+1\leq i\leq n} A_i(Y)X^i)$, where $r+1 = [n/p]$, $a_i(Y), A_i(Y) \in K(Y)$.

- There is a unique α such that $r < p^\alpha < n < p^{\alpha+1}$

- There is $T(Y) \in R[Y]$ with $A_{p^\alpha}(Y) = -\binom{\frac{1}{p}}{p^{\alpha-1}}^p \frac{S_1(Y)^{p^\alpha} + pT(Y)}{S_0(Y)^{p^\alpha}}$.

- $\mathcal{L}(Y) := S_1(Y)^{p^\alpha} + pT(Y)$. This is a polynomial of degree $p^\alpha(m-1)$ which is called the **monodromy polynomial** of $f(Y)$.

4.1 Marked stable model

We mean the R-model $\mathcal{C}_R$ defined by $Z_0^p = f(X_0) = \prod_{1\leq i\leq m}(X_0 - x_i)^{n_i} \in R[X_0]$ (cf. fig 1).

Theorem 4.1. *([Le-Ma 3])*

- *The components with genus > 0 of the marked stable model of C correspond bijectively to the Gauss valuations v_{X_j} with $\rho_j X_j = X_0 - y_j$, where y_j is a zero of the monodromy polynomial $\mathcal{L}(Y)$*

- $\rho_j \in R^{\mathrm{alg}}$ *satisfies* $v(\rho_j) = \max\{\frac{1}{i}v\left(\frac{\lambda^p}{A_i(y_j)}\right)$ *for* $r+1 \leq i \leq n\}$.

- *The dual graph of the special fiber of the marked stable model of C is an oriented tree whose ends are in bijection with the components of genus > 0.*

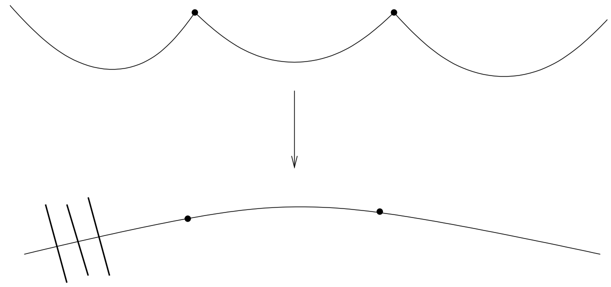


Figure 1: $\mathcal{C}_R \otimes_R k \longrightarrow \mathbb{P}^1_k$ with singularities and branch locus

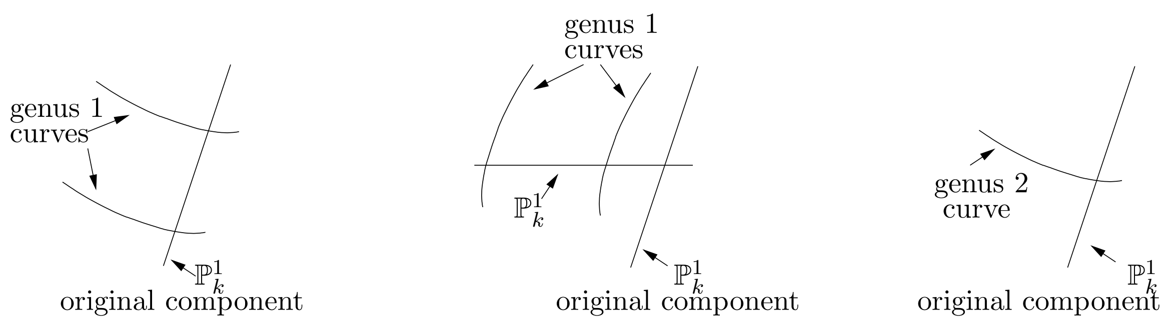
4.2 Potentially good reduction

Theorem 4.2. *([Le-Ma 3])*

- $p > 2$, $q = p^n$, $n \geq 1$, $K = \mathbb{Q}_p^{\mathrm{ur}}(p^{p/(q+1)})$ *and* $C \longrightarrow \mathbb{P}^1_K$ *is birationally defined by the equation* $Z_0^p = f(X_0) = 1 + p^{p/(q+1)} X_0^q + X_0^{q+1}$.
- *Then,* C *has potentially good reduction and* $\mathcal{L}(Y)$ *is irreducible over* K*.*
- *The monodromy* L/K *is the extension of the decomposition field of* $\mathcal{L}(Y)$ *obtained by adjoining the* p*-roots* $f(y)^{1/p}$*, for* y *describing the zeroes of* $\mathcal{L}(Y)$*.*
- *The monodromy group is the extraspecial group with exponent* p^2 *and order* pq^2 *(which is maximal for this conductor).*

4.3 Genus 2

- Case $p = 2$ and $m = 5$ (i.e. curves with genus 2 over a 2-adic field $\subset \mathbb{Q}_2^{\mathrm{tame}}$).
- There are 3 types of degeneration for the marked stable model.



- Type 1: $\mathrm{Gal}(K'/K)_w \hookrightarrow Q_8 \times Q_8$ — Type 2: $\mathrm{Gal}(K'/K)_w \hookrightarrow (Q_8 \times Q_8) \rtimes \mathbb{Z}/2\mathbb{Z}$ — Type 3: $\mathrm{Gal}(K'/K)_w \hookrightarrow Q_8 * D_8$

- $C \longrightarrow \mathbb{P}^1_K$ is birationally defined by the equation $Z_0^p = f(X_0)$ with $f(X_0) = 1 + b_2 X_0^2 + b_3 X_0^3 + b_4 X_0^4 + X_0^5 \in R[X_0]$.

Now, we see that the monodromy can be maximal for the 3 types of degeneration.
a) $f(X_0) = 1 + 2^{3/5} X_0^2 + X_0^3 + 2^{2/5} X_0^4 + X_0^5$ and $K = \mathbb{Q}_2^{\mathrm{ur}}(2^{1/15})$;

- C has a marked stable model of type 1.

- The maximal monodromy group is $\simeq Q_8 \times Q_8$.

b) Let $K = \mathbb{Q}_2^{\mathrm{ur}}(a)$ with $a^9 = 2$ and $f(X_0) = 1 + a^3 X_0^2 + a^6 X_0^3 + X_0^5$.

- C has a marked stable model of type 2.
- The maximal monodromy group is $\simeq (Q_8 \times Q_8) \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ exchanges the 2 factors.

c) $K = \mathbb{Q}_2^{\mathrm{ur}}$ and $f(X_0) = 1 + X_0^4 + X_0^5$.

- C has potentially good reduction (i.e. is of type 3)
- The maximal monodromy group is $\simeq Q_8 * D_8$.

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