Using a Boussinesq-type 1-D model to simulate Favre waves

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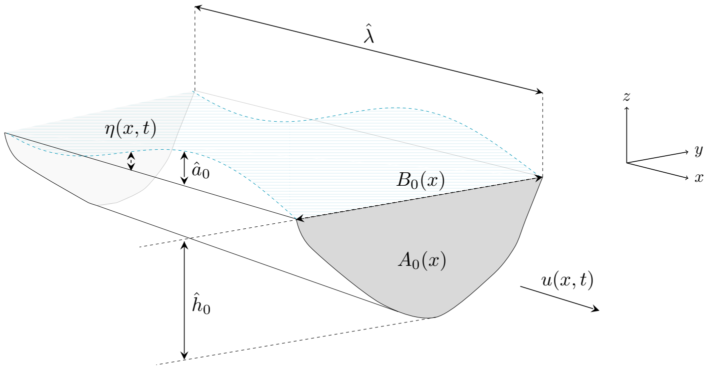
Context

- Fast opening/closing of sluice gates of hydro power plant
- Mobile surge propagating upstream
- Hydraulic jump ($F_r > 1.3$) or Favre waves ($F_r < 1.3$)
- Favre waves are both **nonlinear** and **dispersive**
- 1D modelling for industrial applications

Triggering tests at Sisteron plant (France, 2017, EDF/CIH videos)

Notation



$\epsilon \doteqdot \frac{\hat{a}_0}{\hat{h}_0}$: **nonlinearity** $\mu \doteqdot \frac{\hat{h}_0}{\hat{\lambda}}$: **dispersion**

Winckler–Liu's model in conservative variables

- Winckler–Liu's model (2015) expressed in terms of **wetted area** A and **flow rate** Au

$$\underbrace{\begin{aligned} & A_t + \epsilon\,(Au)_x = 0 \\ & (Au)_t + \left(\epsilon A u^2 + gK(h)\right)_x \end{aligned}}_{\text{Saint-Venant equations}} =$$

with

$$K(h) \doteqdot \int A(h)\,dh$$

Winckler–Liu's model in conservative variables

- Winckler–Liu's model (2015) expressed in terms of **wetted area** A and **flow rate** Au

$$\begin{cases} \left(1+\mu^2\mathcal{T}\right)\Phi = -H_x \\ A_t + \epsilon\,(Au)_x = 0 \\ \underbrace{(Au)_t + \left(\epsilon A u^2 + gK(h)\right)_x}_{\text{Saint-Venant equations}} = \underbrace{A\Phi + AH_{s,x}}_{\text{non-hydrostatic (dispersive) terms}} \end{cases}$$

with

$$K(h) \doteqdot \int A(h)\,dh, \quad H \doteqdot \epsilon\frac{u^2}{2} + g\eta, \quad H_s \doteqdot \epsilon\frac{u^2}{2} + gh$$

$$\mathcal{T} \doteqdot \alpha\mathcal{I} + (\beta - \gamma_x)\,\partial_x + \partial_x\left(\gamma\partial_x\right)$$

$\alpha(x)$, $\beta(x)$, $\gamma(x)$ are channel geometry dependent coefficients

- **Decoupling** the **hyperbolic** problem (shallow water equations) and the **elliptic** problem (non-hydrostatic terms)

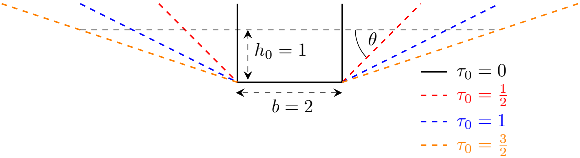
Travelling wave solution

- **Prismatic** channel with **trapezoidal** cross-section ($\alpha = \beta = 0$, $\gamma = cst$ and is computed numerically)

$$\begin{cases} \left(1+\mu^2\mathcal{T}\right)\Phi = -H_x \\ A_t + \epsilon\left(Au\right)_x = 0 \\ \left(Au\right)_t + \left(\epsilon A u^2 + gK(h)\right)_x = A\Phi + AH_{s,x} \end{cases}$$

$$\mathcal{T} = \gamma\partial_{xx}, \qquad \tau(h) \doteq \frac{h}{b}\cot\theta$$

$$A(h) = bh\left(1+\tau\right), \qquad K(h) = bh^2\left(\tfrac{1}{2}+\tfrac{1}{3}\tau\right)$$



$$\tau_0 \doteq \tau\left(h_0\right)$$

Travelling wave solution

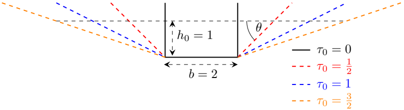
- Semi-analytical solution

$$\frac{x-ct}{h_0} = \mu\sqrt{-3\left(1+\tau_0\right)\frac{\gamma}{h_0^2}}\int_{\frac{h}{h_0}}^{1+\epsilon}\frac{1+2\tau_0\mathrm{H}}{\sqrt{\varphi(\mathrm{H})-\frac{\varphi(1+\epsilon)}{\psi(1+\epsilon)}\psi(\mathrm{H})}}\mathrm{dH}$$

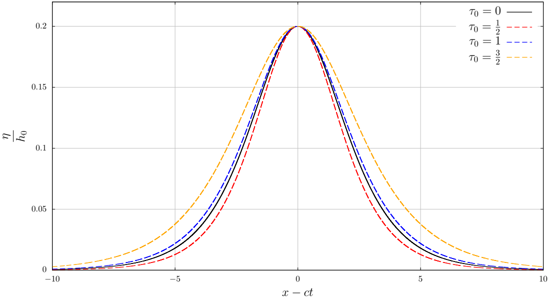
$$\varphi(x) \doteqdot \left[\frac{2}{1+\tau_0}x\left(1+\tau_0 x\right)+1\right]\left[x\left(1+\tau_0 x\right)-1-\tau_0\right]^2 x\left(1+\tau_0 x\right)$$

$$\psi(x) \doteqdot x^4\left(1+\tau_0 x\right)^3\left[\left(1+\tau_0 x\right)\ln\frac{x(1+\tau_0)}{1+\tau_0 x}+\frac{1}{x}-1\right]$$

- Approximation with a quadrature method



$\epsilon = 0.2$



Numerical solving + Verification

$$\begin{cases} \boxed{\left(1+\mu^2\mathcal{T}\right)\Phi = -H_x} \\ A_t + \epsilon\left(Au\right)_x = 0 \\ \left(Au\right)_t + \left(\epsilon A u^2 + gK(h)\right)_x = A\Phi + AH_{s,x} \end{cases}$$

Finite ($\mathbb{P}_1$) Element Method

Numerical solving + Verification

$$\begin{cases} \left(1+\mu^2\mathcal{T}\right)\Phi = -H_x \\ \boxed{\begin{array}{l} A_t + \epsilon\left(Au\right)_x = 0 \\ \left(Au\right)_t + \left(\epsilon A u^2 + gK(h)\right)_x = A\Phi + AH_{s,x} \end{array}} \end{cases}$$

Finite Volume Method
with 2nd-order MUSCL
reconstruction (Roe scheme)

Numerical solving + Verification

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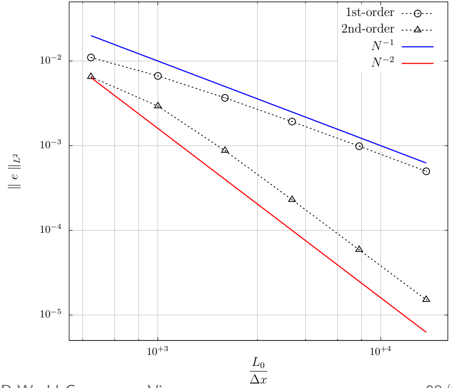
Finite ($\mathbb{P}_1$) Element Method

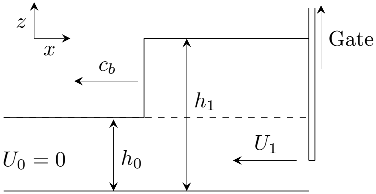
Finite Volume Method
with 2nd-order MUSCL reconstruction (Roe scheme)

- Numerical verification with the soliton case
- $\epsilon = 0.2$
- Bief: $x \in [-12\lambda, 12\lambda]$
- Comparison at $t = \dfrac{6.5\lambda}{c}$



Favre waves applications

- Favre waves experiments of Favre (1935) and Treske (1994)
- Test case: **fast sluice gate opening** on a flow at rest



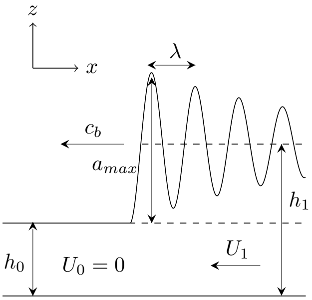
- Initial conditions are smoothed from Rankine–Hugoniot formulae:

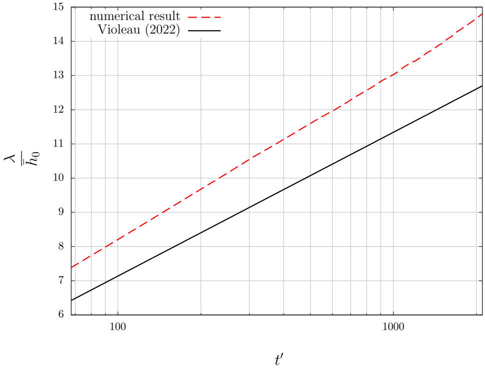
$$h(x, t=0) = \frac{h_0 - h_1}{2} \tanh\left(\frac{x}{5}\right) + h_1 - \tfrac{1}{2}\epsilon h_0$$
$$u(x, t=0) = \frac{U_0 - U_1}{2} \tanh\left(\frac{x}{5}\right) + \tfrac{1}{2}\, U_0$$

- We first look at the evolution of the wavelength λ and the maximum amplitude a_{max} for long propagation times

Time evolution

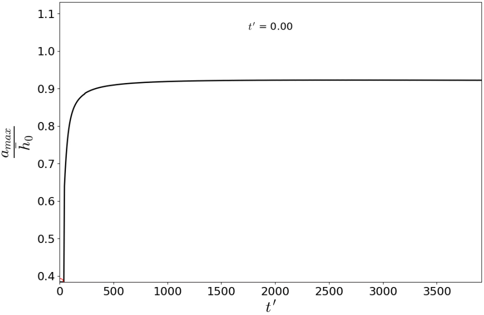
- **λ increases logarithmically with time** as demonstrated in Violeau's IAHR Water Monograph on KdV (2022)
- $\epsilon = \dfrac{h_1 - h_0}{h_0} = 0.2$ and $t' \doteqdot \sqrt{\dfrac{g}{h_0}}\, t$

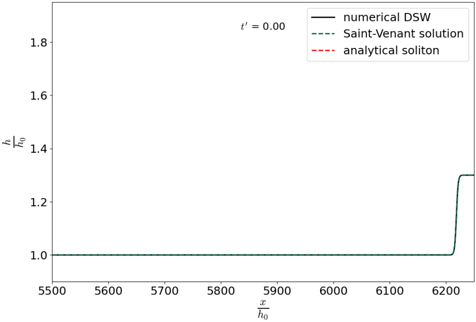




Asymptotic behaviour

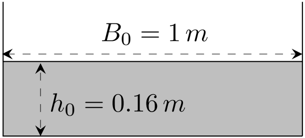
- For long propagation times the maximum amplitude a_{max} reaches an asymptote
- The first wave separates from the following, and propagates at the velocity c of the soliton
- $\epsilon = 0.3$



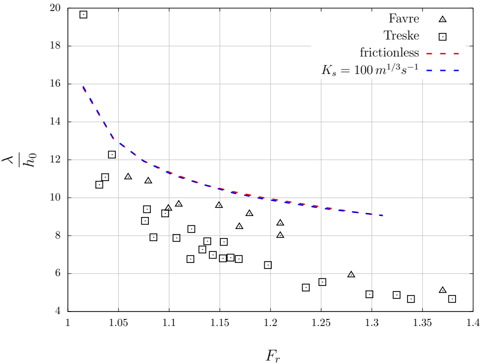


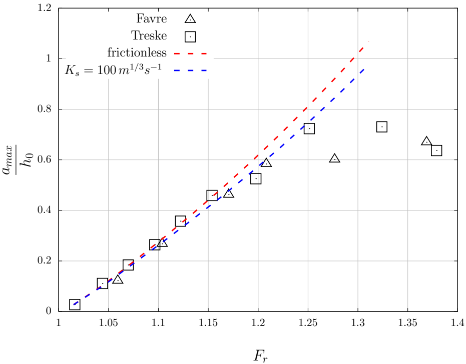
Rectangular case

- Comparison with Favre (1935) and Treske (1994) data
- Measurement **timing unspecified**
- Friction is accounted for through a Strickler coefficient $K_s = 100\, m^{1/3} s^{-1}$



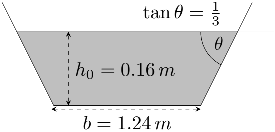
$$F_r \doteqdot \frac{|U_0 - c_b|}{\sqrt{g\bar{h}_0}}, \quad \bar{h}_0 \doteqdot \frac{A_0}{B_0}$$

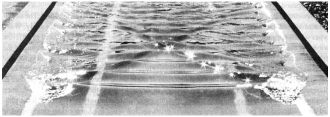




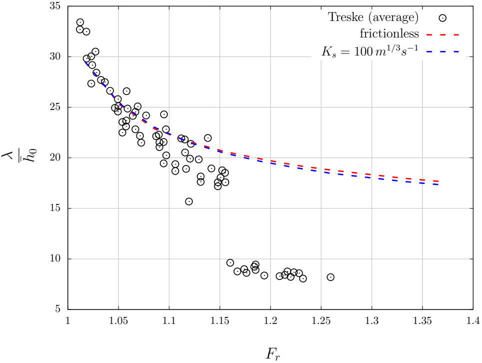
Trapezoidal case

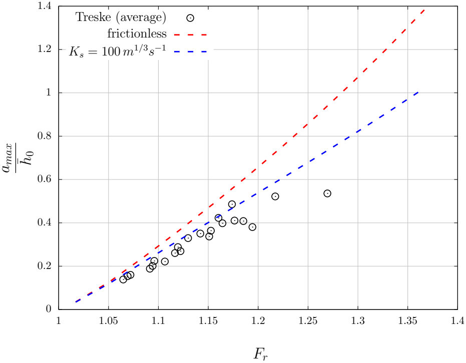
- For $F_r > 1.15$, the flow becomes strongly 3D (transition)
- Refraction on the banks
- Average of banks and channel axis experimental data





$1.15 < F_r < 1.3$, from Treske (1994)





Conclusion

- Reformulation of the Winckler–Liu's equations in terms of conservative variables (+ decoupling hyperbolic–elliptic problems)
- Solitary wave solution for a prismatic channel of trapezoidal cross-section + numerical verification
- Validation with Favre and Treske's experiments

Perspectives

- Deal with variable channel geometry
- Comparison to our own experiments (ongoing)
- Applications to realistic cases

N.B.: Limits of the 1D modelling

Thank you for your attention.

Inria

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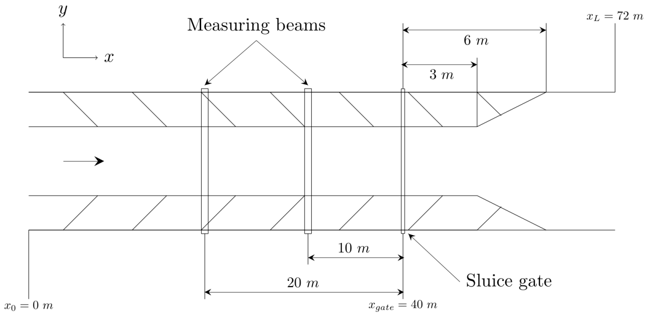
Sizing of Favre waves experiments

- Favre waves experiments planned this summer in channel 5 of the LNHE 72 x 1.5 x 1.2 m (L x W x H)
- Fast closing ($< 1\ s$) of a sluice gate placed at $x_{gate} = 40\ m$
- Measurement of the positive downstream wave over the length and width of the channel
- Sensors: 20 TDH, 8 ultrasonic



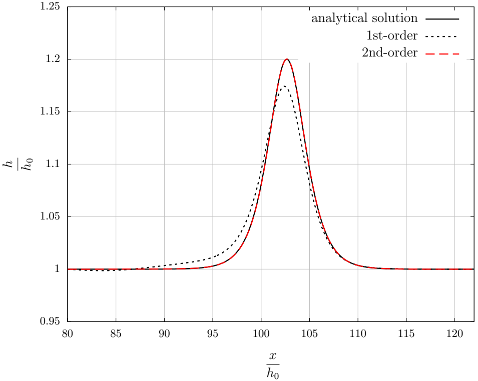
Form of the solutions – Verification case

- The reformulated system for a prismatic channel with trapezoidal cross-section

$$\begin{cases} \left(1+\mu^2\mathcal{T}\right)\Phi = -H_x \\ A_t + \epsilon\left(Au\right)_x = 0 \\ \left(Au\right)_t + \left(\epsilon A u^2 + gK(h)\right)_x = A\Phi + AH_{s,x} \end{cases}$$

$\mathcal{T} \doteqdot \gamma\partial_{xx}$

- Solitary wave case ($\boldsymbol{\epsilon = 0.2}$)
- Domain: $\boldsymbol{x \in [-12\lambda, 12\lambda]}$
- Solutions at $\boldsymbol{t = \frac{6.5\lambda}{c}}$



Winckler–Liu's model reformulated in terms of conservative variables

- First taking the exact continuity equation of the system on slide 5 gives

$$\begin{cases} A_t + \epsilon\left(Au\right)_x = 0 \\ u_t + \epsilon u u_x + g\eta_x + \mu^2\left(\alpha u_t + \beta u_{tx} + \gamma^* u_{txx}\right) = O(\epsilon^2, \epsilon\mu^2, \mu^4) \end{cases}$$

- and defining the elliptic operator

$$\mathcal{T} \doteq \alpha\mathcal{I} + \beta^*\partial_x - \partial_x\left(\gamma^*\partial_x\right)$$

$$\beta' \doteq \beta + \gamma^*_x, \quad \gamma^* \doteq -\gamma > 0$$

- the system becomes

$$\begin{cases} A_t + \epsilon\left(Au\right)_x = 0 \\ \left(1 + \mu^2\mathcal{T}\right) u_t + \epsilon u u_x + g\eta_x = O(\epsilon^2, \epsilon\mu^2, \mu^4) \end{cases}$$

Winckler–Liu's model reformulated in terms of conservative variables

$$\begin{cases} A_t + \epsilon\,(Au)_x = 0 \\ \left(1+\mu^2\mathcal{T}\right)u_t + \epsilon u u_x + g\eta_x = O(\epsilon^2, \epsilon\mu^2, \mu^4) \end{cases}$$

- $u \times$ continuity $+$ $A \times$ momentum

$$\begin{aligned} A\left(1+\mu^2\mathcal{T}\right)u_t + uA_t + \epsilon\left(Au^2\right)_x + gA\eta_x &= O(\epsilon^2, \epsilon\mu^2, \mu^4) \\ (Au)_t + \mu^2 A\mathcal{T}u_t + \epsilon\left(Au^2\right)_x + gA\eta_x &= O(\epsilon^2, \epsilon\mu^2, \mu^4) \end{aligned}$$

- Defining another operator $\mathbb{T}(\cdot) \doteqdot A\mathcal{T}\left[\frac{1}{A}(\cdot)\right]$

$$\begin{aligned} A\mathcal{T}u_t = A\mathcal{T}\left[\frac{1}{A}(Au_t)\right] &= A\mathcal{T}\left[\frac{1}{A}\Big((Au)_t - uA_t\Big)\right] \\ &= \mathbb{T}\Big[(Au)_t\Big] + \mathbb{T}\Big[\epsilon u\,(Au)_x\Big] \end{aligned}$$

- Hence

$$\left(1+\mu^2\mathbb{T}\right)\Big[(Au)_t\Big] + \epsilon\left(Au^2\right)_x + gA\eta_x + \mu^2\mathbb{T}\Big[\epsilon u\,(Au)_x\Big] = 0$$

Winckler–Liu's model reformulated in terms of conservative variables

$$\left(1+\mu^2\mathbb{T}\right)\left[(Au)_t\right]+\epsilon\left(Au^2\right)_x+gA\eta_x+\mu^2\mathbb{T}\left[\epsilon u\left(Au\right)_x\right]=0$$

- Applying $\left(1+\mu^2\mathbb{T}\right)$ to all Saint-Venant terms

$$\begin{aligned}\left(1+\mu^2\mathbb{T}\right)\left[(Au)_t+\epsilon\left(Au^2\right)_x+gA\eta_x\right]&=\mu^2\mathbb{T}\left[-\epsilon u\left(Au\right)_x+\epsilon\left(Au^2\right)_x+gA\eta_x\right]\\&=\mu^2\mathbb{T}\left[\epsilon Auu_x+gA\eta_x\right]\end{aligned}$$

- now defining

$$\begin{aligned}A\phi&\doteqdot(Au)_t+\epsilon\left(Au^2\right)_x+gA\eta_x\\AH_x&\doteqdot A\Big(\epsilon uu_x+g\eta_x\Big)=A\left(\epsilon\frac{u^2}{2}+g\eta\right)_x\end{aligned}$$

- we have

$$\begin{aligned}\left(1+\mu^2\mathbb{T}\right)(A\phi)&=\mu^2\mathbb{T}\left(AH_x\right)\\\left(1+\mu^2\mathcal{T}\right)\phi&=\mu^2\mathcal{T}\left(H_x\right)\end{aligned}$$

Winckler–Liu's model reformulated in terms of conservative variables

$$\begin{cases} \left(1+\mu^2\mathcal{T}\right)\phi = \mu^2\mathcal{T}\left(H_x\right) \\ A_t + \epsilon\left(Au\right)_x = 0 \\ \left(Au\right)_t + \epsilon\left(Au^2\right)_x + gA\eta_x = A\phi \end{cases}$$

- making the following change of variable: $\Phi = \phi - H_x$

$$\begin{cases} \left(1+\mu^2\mathcal{T}\right)\Phi = -H_x \\ A_t + \epsilon\left(Au\right)_x = 0 \\ \underbrace{\left(Au\right)_t + \epsilon\left(Au^2\right)_x + gA\eta_x}_{\text{Saint-Venant equations}} = \underbrace{A\Phi + AH_x}_{\text{non-hydrostatic (dispersive) terms}} \end{cases}$$

with

$$\mathcal{T} \doteqdot \alpha\mathcal{I} + \beta^*\partial_x - \partial_x\left(\gamma^*\partial_x\right)$$

$$\beta^* \doteqdot \beta + \gamma^*_x, \quad \gamma^* \doteqdot -\gamma, \quad H \doteqdot \epsilon\frac{u^2}{2} + g\eta$$

- **Decoupling** the **hyperbolic** problem (shallow water equations) and the **elliptic** problem (non-hydrostatic terms)

Winckler–Liu's model in conservative variables

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- Invertibility of $1+\mu^2\mathcal{T}$ requires **coercivity**

- **Coercivity** of the **bilinear form** $a(v,w)$, associated to the weak formulation of the elliptic equation, is based on the **positivity** of its **quadratic form** $a(v,v)$:

$$a(v,v) = \mu^2\int_{\mathbb{R}}\gamma^* v_x^2\,\mathrm{d}x + \int_{\mathbb{R}}\left(\alpha^* - \frac{1}{2}\mu^2\beta_x^*\right)v^2\,\mathrm{d}x$$

$\alpha^* = 1+\mu^2\alpha$

Winckler–Liu's model in conservative variables

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$\alpha^* = 1+\mu^2\alpha$

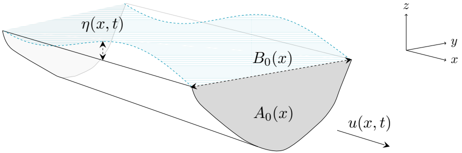
- For a **prismatic channel**: $\alpha=\beta=0$, $\gamma^*=cte$, hence $\boldsymbol{1+\mu^2\mathcal{T}}$ is **coercive** if $\boldsymbol{\gamma^*>0}$

Positivity of γ^*

- γ^* stems from a boundary value problem (Winckler and Liu, 2015)

$$\begin{aligned} \Delta\chi &= -1, \\ \boldsymbol{\nabla}\chi \cdot \boldsymbol{n} &= -\frac{A_0}{B_0} \quad \text{on} \quad B_0, \\ &= 0 \quad \text{on} \quad \partial A_0 \backslash B_0 \end{aligned}$$

$$\text{where} \quad \gamma^* = \frac{1}{A_0}\int_{A_0} \chi \,\mathrm{d}S - \frac{1}{B_0}\int_{B_0} \chi \,\mathrm{d}\ell = \overline{\chi} - \widetilde{\chi}$$



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$$\boldsymbol{\nabla}\chi \cdot \boldsymbol{n} = -\frac{A_0}{B_0} \quad \text{on} \quad B_0, \qquad = \overline{\chi} - \widetilde{\chi}$$

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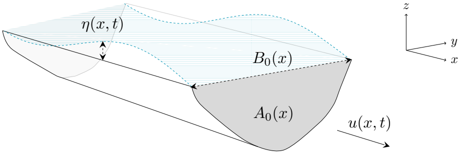


- Proof of the positivity

$$\gamma^* = -\frac{1}{A_0}\int_{A_0} \chi\Delta\chi \,\mathrm{d}S - \frac{1}{B_0}\int_{B_0} \chi \,\mathrm{d}\ell$$

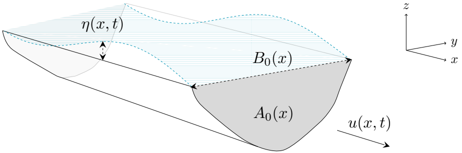
Positivity of γ^*

- γ^* stems from a boundary value problem (Winckler and Liu, 2015)

$$\Delta\chi = -1, \qquad \text{where } \gamma^* = \frac{1}{A_0}\int_{A_0}\chi\,\mathrm{d}S - \frac{1}{B_0}\int_{B_0}\chi\,\mathrm{d}\ell$$

$$\boldsymbol{\nabla}\chi\cdot\boldsymbol{n} = -\frac{A_0}{B_0} \quad \text{on} \quad B_0, \qquad = \overline{\chi} - \widetilde{\chi}$$

$$= 0 \quad \text{on} \quad \partial A_0 \backslash B_0$$



- Proof of the positivity

$$\gamma^* = -\frac{1}{A_0}\int_{A_0}\boldsymbol{\nabla}\cdot(\chi\boldsymbol{\nabla}\chi)\,\mathrm{d}S + \frac{1}{A_0}\int_{A_0}|\boldsymbol{\nabla}\chi|^2\,\mathrm{d}S - \frac{1}{B_0}\int_{B_0}\chi\,\mathrm{d}\ell$$

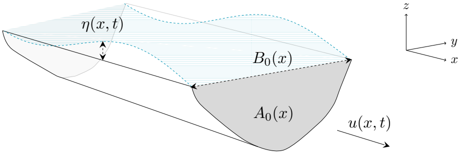
Positivity of γ^*

❑ γ^* stems from a boundary value problem (Winckler and Liu, 2015)

$$\Delta\chi = -1, \qquad \text{where} \quad \gamma^* = \frac{1}{A_0}\int_{A_0}\chi\,\mathrm{d}S - \frac{1}{B_0}\int_{B_0}\chi\,\mathrm{d}\ell$$

$$\boldsymbol{\nabla}\chi\cdot\boldsymbol{n} = -\frac{A_0}{B_0} \quad \text{on} \quad B_0, \qquad = \overline{\chi} - \widetilde{\chi}$$

$$= 0 \quad \text{on} \quad \partial A_0 \backslash B_0$$



❑ Proof of the positivity

$$\gamma^* = -\frac{1}{A_0}\int_{\partial A_0}\chi\boldsymbol{\nabla}\chi\cdot\boldsymbol{n}\,\mathrm{d}\ell + \overline{|\boldsymbol{\nabla}\chi|^2} - \frac{1}{B_0}\int_{B_0}\chi\,\mathrm{d}\ell$$

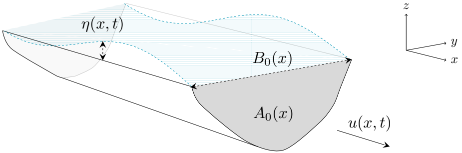
Positivity of γ^*

- γ^* stems from a boundary value problem (Winckler and Liu, 2015)

$$\Delta\chi = -1, \qquad \text{where} \quad \gamma^* = \frac{1}{A_0}\int_{A_0} \chi \,\mathrm{d}S - \frac{1}{B_0}\int_{B_0} \chi \,\mathrm{d}\ell$$

$$\nabla\chi \cdot \boldsymbol{n} = -\frac{A_0}{B_0} \quad \text{on} \quad B_0, \qquad = \overline{\chi} - \widetilde{\chi}$$

$$= 0 \quad \text{on} \quad \partial A_0 \backslash B_0$$



- Proof of the positivity

$$\gamma^* = \overline{|\nabla\chi|^2} > 0$$

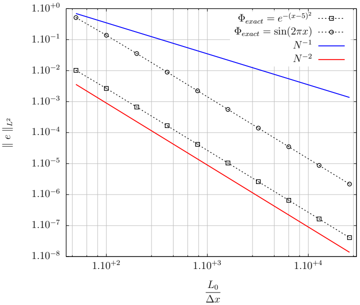
Convergence study of the elliptic module

- We want to check the resolution of the elliptic module using the MMS (Method of Manufactured Solution)
- An exact solution Φ_{exact} is imposed to determine the right-hand side f, which is then imposed to check the code

$$f = (1+\alpha)\,\Phi_{exact} + (\beta + \gamma^*_x)\,\Phi_{exact,x} - \left(\gamma^*\,\Phi_{exact,x}\right)_x$$

- For the convergence study, we look at the L^2-norm error $\parallel e \parallel_{L^2}$

$$\parallel e \parallel_{L^2} = \sqrt{\sum_{e=1}^{N_e} \int_{x_e}^{x_{e+1}} \left(\Phi_{exact} - \Phi_{num}\right)^2 \,\mathrm{d}x}$$



Numerical scheme of the hyperbolic part

- The system to solve

$$U_t + F(U)_x = S(U)$$

with

$$U = \begin{pmatrix} A \\ Au \end{pmatrix},\ F = \begin{pmatrix} Au \\ Au^2 + gK(h) \end{pmatrix} \text{ et } S = \begin{pmatrix} 0 \\ A\Phi + AH_{s,x} \end{pmatrix}$$

- Trapezoidal cross-section

$$A(h) = hb + \frac{h^2}{\tan\theta} = hb\,(1+\tau)$$
$$K(h) = bh^2\left(\tfrac{1}{2} + \tfrac{1}{3}\tau\right)$$

avec $\tau \doteqdot \frac{h}{b}\cot\theta$

- average water height $\bar{h}$

$$\bar{h} \doteqdot \frac{1+\tau}{1+2\tau}h$$

Numerical scheme of the hyperbolic part

- Finite Volume Method

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right) + \frac{\Delta t}{\Delta x}\Delta S_{H_s,i} + \Delta t \bar{S}_\Phi$$

$$F_{i+\frac{1}{2}} = \tfrac{1}{2}\left[F\left(U^L_{i+\frac{1}{2}}\right) + F\left(U^R_{i+\frac{1}{2}}\right)\right] - \tfrac{1}{2}\left|\widetilde{J}\right|_{i+\frac{1}{2}} \Delta \widetilde{U}_{i+\frac{1}{2}}$$

$$\widetilde{J}_{i+\frac{1}{2}} = \left(\widetilde{P}\widetilde{\Lambda}\widetilde{P}^{-1}\right)_{i+\frac{1}{2}}$$

$$\widetilde{P}_{i+\frac{1}{2}} = \begin{pmatrix} 1 & 1 \\ \widetilde{\lambda}_1 & \widetilde{\lambda}_2 \end{pmatrix}, \quad \widetilde{\Lambda}_{i+\frac{1}{2}} = \begin{pmatrix} \widetilde{\lambda}_1 & 0 \\ 0 & \widetilde{\lambda}_2 \end{pmatrix}$$

$$\widetilde{\lambda}_{1,2} = \widetilde{u} \pm \widetilde{c}$$

$$\widetilde{u}_{i+\frac{1}{2}} = \frac{u^L\sqrt{A^L} + u^R\sqrt{A^R}}{\sqrt{A^L} + \sqrt{A^R}}, \quad \widetilde{c}_{i+\frac{1}{2}} = \sqrt{\tfrac{1}{2} g\left(\bar{h}^L + \bar{h}^R\right)}$$

Numerical scheme of the hyperbolic part

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right) + \frac{\Delta t}{\Delta x}\Delta S_{H_s,i} + \Delta t \bar{S}_\Phi$$

$$\Delta S_{H_s,i} = S^-_{H_s,i+\frac{1}{2}}\left(U^L_{i+\frac{1}{2}}, U^R_{i+\frac{1}{2}}\right) + S^+_{H_s,i-\frac{1}{2}}\left(U^L_{i-\frac{1}{2}}, U^R_{i-\frac{1}{2}}\right) + S^*_{H_s}\left(U^L_{i+\frac{1}{2}}, U^R_{i-\frac{1}{2}}\right)$$

with

$$S^{\mp}_{H_s,i\pm\frac{1}{2}}\left(U^L_{i\pm\frac{1}{2}}, U^R_{i\pm\frac{1}{2}}\right) = \frac{1}{2}\left[\widetilde{P}\left(I \mp \widetilde{\Lambda}^{-1}\left|\widetilde{\Lambda}\right|\right)\widetilde{P}^{-1}\right]_{i\pm\frac{1}{2}} \widetilde{S}_{H_s,i\pm\frac{1}{2}}$$

$$\widetilde{S}_{H_s,i\pm\frac{1}{2}} = \begin{bmatrix} 0 \\ \frac{A^L + A^R}{2}\left(H_s^R - H_s^L\right) \end{bmatrix}_{i\pm\frac{1}{2}}$$

$$S^*_{H_s}\left(U^L_{i+\frac{1}{2}}, U^R_{i-\frac{1}{2}}\right) = \begin{bmatrix} 0 \\ \frac{A^L_{i+\frac{1}{2}} + A^R_{i-\frac{1}{2}}}{2}\left(H^L_{s,i+\frac{1}{2}} - H^R_{s,i-\frac{1}{2}}\right) \end{bmatrix}$$

and

$$\bar{S}_\Phi = \frac{1}{8}\begin{bmatrix} 0 \\ A^L_{i+\frac{1}{2}}\left(\Phi_{i+1} + 3\Phi_i\right) + A^R_{i-\frac{1}{2}}\left(3\Phi_i + \Phi_{i-1}\right) \end{bmatrix}$$

Friction treatment

- The system to be solved considering the friction term $S_f \neq 0$

$$U_t + F(U)_x = S(U)$$

with

$$U = \begin{pmatrix} A \\ Au \end{pmatrix}, \; F = \begin{pmatrix} Au \\ Au^2 + gK(h) \end{pmatrix} \text{ et } S = \begin{pmatrix} 0 \\ A\Phi + AH_{s,x} - gAS_f \end{pmatrix}$$

where S_f is evaluated using the Gauckler–Mannig–Strickler formula

$$S_f = \frac{Q^2}{K_s^2 A^2 R_h^{4/3}}$$

with $Q \doteqdot Au$ the flow rate and R_h the hydraulic radius

Friction treatment

- This friction term can be treated numerically in a semi-implicit way, Bristeau (2001)
- First step : frictionless

$$U_i^* = U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right) + \frac{\Delta t}{\Delta x}\Delta S_{H_s,i} + \Delta t \bar{S}_\Phi$$

- Second step :

$$U_t = \begin{pmatrix} 0 \\ -g\frac{Q^2}{K_s^2 A R_h} \end{pmatrix} \Rightarrow \begin{aligned} A_i^{n+1} &= A_i^* \\ \frac{Q_i^{n+1} - Q_i^*}{\Delta t} &= -g\frac{Q_i^{n+1}\,|Q_i^n|}{K_s^2 A_i^* \left(R_h^{4/3}\right)_i^n} \end{aligned}$$

- factoring by Q_i^{n+1}.

$$Q_i^{n+1} = \frac{Q_i^*}{1 + \Delta t g \frac{|Q_i^n|}{K_s^2 A_i^* \left(R_h^{4/3}\right)_i^n}}$$