The Shifted Boundary Method:
A Framework for Embedded Computational Mechanics

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Motivation I

Complex geometry is still a key challenge in engineering simulations

- Engineering simulations are dominated by geometric complexity
- The merging of topology optimization and advanced manufacturing (e.g., additive manufacturing) exacerbates geometric complexity
Motivation II

Imaging-to-computing, an *emerging* field:

An efficient transition from geometries reconstructed from images to computation may impact and transform many fields of application:

- **Biomedical engineering**: CT-scans are given as pixilated data or STL format (collections of triangular facets and their nodal coordinates). Body-fitted meshing can be quite hard to perform.

- **Subsurface imaging and computing** (meshing requires considerable effort in reservoir engineering applications)

- **Additive manufacturing simulations** (e.g., 3D-printing). The typical file format for 3D-printers is again STL

In these examples the geometric information is not very precise and/or consistent (surfaces with *gaps* and *overlaps*, typical of computer graphics, STL = set of disconnected triangular faces)
Overview

Two commonly used computational strategies:

1. *Body-fitted grids.* The grid conforms to the boundary geometry of the shape to be simulated.
   - **Advantages:** Easier treatment of the boundary conditions (and boundary layers)
   - **Limitations:** Requires more advanced meshing for complex geometry, or re-meshing in problems with large deformations

2. *Embedded/immersed grids.* The shape to be simulated is fully or partially embedded (or immersed) into a regular background grid.
   - **Advantages:** Generality of the method, especially if coupling heterogeneous computational frameworks, rapid prototyping
   - **Limitations:** More complex enforcement of boundary conditions
Existing Embedded Boundary Methods

Unfitted/Embedded Finite Element Methods

- Embedded methods of finite element type (a.k.a. cutFEMs, unfitted FEMs, Finite Cell Method, Embedded Splines, IGA-Immersogeometric etc.) often rely on XFEM methodologies to integrate on cut cells, Inverse Lax-Wendroff procedure (DG) [Burman, Hansbo, Larson, Massing, Cirak, Kamenski, Schillinger, Parvizian, Düster, Rank, Wall, Annavarapu, Dolbow, Harari, Moës, Badia, Rossi, C-W. Shu etc.]

- Unfitted/embedded FEMs typically utilize Lagrange multipliers or Nitsche variational formulations

- CutFEMs/unfitted FEMs require data structures and special quadratures to integrate on geometrically complex cut cells

- The small cut-cell problem: Integration over cut cells introduces additional interface degrees of freedom that may yield stability problems, very small time-steps or poor matrix conditioning. [Burman & Hansbo Appl. Num. Math. (2012)]. Solution: ghost penalty, and related methods
Overview of the Shifted Boundary Method

Key ideas:

- Use a *purely* embedded approach
- Use the Nitsche framework to impose boundary conditions weakly
- Apply boundary conditions on a surrogate boundary, near the true boundary
- Appropriately modify the boundary condition to account for the discrepancy between surrogate and true boundary
The Shifted Boundary Method: Key Ideas

The extension map $M$ & a distance vector function $d$

$M : \Gamma \rightarrow \Gamma$

$\tilde{x} \mapsto x$

$d_M(\tilde{x}) = x - \tilde{x} = [M - I](\tilde{x})$

Extension of functions defined on boundaries: $\tilde{\psi}(\tilde{x}) \equiv \psi(M(\tilde{x}))$

Important assumption (resolution): $n \cdot \tilde{n} > 0$

Surrogate domain geometric representation: via Physbam library (Fedkiw, Stanford U.), and the improved version developed in the Farhat research group (Stanford U.)
The (Base) Nitsche Method

A prototypical example: The Poisson problem

Strong form of the equation

\[
\Delta u + f = 0 \quad \text{on } \Omega \\
u = g \quad \text{on } \Gamma = \partial \Omega
\]

Weak form of the equation with weak boundary conditions (Nitsche method):

\[
\int_{\Omega} w_i u_{,i} = \int_{\Omega} w f
\]
The Shifted Nitsche Method

A prototypical example: The Poisson problem

Weak form of the equation with weak boundary conditions (Nitsche method):

\[ \int_{\Omega} w_i u_i - \int_{\Gamma} w u_i n_i - \int_{\Gamma} w_i (u - g) n_i + \alpha \int_{\Gamma} (u - g) w = \int_{\Omega} w f \]

Shifted Nitsche method:

\[ \int_{\tilde{\Omega}} w_i u_i - \int_{\tilde{\Gamma}} w u_i n_i - \int_{\tilde{\Gamma}} w_i (u - \tilde{g}) n_i + \alpha \int_{\tilde{\Gamma}} (u - \tilde{g}) \tilde{w} = \int_{\Omega} w f \]
The Shifted Nitsche Method

A prototypical example: The Poisson problem

Weak form of the equation with weak boundary conditions (Nitsche method):

\[ \int_{\Omega} w_i u_i - \int_{\Gamma} w u_i n_i - \int_{\Gamma} w_i (u - \tilde{g}) n_i + \alpha \int_{\Gamma} (u - \tilde{g}) \tilde{w} = \int_{\Omega} w f \]

\[ \int_{\Omega} w_i u_i - \int_{\Gamma} w u_i \tilde{n}_i - \int_{\Gamma} w_i (u + u_i d_j - g) \tilde{n}_i + \alpha \int_{\Gamma} (u + u_i d_j - g) (w + w_i d_k) = \int_{\Omega} w f \]
Numerical Results: Poisson Problem

Numerical convergence test with an exact solution
\[ \Delta u + 1 = 0 \quad \text{on } \Omega \]
\[ u|_{\Gamma} = 0 \]

Exact solution:
\[ u = \frac{1}{4} (R^2 - r^2) \]

Convergence of the $L^2$ norm of the error

With Taylor expansion

Without Taylor expansion
Condition number trends
For the previous problem, the algebraic condition number has the following properties:

- About 3 times larger than for the body-fitted method
- Same scaling with $h^{-2}$ has for the body-fitted method
**Numerical Analysis: Poisson Problem**

**Coercivity of the SBM variational form** *(sketch of proof for Poisson operator)*

\[
a^h(u^h, w^h) = l^h(w^h)
\]

\[
a^h(u^h, w^h) = (\nabla w^h, \nabla u^h)_\Omega - \langle w^h + \nabla w^h \cdot d, \nabla u^h \cdot \tilde{n} \rangle_{\Gamma_D} - \langle \nabla w^h \cdot \tilde{n}, u^h + \nabla u^h \cdot d \rangle_{\Gamma_D} + \langle \nabla w^h \cdot d, \nabla u^h \cdot \tilde{n} \rangle_{\Gamma_D}
\]

\[
+ \langle \alpha/h (w^h + \nabla w^h \cdot d), u^h + \nabla u^h \cdot d \rangle_{\Gamma_D} + \langle \beta \|d\| w^h, \tilde{u}^h \rangle_{\Gamma_D}
\]

\[
l^h(w^h) = (w^h, f)_\Omega - \langle \nabla w^h \cdot \tilde{n}, \tilde{u}_D \rangle_{\Gamma_D} + \langle \alpha/h (w^h + \nabla w^h \cdot d), \tilde{u}_D \rangle_{\Gamma_D} + \langle \beta \|d\| w^h, \tilde{u}_D \rangle_{\Gamma_D}.
\]

... and replace \( w^h = u^h \):

\[
a^h(u^h, u^h) = \|\nabla u^h\|_{0,\Omega}^2 - 2\langle u^h + \nabla u^h \cdot d, \nabla u^h \cdot \tilde{n} \rangle_{\Gamma_D} + \langle \nabla u^h \cdot d, \nabla u^h \cdot \tilde{n} \rangle_{\Gamma_D}
\]

\[
+ \alpha \|\sqrt{1/h} (u^h + \nabla u^h \cdot d)\|_{0,\Gamma_D}^2 + \beta \|\sqrt{h} u^h\|_{0,\Gamma_D}^2,
\]

Use the following decomposition

\[
\nabla u^h \cdot \tilde{n} = \left((\nabla u^h \cdot n)n + (\nabla u^h \cdot \tau_i)\tau_i\right)\cdot \tilde{n},
\]

\[
\nabla u^h \cdot d = \nabla u^h \cdot n \|d\|.
\]
Numerical Analysis: Poisson Problem

... sketch of coercivity proof continued

\[
\langle \nabla u^h \cdot d, \nabla u^h \cdot \vec{n} \rangle_{\Gamma_D} \geq \| (n \cdot \vec{n}) \| \| \nabla u^h \cdot n \|_{0, \Gamma_D}^2 - |\langle \partial_{\tau_j} u^h \cdot \tau_j \cdot \vec{n} \rangle_{\Gamma_D} | \\
\geq \| (n \cdot \vec{n}) \| \| \nabla u^h \cdot n \|_{0, \Gamma_D}^2 - \| \| \nabla \nabla u^h \|_{0, \Gamma_D} \| \| \nabla u^h \|_{0, \Gamma_D} \| \| \vec{n} \|_{0, \Gamma_D} \| \| \vec{d} \| \| \vec{d} \|_{0, \Gamma_D}^2 \\
\geq \| (n \cdot \vec{n}) \| \| \nabla u^h \cdot n \|_{0, \Gamma_D}^2 - \left( \frac{\epsilon_1}{2} \| \| \nabla \nabla u^h \|_{0, \Gamma_D} \| \| \vec{n} \|_{0, \Gamma_D}^2 + \frac{1}{2\epsilon_1} \| \| \nabla \nabla u^h \|_{0, \Gamma_D} \| \| \vec{d} \|_{0, \Gamma_D}^2 \right)
\]

and

\[
2 \langle u^h + \nabla u^h \cdot d, \nabla u^h \cdot \vec{n} \rangle_{\Gamma_D} \leq \frac{1}{\epsilon} \| \| \sqrt{1/h^}\perp (u^h + \nabla u^h \cdot d) \|_{0, \Gamma_D}^2 + \epsilon \| \| \sqrt{h^}\perp \nabla u^h \cdot \vec{n} \|_{0, \Gamma_D}^2
\]

In conclusion:

\[
\| u^h \|_{SBa}^2 = \| \nabla u^h \|_{0, \Omega}^2 + \| (n \cdot \vec{n}) \| \| \nabla u^h \cdot n \|_{0, \Gamma_D}^2 + \| \| \sqrt{1/h^}\perp (u^h + \nabla u^h \cdot d) \|_{0, \Gamma_D}^2 + \| \| \nabla \nabla u^h \|_{0, \Gamma_D} \| \| \vec{n} \|_{0, \Gamma_D}^2 \\
C_{SBa} = \min \left( 1 - C_I \left( \epsilon + \frac{\epsilon_1}{2} \right), \alpha - \frac{1}{\epsilon}, \beta - \frac{1}{2\epsilon_1} \right).
\]
Sketch of the proof of convergence

Estimate the consistency error:

\[
\alpha_h(u, w^h) - l^h(u^h) = \frac{\tilde{a}_h(u, w^h) - l^h(w^h)}{0} + \left( \sqrt{h^3 \nabla w^h \cdot \tilde{n}} + \sqrt{1/h} \left( u + \nabla u \cdot d - \tilde{u}_D \right) \right)_{\Gamma_D} O(h^3) + \left( \alpha \sqrt{1/h} \left( w^h + \nabla w^h \cdot d \right), \sqrt{1/h} \left( u + \nabla u \cdot d - \tilde{u}_D \right) \right)_{\Gamma_D} + \left( \beta \sqrt{\|d\|} w^h_{\tau, \tilde{r}} + \sqrt{\|d\|} \left( u_{\tau, \tilde{r}} - \tilde{u}_D_{\tau, \tilde{r}} \right) \right)_{\Gamma_D} O(h^{3/2})
\]

\[
\leq \left\| w^h \right\|_{SB_u} O(h^{3/2})
\]

Apply second Strang’s lemma:

\[
\left\| u - u^h \right\|_{SB_u} \leq \left\| u - u^h \right\|_{V(h)} \leq \left( 1 + \frac{\|a'_h\|_{V(h) \times V(h)} \times V(h)}{C_{SB_u}} \right) \inf_{w^h \in V(h)} \left\| u - w^h \right\|_{V(h)} + \frac{1}{C_{SB_u}} \sup_{v^h \in V(h)} \left\| v^h \right\|_{SB_u} O(h) + \frac{1}{C_{SB_u}} \sup_{v^h \in V(h)} \left\| v^h \right\|_{SB_u} O(h^{3/2})
\]

\[
\leq Ch
\]
Numerical Analysis: Poisson Problem

**Sketch of (duality) L^2-estimates** [a new/harder version of Nitsche-Aubin trick]

\[
a^h_{sc}(u, w) = a^h_{sc}(u, w) + a^h_{sd}(u, w) + a^h_{ud}(u, w),
\]

\[
a^h_{sc}(u, w) = \langle \nabla w, \nabla u \rangle_{\tilde{\Omega}} - \langle w, \nabla u \cdot \tilde{n} \rangle_{\tilde{\Gamma}_D} - \langle \nabla w \cdot \tilde{n}, u \rangle_{\tilde{\Gamma}_D} + \langle \alpha / h^\perp w, u \rangle_{\tilde{\Gamma}_D}, \quad \text{(Body-fitted)}
\]

\[
a^h_{sd}(u, w) = -\langle \nabla w \cdot d, \nabla u \cdot \tilde{n} \rangle_{\tilde{\Gamma}_D} - \langle \nabla w \cdot \tilde{n}, \nabla u \cdot d \rangle_{\tilde{\Gamma}_D} + \langle \alpha / h^\perp \nabla w \cdot d, u \rangle_{\tilde{\Gamma}_D} + \langle \alpha / h^\perp w, \nabla u \cdot d \rangle_{\tilde{\Gamma}_D}, \quad \text{(SBM-term, symmetric)}
\]

\[
a^h_{ud}(u, w) = \langle \nabla w \cdot d, \nabla u \cdot \tilde{n} \rangle_{\tilde{\Gamma}_D}. \quad \text{(SBM-term, unsymmetric)}
\]

**SBM terms**

\[
\left\{ \begin{array}{l}
-\Delta \psi = u - u^h, \quad \text{in} \ \tilde{\Omega}, \\
\psi = 0, \quad \text{in} \ \tilde{\Gamma} = \partial \tilde{\Omega}.
\end{array} \right. \quad \text{(Auxiliary (dual/adjoint) problem)}
\]

\[
a^h_{sc}(w, \psi) = a^h_{sc}(w, \psi) - a^h_{sd}(w, \psi) - a^h_{ud}(w, \psi) = \langle w, u - u^h \rangle_{\tilde{\Omega}}, \quad \text{(Weak dual problem)}
\]

Taking \( w = u - u^h \), we obtain:

\[
\|u - u^h\|_{0, \tilde{\Omega}}^2 = a^h_{u}(u - u^h, \psi - \psi_1) + a^h_{u}(u - u^h, \psi_1) + R_c(u - u^h, \psi),
\]
Numerical Analysis: Poisson Problem

**Sketch of (duality) L^2-estimates** [a new/harder version of Nitsche-Aubin trick]

\[ \| u - u^h \|_{0;\Omega}^2 = a_u^h(u - u^h, \psi - \psi_I) + a_u^h(u - u^h, \psi_I) + R_c(u - u^h, \psi), \]

Every term can be bounded:

\[ a_u^h(u - u^h, \psi - \psi_I) \leq C_{b_u} \| u - u^h \|_{SB_3} \| \psi - \psi_I \|_{SB_3} \leq C_{b_u} C_{int} \| u - u^h \|_{V_s(h)} h \| \psi \|_{2;\Omega}, \]

\[ a_u^h(u - u^h, \psi_I) = a_u^h(u, \psi_I) - l_u^h(\psi_I) \]

\[ = \langle \alpha / h^1 \nabla \psi_I \cdot d - \nabla \psi_I \cdot \tilde{n}, u + \nabla u \cdot d - \tilde{u}_D \rangle_{\Gamma_D} + o(h^2) \]

\[ \leq C'_{int} \| \psi \|_{2;\Omega} O(h^2), \]

\[ R_c(\psi, u - u^h) = \langle \sqrt{\| d \|} (n \cdot \tilde{n}) \nabla \psi \cdot \tilde{n}, \sqrt{(n \cdot \tilde{n})} \| d \| \nabla (u - u^h) \cdot n \rangle_{\Gamma_D} \]

\[ - \langle \sqrt{\alpha / h^1} \nabla \psi \cdot d, \sqrt{\alpha / h^1} (u - u^h + \nabla (u - u^h) \cdot d) \rangle_{\Gamma_D} \]

\[ + \langle \sqrt{\beta} \| d \| \psi, \tilde{\tau}, \sqrt{\beta} \| d \| (u - u^h), \tilde{\tau} \rangle_{\Gamma_D} \]

\[ \leq C_{\Gamma} \| u - u^h \|_{SB_3} h^{1/2} \| \psi \|_{2;\Omega}. \]

Finally:

\[ \| u - u^h \|_{0;\Omega} \leq C_{NA} h^{3/2} \]

(Possibly, this estimate is not sharp, since, numerically, we observe second order!)
Numerical Analysis: Stokes Flow Problem

[ Collaboration with Claudio Canuto, Mathematics Dept., Politecnico di Torino ]

Stability (LBB)
\[ \mathcal{B}([u^h, p^h]; [w^h, q^h]) \geq \alpha_{LBB} \|[u^h, p^h]\|_\mathcal{B} \|[w^h, q^h]\|_\mathcal{B} \]

An LBB \textit{inf-sup} condition can be derived in the case of the Stokes’ operator.

Convergence (in natural norm)
\[ \|[u, p] - [u^h, p^h]\|_{W(\tilde{\Omega}; h)} \leq C h_{\tilde{\Omega}} \left( \|\nabla(\nabla u)\|_{0, \Omega} + \|\nabla p\|_{0, \tilde{\Omega}} \right) \]

The proof is analogous to the one for the Poisson problem, using the \textit{inf-sup} LBB condition and Strang’s second lemma.

Duality estimates (\(L^2\)-estimates for the velocity field)
\[ \|u - u^h\|_{0; \tilde{\Omega}} \leq C_0 h_{\tilde{\Omega}}^{3/2} \left( \|\nabla(\nabla u)\|_{0, \tilde{\Omega}} + \|\nabla p\|_{0, \tilde{\Omega}} \right) \]

Analogous but more complicated proof than in the Poisson case. We observe quadratic convergence for the velocity, in practical calculations.
Advection-Diffusion Problem

Strong form of the equations

\[ \nabla \cdot (\mathbf{a}u - \kappa \nabla u) = f \] on \( \Omega \),

\[ u = g \] on \( \Gamma \),

\[ -(\mathbf{a}u - \kappa \nabla u) \cdot \mathbf{n} = h \] on \( \Gamma_h^- \),

\[ \kappa \nabla u \cdot \mathbf{n} = h \] on \( \Gamma_h^+ \),

\( \mathbf{a} \) is a solenoidal vector field (\( \nabla \cdot \mathbf{a} = 0 \))

\[ \mathbf{a} \cdot \nabla u - \kappa \Delta u = f \]
Numerical Analysis: Advection-Diffusion

Variational formulation (shifted Nitsche-type + SUPG)

Find $u^h \in V^h(\Omega)$ such that, $\forall w^h \in V^h(\Omega)$,

$$a^h(u^h, w^h) = l^h(w^h),$$

$$a^h(u^h, w^h) = \quad - (\nabla w^h, \alpha u^h)_{\Omega} + (\nabla w^h, \kappa \nabla u^h)_{\Omega} + (\tau \alpha \cdot \nabla w^h, \alpha \cdot \nabla u^h - \kappa \Delta u^h)_{\Omega} +$$

$$+ \langle w^h, u^h \alpha \cdot \tilde{n} \rangle_{\Gamma^+} - \langle w^h, \kappa \nabla u^h \cdot \tilde{n} \rangle_{\Gamma_D} - \langle \kappa \nabla w^h \cdot \tilde{n}, u^h + \nabla u^h \cdot \tilde{d} \rangle_{\Gamma_D^+} - \langle \kappa \nabla w^h \cdot \tilde{n}, u^h \rangle_{\Gamma_D^-}$$

$$+ \langle \alpha \kappa/h \perp w^h, u^h \rangle_{\Gamma_D^-} + \langle \alpha \kappa/h \perp (w^h + \nabla u^h \cdot \tilde{d}), u^h + \nabla u^h \cdot \tilde{d} \rangle_{\Gamma_D^-} + \langle \beta \kappa h \perp w^h, u^h \rangle_{\Gamma_D^-}$$

$$l^h(w^h) = \quad (w^h, f)_{\Omega} + (\tau \alpha \cdot \nabla w^h, f)_{\Omega}$$

$$+ \langle w^h, t_N \rangle_{\Gamma_N} - \langle \kappa \nabla w^h \cdot \tilde{n}, u_D \rangle_{\Gamma_D} - \langle w^h, u_D \alpha \cdot \tilde{n} \rangle_{\Gamma_D^-}$$

$$+ \langle \alpha \kappa/h \perp w^h, u_D \rangle_{\Gamma_D^-} + \langle \alpha \kappa/h \perp (w^h + \nabla u^h \cdot \tilde{d}), u_D \rangle_{\Gamma_D^{-}} + \langle \beta \kappa h \perp w^h, u_D \rangle_{\Gamma_D^-}$$

Conservation statement (selecting a constant test function)

(Dirichlet outflow)

$$\langle 1, u_D \alpha \cdot \tilde{n} - \kappa \nabla u^h \cdot \tilde{n} + \alpha \kappa/h \perp (u^h - u_D) \rangle_{\Gamma_D^-} = \langle 1, u^h \alpha \cdot \tilde{n} - \kappa \nabla u^h \cdot \tilde{n} + \alpha \kappa/h \perp (u^h - u_D) \rangle_{\Gamma_D^-}$$

(Dirichlet inflow)

$$\langle 1, t_N \rangle_{\Gamma_N} + \langle 1, u^h \alpha \cdot \tilde{n} - t_N \rangle_{\Gamma_N} = (1, f)_{\Omega}$$

(Neumann inflow)

(Neumann outflow)
Numerical Analysis: Advection-Diffusion

Stability

\[ a^h(u^h, u^h) \geq C_{SB} \| u^h \|_{SB}^2 , \]

\[ \| u^h \|_{SB}^2 = \| a \cdot \tilde{n} \|_{0, \tilde{\Gamma}}^2 u^h + \kappa \| \nabla u^h \|_{0, \tilde{\Omega}}^2 + \| \tau^{1/2} a \cdot \nabla u^h \|_{0, \tilde{\Omega}}^2 \]

\[ + \kappa \| \sqrt{1/h^\perp} u^h \|_{0, \tilde{\Gamma}_D}^2 + \kappa \| \sqrt{1/h^\perp} (u^h + \nabla u^h \cdot d) \|_{0, \tilde{\Gamma}^+_D}^2 + \kappa \| \sqrt{h^\perp} u^h \|_{0, \tilde{\Gamma}_D}^2 , \]

\[ C_{SB} = \min \left( \frac{1}{2} - C_I \left( 2\epsilon_1 + \frac{\epsilon_2}{2} \right) , \alpha - \frac{1}{\epsilon_1} , \beta - \frac{1}{2\epsilon_2} \right) . \]

Convergence

Analogous to the one for the Poisson problem, using again Strang’s second lemma.
Incompressible Navier-Stokes Equations

**Strong form**

\[
\rho(u_t + u \cdot \nabla u) + \nabla p - \nabla \cdot (2\mu \varepsilon(u)) - \rho b = 0, \quad \forall x \in \Omega, \quad \Gamma_g^- = \{x \in \Gamma_g \mid g \cdot n < 0\},
\]

\[
\nabla \cdot u = 0, \quad \forall x \in \Omega, \quad \Gamma_h^- = \{x \in \Gamma_h \mid u \cdot n < 0\},
\]

\[
u = g, \quad \forall x \text{ on } \Gamma_g, \quad \Gamma_g^+ = \Gamma_g \setminus \Gamma_g^- \text{ and } \Gamma_h^+ = \Gamma_g \setminus \Gamma_h^-.
\]

\[
- (\rho u \otimes u \chi_{\Gamma_h^-} + \mu I - 2\mu \varepsilon(u)) n = h, \quad \forall x \text{ on } \Gamma_h.
\]

**Shifted boundary method (+ SUPG/PSPG/VMS stabilization)**

Find \( u \in V^h(\tilde{\Omega}) \) and \( p \in Q^h(\tilde{\Omega}) \) such that, \( \forall u \in V^h(\tilde{\Omega}) \) and \( \forall q \in Q^h(\tilde{\Omega}) \),

\[
0 = \text{NS}[\mu](u, p; w, q)
\]

\[
= \langle w, \rho (u_t + u \cdot \nabla u - b) \rangle_{\tilde{\Omega}} - \langle w, \rho u \cdot \tilde{n} (u - g) \rangle_{\Gamma_h^-} - \langle w, h \rangle_{\Gamma_h} - \langle w, (u \cdot \tilde{n}) \rho u \rangle_{\Gamma_h^-} - (\nabla \cdot w, p)_{\tilde{\Omega}} - (q, \nabla \cdot u)_{\tilde{\Omega}}
\]

\[
+ \langle \varepsilon(w), 2\mu \varepsilon(u) \rangle_{\tilde{\Omega}} - \langle w \otimes \tilde{n}, 2\mu \varepsilon(u) - \mu I \rangle_{\Gamma_h^-} - (2\mu \varepsilon(w), u \otimes \tilde{n} \chi_{\Gamma_h^-}(\nabla u d - g))_{\Gamma_h^-}
\]

\[
+ \langle w + \chi_{\Gamma_h^-}(\nabla w) d, \mu / h (\alpha_1 I + \alpha_2 \text{Re}[\mu] n \otimes n) (u + \chi_{\Gamma_h^-}(\nabla u d - g)) \rangle_{\Gamma_h^-}
\]

\[
+ \beta \langle \nabla_{\tilde{\tau}} w, 2\mu h (\nabla_{\tilde{\tau}} u - \nabla_{\tilde{\tau}} g) \rangle_{\Gamma_h^-}.
\]

Euler-Lagrange and conservation statements analogous to the advection-diffusion case.
The Shifted Boundary Method

Convergence test (Navier-Stokes): Decelerating cylinder test

Exact solution:

\[ u_\phi(r, t) = \frac{C}{J_1 \left( \sqrt{\frac{i \omega \rho}{\mu}} r \right)} J_1 \left( \sqrt{\frac{i \omega \rho}{\mu}} R \right) e^{-i \omega t} \]

<table>
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<th>Mesh Size</th>
<th>With Extrapolation</th>
<th>Without Extrap.</th>
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<tr>
<td>K</td>
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<td>1</td>
</tr>
</tbody>
</table>
Turbulent Flows

Formulations based on turbulent viscosities:

\[ \text{Find } u \in V^h(\tilde{\Omega}) \text{ and } p \in Q^h(\tilde{\Omega}) \text{ such that, } \forall u \in V^h(\tilde{\Omega}) \text{ and } \forall q \in Q^h(\tilde{\Omega}), \]
\[ 0 = \text{NS}[\mu + \mu_T](u, p; w, q) + \text{STAB}[\mu + \mu_T](u, p; w, q) \]

- Spalart-Allmaras (SA) model with the Shifted Boundary Method are very similar to the Navier-Stokes equations
- Implicit LES is performed through the VMS stabilization/modeling approach

Wall model for the velocity boundary conditions:
\[ u = g - \nabla u \cdot d \quad \Rightarrow \quad u = g - u_{\text{wall}}(d, \nabla u, \ldots) \]
\[ u^+ = \frac{1}{\kappa} \log y^+ + C^+ \]
\[ u_\tau = \sqrt{\frac{\tau_w}{\rho}}, \]
\[ y^+ = \frac{u_\tau y}{\nu}, \]
\[ u^+ = \frac{u}{u_\tau}, \]
Flow Over a Circular Cylinder

A classical test to validate algorithms for laminar/turbulent flow

<table>
<thead>
<tr>
<th>$Re$</th>
<th>$St$ (vs. reference)</th>
<th>$C_D$ (vs. reference)</th>
<th>Reference source</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>-</td>
<td>2.09 (1.99)</td>
<td>[7]</td>
</tr>
<tr>
<td>100</td>
<td>0.167 (0.164, 0.157)</td>
<td>1.35 (1.34)</td>
<td>[7, 39]</td>
</tr>
<tr>
<td>300</td>
<td>0.211 (0.203, 0.215)</td>
<td>1.38 (1.37)</td>
<td>[39]</td>
</tr>
<tr>
<td>3900</td>
<td>0.203 (0.203)</td>
<td>1.04 (1.00)</td>
<td>[7]</td>
</tr>
</tbody>
</table>


Lift coefficient, $Re = 250$

\[
C_D = 1.43 \\
St = \frac{fD}{U} = 0.206
\]
Flow Over a Circular Cylinder at Re=3,900

A classical test to validate algorithms for turbulent flow simulation

Q-criterion isosurfaces (used to visualize vortex structures)
Flow Over a Circular Cylinder at Re=3,900

A classical test to validate algorithms for turbulent flow simulation

Q-criterion isosurfaces (used to visualize vortical structures)
A More Complicated Shape at Re=3,900

A differential geometry monster (the Monkey Trefoil)
A More Complicated Shape at Re=3,900

A differential geometry monster (the Monkey Trefoil)

Q-criterion isosurfaces (used to visualize vortical structures)

About 25 million elements

Setup (in parallel) ~ 1 minute!
Porous Media Flow (Mixed Formulation)

Original geometry
Surrogate geometry
Solution (flux)

$\Lambda^{-1} \beta + \nabla p = 0$ in $\Omega$
$\nabla \cdot \beta = \phi$ in $\Omega$
$p = p_D$ on $\Gamma_D$
$\beta \cdot n = h_N$ on $\Gamma_N$

~20 million elements
setup time < 1 minute
Shifted *Interface* Method (Mixed Form)

A general (mixed) framework (Darcy-like):

\[
\nabla \cdot \beta = f , \\
\beta = -\kappa \nabla u , \\
\llbracket u \rrbracket = J_1 , \\
\llbracket \beta \rrbracket \cdot n = -J_2 ,
\]

Taylor prolongation operators:

\[
J_1 = \llbracket u + \nabla u \cdot d \rrbracket + O\left(\|d(\bar{x})\|^2\right) , \\
J_2 = -\llbracket \beta + \nabla \beta d \rrbracket \cdot n + O\left(\|d(\bar{x})\|^2\right) .
\]
Shifted *Interface* Method (Mixed Form)

Three-dimensional numerical examples: Complex geometry

- Diffusivity ratio = 1000
- Manufactured solution
- Finest mesh: 23 million el.
- Setup time < 1 minute
- Primary variable 2\textsuperscript{nd} – order accurate
- Flux variable accuracy of order 3/2
Static Linear Elasticity

Shifted boundary formulation for static linear elasticity (work with N. Atallah)

Base Nitsche method:

\[
\int_{\Omega} \sigma_{ij}^u w_{i,j} - \int_{\Gamma} \sigma_{ij}^u w_i n_j - \int_{\Omega} \sigma_{ij}^w n_j u_i - g_i + \int_{\Gamma} \kappa \frac{\gamma}{h} ((n \otimes n) u_i - g_i) w_i \\
+ \int_{\Gamma} \mu \frac{\gamma}{h} ((I - (n \otimes n)) u_i - g_i) (w_i) = \int_{\Omega} b_i w_i
\]

Shifted Nitsche method:

\[
\int_{\Omega} \sigma_{ij}^u w_{i,j} - \int_{\Gamma} \sigma_{ij}^u w_i n_j - \int_{\Gamma} \sigma_{ij}^w n_j (u_i + u_{i,k} d_k - g_i) + \int_{\Gamma} \kappa \frac{\gamma}{h} ((n \otimes n) (u_i + u_{i,k} d_k) - g_i) (w_i + w_{i,k} d_k) \\
+ \int_{\Gamma} \mu \frac{\gamma}{h} ((I - (n \otimes n)) (u_i + u_{i,k} d_k) - g_i) (w_i + w_{i,k} d_k) = \int_{\Omega} b_i w_i
\]
Problem Statement

- Domain ($\Omega$): Unit square
- Zero Dirichlet boundary condition on $\Gamma$

Exact Solution:

$$\begin{bmatrix}
sin(\pi x)sin(\pi y) \\
Csin(\pi x)sin(\pi y)
\end{bmatrix}$$
The Shifted Boundary Method in Action

Example: A linearly elastic compressible solid
The Shifted Boundary Method in Action

Example: A linearly elastic compressible solid

The background domain
The Shifted Boundary Method in Action

Example: A linearly elastic compressible solid

The background domain and the immersed original geometry
The Shifted Boundary Method in Action

Example: A linearly elastic compressible solid

The initial set of active elements (with boundary conditions sidesets)
The Shifted Boundary Method in Action

Deformed configuration of the set of active elements
The Shifted Boundary Method in Action

The intersection of the immersed geometry with the grid
The Shifted Boundary Method in Action

Deformation of the intersected immersed geometry
Problem Statement:

- Domain ($\Omega$): Stent ($r = 1.8$, $L = 14.4$).
- Zero forcing term

\[
\sigma_{ij} n_j = 0 \quad u_i = 0
\]

\[
u_i = -1
\]
Acknowledgments & Future Directions

Ongoing work [contact guglielmo.scovazzi@duke.edu for drafts]

- **Numerical analysis of SBM for Stokes and Darcy operators**
  *(collaboration with C. Canuto)*
- **High-order approximations** *(collaborators: M. Ricchiuto & C. Canuto)*
- **CFD + ROM** *(collaboration with E. Karatzas, G. Stabile, G. Rozza)*
- **Free surface flows and multiphase flows**

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- ONR [Navier-Stokes, free surface flows, acoustics, shallow water flows]
- ARO [solid mechanics]
- ExxonMobil Upstream Research Company [mechanics & geomechanics]