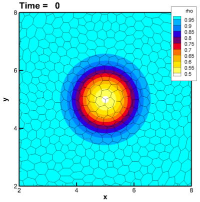
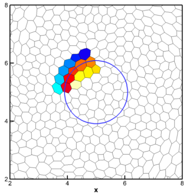
Numerical schemes for complex PDEs & MSCA-IF: SuPerMan

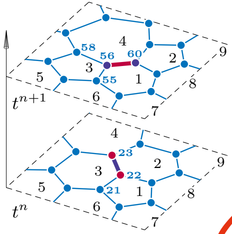
Elena GABURRO

REDEP - Équipe CARDAMOM

21/04/2021





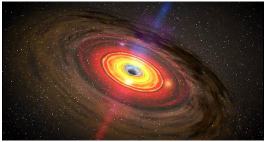


Research domain

New robust and effective numerical methods
for geophysical and astrophysical flows

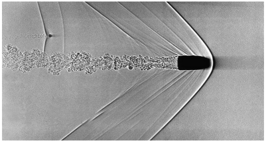
- **Vortical flows: new energies, astrophysics**





- **Multiphase flows: instabilities, fluid-structure interaction**



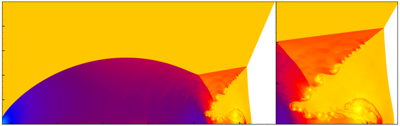


$\Longrightarrow$ **Common mathematical description**

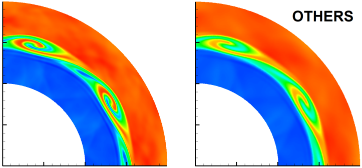
Hyperbolic PDEs: $\partial_t \mathbf{Q} + \nabla \cdot \mathbf{F}(\mathbf{Q}) + \mathbf{B}(\mathbf{Q}) \cdot \nabla \mathbf{Q} = \mathbf{S}(\mathbf{Q})$

Difficulties & corresponding numerical schemes capabilities

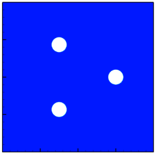
- **Multi-scale phenomena**



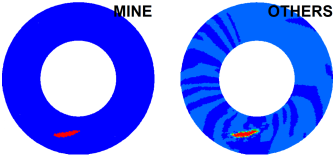
- **Equilibria and physical instabilities**



- **Interface tracking**



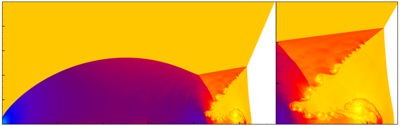
- **Convection without dissipation**



Difficulties & corresponding numerical schemes capabilities

High order FV-DG methods + HPC (Fortran MPI or CUDA)

- **Multi-scale phenomena**



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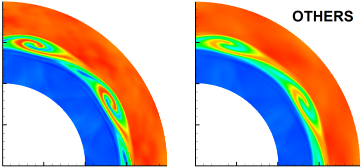

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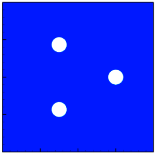

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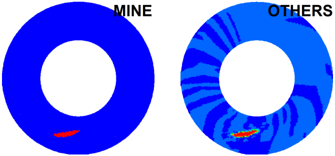
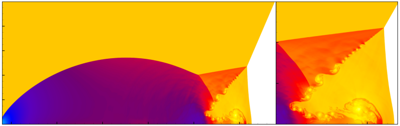


Difficulties & corresponding numerical schemes capabilities

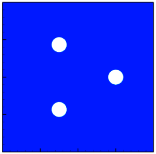
High order FV-DG methods + HPC (Fortran MPI or CUDA)

- **Multi-scale phenomena**



AMR3D

- **Interface tracking**



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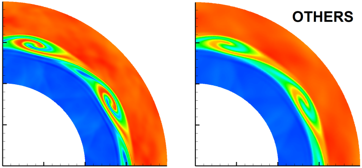
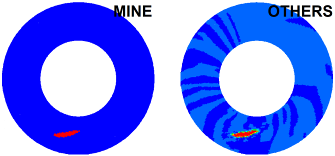


$\Rightarrow$ well balanced methods

Nonconforming-WB CUDA-WB WB-GR

- **Convection without dissipation**



Lagrangian schemes

ALEVoronoi

1. Multiphase: interfaces and fluid structure interaction

- **Diffuse interface methods**: **Idea**: Model + high order

Volume fraction α transported with the interface velocity $\mathbf{v_I}$ $\partial_t \alpha + \mathbf{v_I} \cdot \nabla \alpha = 0$ coupled with the multiphase model

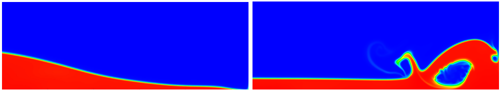
AMR3D

$$\frac{\partial(\alpha\rho)}{\partial t} + \frac{\partial(\alpha\rho v_k)}{\partial x_k} = 0$$

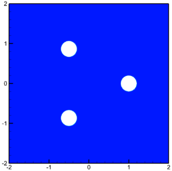
$$\frac{\partial(\alpha\rho v_i)}{\partial t} + \frac{\partial\left(\alpha\rho v_i v_k + \alpha p \delta_{ik}\right)}{\partial x_k} = \rho g_i$$

CUDA-WB

Fluids: dambreak and **non hydrostatic breaking waves**



Rotating solids 2D



Rotating solids 3D

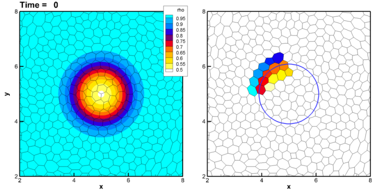
- **Sharp interface methods**: **Idea**: Geometry + high order

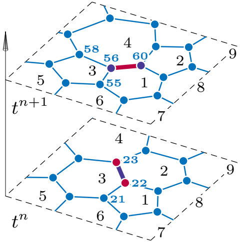
CARDAMOM, in particular: MR & M. Ciallella (PhD)

HOTHYPE

2. Direct ALE on moving meshes with **topology changes**

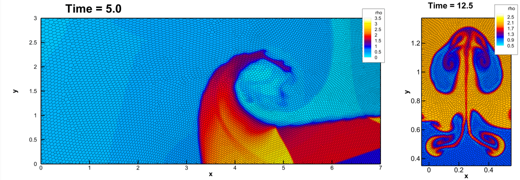
Euler: vortex on moving meshes with reconnection



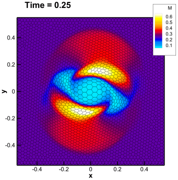


DG $P_2P_2 \to \mathcal{O}3$		
h_f	$\epsilon(\rho)_{L_1}$	$\mathcal{O}$
7.5E-01	1.4E-02	-
6.1E-01	7.2E-03	3.4
3.2E-01	9.3E-04	3.2
2.2E-01	2.8E-04	3.0
1.6E-01	1.2E-04	3.0
DG $P_3P_3 \to \mathcal{O}4$		
h_f	$\epsilon(\rho)_{L_1}$	$\mathcal{O}$
6.1E-01	1.4E-03	-
5.2E-01	7.4E-04	3.7
4.7E-01	4.1E-04	5.9
3.2E-01	7.7E-05	4.4
2.2E-01	1.6E-05	4.0
DG $P_4P_4 \to \mathcal{O}5$		
h_f	$\epsilon(\rho)_{L_1}$	$\mathcal{O}$
1.4E-00	1.1e-02	-
1.0E-00	2.0e-03	5.9
9.8E-01	1.6e-03	4.7
8.9E-01	9.0e-04	5.9
8.5E-01	7.0e-04	5.1

Multiphase: triple point and Rayleigh-Taylor

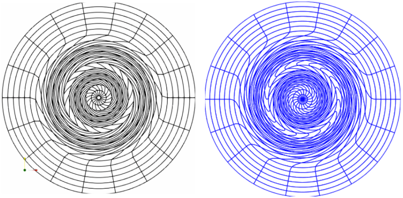


MHD: rotor

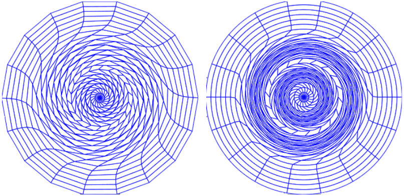


ALEVoronoi to be improved by coupling with **MMG**

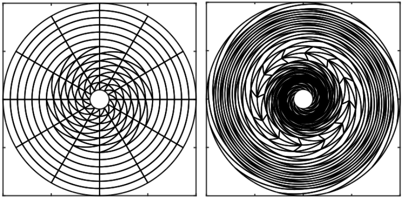
- **Shashkov & Morgan - USA**



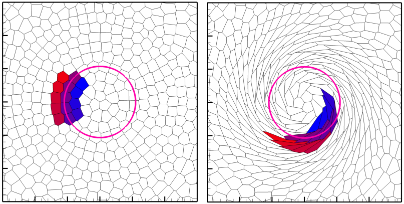
- **Maire & Vilar - CEA**



- **Standard ALE approach**



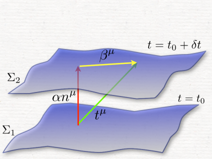
- **Voronoi meshes**



3. MSCA-IF funded project (184.700€)

SuPerMan

Structure Preserving schemes for conservation laws on space time Manifolds



$$\partial_t \mathbf{Q} + \nabla \cdot \mathbf{F}(\mathbf{Q}) + \mathbf{B}(\mathbf{Q}) \cdot \nabla \mathbf{Q} = \mathbf{S}(\mathbf{Q})$$

$\boxed{\mathbf{Q} \leftarrow \gamma_{ij}}$ with $ds^2 = -(\alpha^2 - \beta^i \beta_i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$

Schemes

- **HPC** & **Lagrangian** of very high order
- **asymptotic preserving**, **well balanced** and **metric independent**

for **applications** in

- **Geophysics**: tsunami/hurricane on the geoid
- **Astrophysics**: vortical fluxes around black holes/neutron stars

National/international network:

INRIA (MR + 1 postdoc from 2022) + Uni Trento + Max Planck + Uni Malaga

$\rightarrow$ fundation for **ERC StG** application

$$
\begin{aligned}
\partial_t \tilde{\gamma}_{ij} &= \beta^k 2D_{kij} + \tilde{\gamma}_{ki} B_j^k + \tilde{\gamma}_{kj} B_i^k - 2/3 \tilde{\gamma}_{ij} B_k^k - 2\alpha \left(\tilde{A}_{ij} - 1/3 \tilde{\gamma}_{ij} \mathrm{tr} \tilde{A} \right) - \tau^{-1} (\tilde{\gamma} - 1)\, \tilde{\gamma}_{ij}, \\
\partial_t \ln \alpha &= \beta^k A_k - \alpha g(\alpha)(K - K_0 - 2\Theta c)\,, \\
\partial_t \beta^i &= s \beta^k B_k^i + s f b^i \\
\partial_t \ln \phi &= \beta^k P_k + 1/3 \left(\alpha K - B_k^k \right)\,, \\
\partial_t \tilde{A}_{ij} - \beta^k \partial_k \tilde{A}_{ij} &- \phi^2 \left[-\nabla_i \nabla_j \alpha + \alpha \left(R_{ij} + \nabla_i Z_j + \nabla_j Z_i \right) \right] + \phi^2 1/3 \frac{\tilde{\gamma}_{ij}}{\phi^2} \left[-\nabla^k \nabla_k \alpha + \alpha (R + 2\nabla_k Z^k) \right] \\
&= \tilde{A}_{ki} B_j^k + \tilde{A}_{kj} B_i^k - 2/3 \tilde{A}_{ij} B_k^k + \alpha \tilde{A}_{ij} (K - 2\Theta c) - 2\alpha \tilde{A}_{il} \tilde{\gamma}^{lm} \tilde{A}_{mj} - \tau^{-1}\, \tilde{\gamma}_{ij}\, \mathrm{tr} \tilde{A}\,, \\
\partial_t K - \beta^k \partial_k K &+ \nabla^i \nabla_i \alpha - \alpha (R + 2\nabla_i Z^i) = \alpha K (K - 2\Theta c) - 3\alpha \kappa_1 (1 + \kappa_2) \Theta \\
\partial_t \Theta - \beta^k \partial_k \Theta &- 1/2 \alpha e^2 (R + 2\nabla_i Z^i) = 1/2 \alpha e^2 \left(2/3 K^2 - \tilde{A}_{ij} \tilde{A}^{ij} \right) - \alpha \Theta K c - Z^i \alpha A_i - \alpha \kappa_1 (2 + \kappa_2) \Theta\,, \\
\partial_t \hat{\Gamma}^i - \beta^k \partial_k \hat{\Gamma}^i &+ 4/3 \alpha \tilde{\gamma}^{ij} \partial_j K - 2\alpha \tilde{\gamma}^{ki} \partial_k \Theta - s \tilde{\gamma}^{kl} \partial_{(k} B_{l)}^i - s 1/3 \tilde{\gamma}^{ik} \partial_{(k} B_{l)}^l - s 2\alpha \tilde{\gamma}^{ik} \tilde{\gamma}^{nm} \partial_k \tilde{A}_{nm} \\
&= 2/3 \tilde{\Gamma}^i B_k^k - \tilde{\Gamma}^k B_k^i + 2\alpha \left(\tilde{\Gamma}^i_{jk} \tilde{A}^{jk} - 3 \tilde{A}^{ij} P_j \right) - 2\alpha \tilde{\gamma}^{ki} \left(\Theta A_k + 2/3 K Z_k \right) - 2\alpha \tilde{A}^{ij} A_j \\
&\quad - 4s\, \alpha \tilde{\gamma}^{ik} D_k{}^{nm} \tilde{A}_{nm} + 2\kappa_3 \left(2/3 \tilde{\gamma}^{ij} Z_j B_k^k - \tilde{\gamma}^{jk} Z_j B_k^i \right) - 2\alpha \kappa_1 \tilde{\gamma}^{ij} Z_j \\
\partial_t b^i - s\beta^k \partial_k b^i &= s \left(\partial_t \hat{\Gamma}^i - \beta^k \partial_k \hat{\Gamma}^i - \eta b^i \right)\,,
\end{aligned}
$$

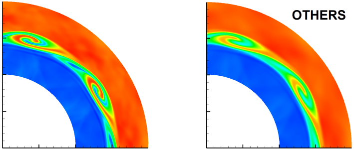
avec les EDP suivantes pour les variables auxiliaires

$$
\begin{aligned}
\partial_t A_k - \beta^l \partial_l A_k &+ \alpha g(\alpha) \left(\partial_k K - \partial_k K_0 - 2c \partial_k \Theta \right) + s \alpha g(\alpha) \tilde{\gamma}^{nm} \partial_k \tilde{A}_{nm} \\
&= +2s\, \alpha g(\alpha) D_k{}^{nm} \tilde{A}_{nm} - \alpha A_k \left(K - K_0 - 2\Theta c \right) \left(g(\alpha) + \alpha g'(\alpha) \right) + B_k^l\, A_l\,, \\
\partial_t B_k^i - s\beta^l \partial_l B_k^i &- s \left(f \partial_k b^i + \alpha^2 \mu\, \tilde{\gamma}^{ij} \left(\partial_k P_j - \partial_j P_k \right) - \alpha^2 \mu\, \tilde{\gamma}^{ij} \tilde{\gamma}^{nl} \left(\partial_k D_{ljn} - \partial_l D_{kjn} \right) \right) = s B_k^l\, B_l^i\,, \\
\partial_t D_{kij} - \beta^l \partial_l D_{kij} &+ s \left(-1/2 \tilde{\gamma}_{mi} \partial_{(k} B_{j)}^m - 1/2 \tilde{\gamma}_{mj} \partial_{(k} B_{i)}^m + 1/3 \tilde{\gamma}_{ij} \partial_{(k} B_{m)}^m \right) + \alpha \partial_k \tilde{A}_{ij} - \alpha 1/3 \tilde{\gamma}_{ij} \tilde{\gamma}^{nm} \partial_k \tilde{A}_{nm} \\
&= B_k^l D_{lij} + B_j^l D_{kli} + B_i^l D_{klj} - 2/3 B_l^l D_{kij} - \alpha 2/3 \tilde{\gamma}_{ij} D_k{}^{nm} \tilde{A}_{nm} - \alpha A_k \left(\tilde{A}_{ij} - 1/3 \tilde{\gamma}_{ij} \mathrm{tr} \tilde{A} \right)\,, \\
\partial_t P_k - \beta^l \partial_l P_k &- 1/3 \alpha \partial_k K + s 1/3 \partial_{(k} B_{i)}^i - s 1/3 \alpha \tilde{\gamma}^{nm} \partial_k \tilde{A}_{nm} \\
&= 1/3 \alpha A_k K + B_k^l P_l - s 2/3 \alpha\, D_k{}^{nm} \tilde{A}_{nm}.
\end{aligned}
$$

3. Preliminary results in astrophysics

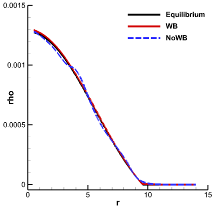
- **Well balancing** for Euler equations (2D)

  Nonconforming-WB

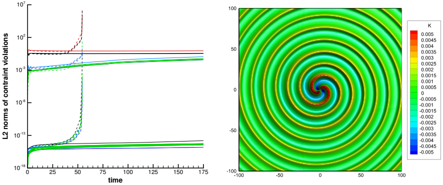


- **Well balancing** for General Relativistic Magnethydrodynamics 1D (GRMHD)

  WB-GR



- **GLM curl cleaning** for Einstein field equations 3D (CCZ4)



Merci beaucoup pour votre attention !