Boundary conditions for Boussinesq-type models in elevation-flux form

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Objectives

Postdoc supported by **Institut des Mathématiques de la Planète Terre**

Supervision : David Lannes and Philippe Bonneton

Long term goal: study extreme waves in littoral area

- Need accurate dispersive model: Boussinesq-type systems
- Boundary conditions are difficult to deal with

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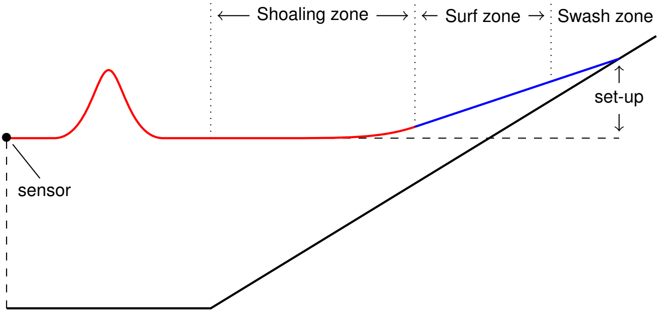

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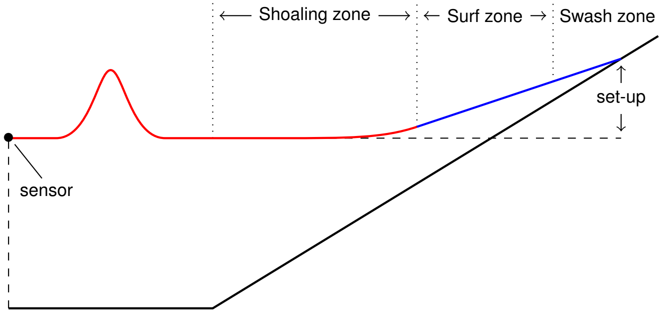
→ We propose a new and efficient method for boundary conditions.



Consider the Boussinesq-Abbott system

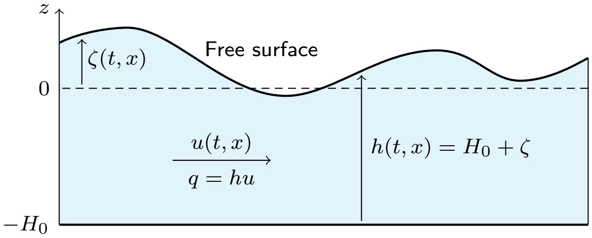
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_{xx}^2)\partial_t q + \partial_x f_{\mathrm{NSW}}(\zeta, q) = 0 \end{cases} \quad \text{in } (0, \ell) \tag{BA}$$

with *generating boundary conditions*

$$\zeta(t, 0) = g_0(t), \qquad \zeta(t, \ell) = g_\ell(t), \tag{1}$$

where $g_0, g_\ell \in C(0, T)$ and

$$\kappa^2 = H_0^2/3, \qquad f_{\mathrm{NSW}}(\zeta, q) = hu^2 + gh^2/2$$



Reformulation of the model

How to account for boundary conditions? How to recover $q_{|_{x=0,\ell}}$?

- Hyperbolic case ($\kappa = 0$) : Riemann invariants
- Dispersive case ($\kappa > 0$) : need to invert $(1 - \kappa^2 \partial_{xx}^2)$ $\rightarrow$ requires knowledge on $\partial_t q_{|_{x=0,\ell}}$
  Lannes and Weynans 2020 "Generating boundary conditions for a Boussinesq system"

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Fix $0 \leq t \leq T$, then $y(x) = \partial_t q(t,x)$ satisfies an ODE of the form

$$\begin{cases} y - \kappa^2 y'' = \phi(x) \\ y(0) = \dot{q}_{|_0}, \quad y(\ell) = \dot{q}_{|_\ell} \end{cases}$$

Equivalently: $y = y_h + y_b$ with $\begin{cases} y_h - \kappa^2 y_h'' = 0 \\ y_h(0) = \dot{q}_{|_0}, \quad y_h(\ell) = \dot{q}_{|_\ell} \end{cases}$ and $\begin{cases} y_b - \kappa^2 y_b'' = \phi(x) \\ y_b(0) = y_b(\ell) = 0 \end{cases}$

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Define R^0 as the inverse of $(1-\kappa^2\partial_{xx}^2)$ with **homogeneous Dirichlet** conditions. Then

$$\partial_t q = \underbrace{-R^0 \partial_x f_{\mathrm{NSW}}}_{y_b} + \underbrace{\mathfrak{s}_{(0)} \dot{q}_{|0} + \mathfrak{s}_{(\ell)} \dot{q}_{|\ell}}_{y_h}$$

where $\begin{cases} (1-\kappa^2\partial_x^2)\mathfrak{s}_{(0)} = 0 \\ \mathfrak{s}_{(0)}(0) = 1, \quad \mathfrak{s}_{(0)}(\ell) = 0 \end{cases}$ and $\begin{cases} (1-\kappa^2\partial_x^2)\mathfrak{s}_{(\ell)} = 0 \\ \mathfrak{s}_{(\ell)}(0) = 0, \quad \mathfrak{s}_{(\ell)}(\ell) = 1 \end{cases}$. (2)

Reformulation of the model

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Proposition 1 (Equivalent formulation with nonlocal flux)

Let (ζ, q) *initially equal to* $(\zeta^{\text{in}}, q^{\text{in}})$. *The two assertions are equivalent:*

1. *The pair* (ζ, q) *satisfies* (BA) *with generating conditions* $\zeta(\cdot, 0) = g_0$ *and* $\zeta(\cdot, \ell) = g_\ell$
2. *The pair* (ζ, q) *satisfies*

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x (R^1 f_{\text{NSW}}) = \mathfrak{s}_{(0)} \dot{q}_{|_0} + \mathfrak{s}_{(\ell)} \dot{q}_{|_\ell} \end{cases} \quad \text{in } (0, \ell), \tag{3}$$

with

$$\begin{pmatrix} \mathfrak{s}'_{(0)}(0) & \mathfrak{s}'_{(\ell)}(0) \\ \mathfrak{s}'_{(0)}(\ell) & \mathfrak{s}'_{(\ell)}(\ell) \end{pmatrix} \begin{pmatrix} \dot{q}_0 \\ \dot{q}_\ell \end{pmatrix} = \frac{1}{\kappa^2} \begin{pmatrix} (R^1 - \text{id})_{|_0} f_{\text{NSW}} \\ (R^1 - \text{id})_{|_\ell} f_{\text{NSW}} \end{pmatrix} - \begin{pmatrix} \ddot{g}_0 \\ \ddot{g}_\ell \end{pmatrix} \tag{4}$$

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Sketch of the proof:

- To get (3), check that $R^0 \partial_x = \partial_x R^1$.
- Apply ∂_x to the discharge eq. from (3); take the trace at $x = 0, \ell$ to get (4).

$$\underbrace{\partial_{xt}^2 q}_{-\partial_{tt}^2 \zeta} + \underbrace{(\partial_{xx} R^1 f_{\text{NSW}})}_{\frac{1}{\kappa^2}(\text{id} - R^1) f_{\text{NSW}}} = \mathfrak{s}'_{(0)} \dot{q}_0 + \mathfrak{s}'_{(\ell)} \dot{q}_\ell$$

Other boundary conditions

Possibility to enforce general boundary conditions

$$\xi^{+}[\zeta, q](t, 0) = g_0(t), \qquad \xi^{-}[\zeta, q](t, \ell) = g_\ell(t). \tag{5}$$

For instance, $\xi^{\pm}$ given by q or Riemann invariants

$$\mathcal{R}_{\pm}(U) = u \pm 2\sqrt{gh}.$$

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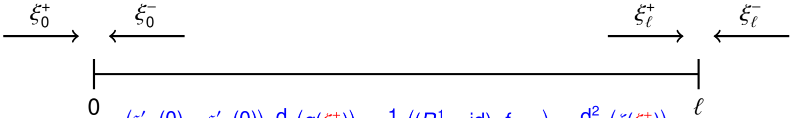
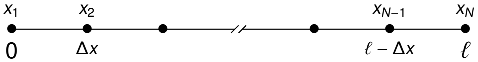
Adapt trace ODE in terms of missing data (outgoing information ξ_0^- and ξ_ℓ^+)



$$\begin{pmatrix} \mathfrak{s}'_{(0)}(0) & \mathfrak{s}'_{(\ell)}(0) \\ \mathfrak{s}'_{(0)}(\ell) & \mathfrak{s}'_{(\ell)}(\ell) \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} q(\xi_0^\pm) \\ q(\xi_\ell^\pm) \end{pmatrix} = \frac{1}{\kappa^2} \begin{pmatrix} (R^1 - \mathrm{id})_{|_0} f_{\mathrm{NSW}} \\ (R^1 - \mathrm{id})_{|_\ell} f_{\mathrm{NSW}} \end{pmatrix} - \frac{\mathrm{d}^2}{\mathrm{d}t^2} \begin{pmatrix} \zeta(\xi_0^\pm) \\ \zeta(\xi_\ell^\pm) \end{pmatrix}$$

Numerical scheme for the reformulated system

Discretize $(0, \ell)$ as follows:



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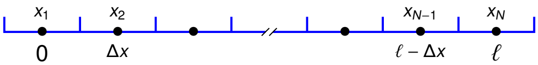

Note $U_i^n = (\zeta_i^n, q_i^n)^T$ the approximation of $\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \begin{pmatrix}\zeta\\ q\end{pmatrix}(t^n, s)\,ds$.

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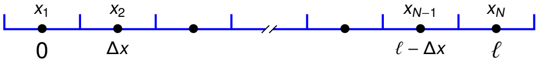
Discretize $(0, \ell)$ as follows:



Note $U_i^n = (\zeta_i^n, q_i^n)^T$ the approximation of $\dfrac{1}{\Delta x} \displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} \begin{pmatrix} \zeta \\ q \end{pmatrix} (t^n, s)\, ds$.

Time stepping procedure

Step 1: Define $\underline{R}^1 f_{\text{NSW}}^n$ as the vector $v \in \mathbb{R}^N$ satisfying

$$\begin{cases} v_i - \kappa^2 \dfrac{v_{i+1} - 2v_i + v_{i-1}}{\Delta x^2} = f_{\text{NSW}}(U_i^n) \ \text{ for } \ 2 \leq i \leq N-1 \\ \dfrac{v_2 - v_1}{\Delta x} = \dfrac{v_N - v_{N-1}}{\Delta x} = 0 \end{cases}.$$

Similar definition for $\underline{\mathfrak{s}}_{(0)}$ and $\underline{\mathfrak{s}}_{(\ell)}$.

Numerical scheme for the reformulated system

Time stepping procedure

Step 2: Approx. trace ODEs using FD scheme to get $\delta_t q_1^n, \delta_t q_N^n$; Update border values

$$\begin{cases} q_1^{n+1} = q_1^n + \Delta t \delta_t q_1^n \\ q_N^{n+1} = q_N^n + \Delta t \delta_t q_N^n \end{cases} \quad \text{and} \quad \begin{cases} \zeta_1^{n+1} = g_0(t^{n+1}) \\ \zeta_N^{n+1} = g_\ell(t^{n+1}) \end{cases} \quad .$$

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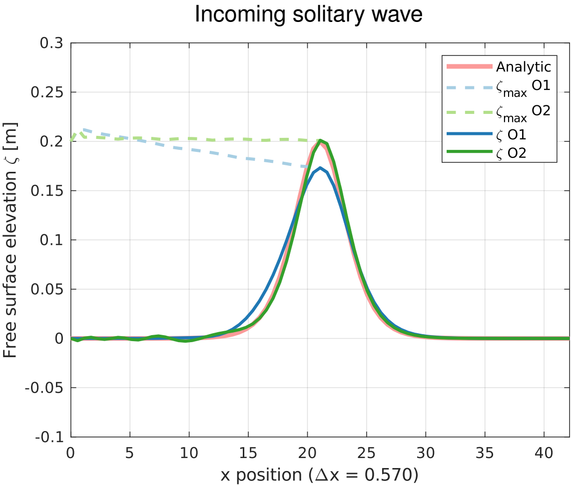
Step 3: For $2 \leq i \leq N$, finite volume update with Lax-Friedrichs numerical flux

$$\begin{cases} \dfrac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} + \dfrac{1}{\Delta x}\left(q_{i+1/2}^n - q_{i-1/2}^n\right) = 0 \\ \dfrac{q_i^{n+1} - q_i^n}{\Delta t} + \dfrac{1}{\Delta x}\left((\underline{R}^1 f_{\mathrm{NSW}}^n)_{i+1/2} - (\underline{R}^1 f_{\mathrm{NSW}}^n)_{i-1/2}\right) = (\mathfrak{s}_{(0)})_i \delta_t q_1^n + (\mathfrak{s}_{(\ell)})_i \delta_t q_N^n \end{cases} .$$

- Second order extension: MacCormack (prediction-correction)
- Advantage: no sponge layer required



Numerical scheme for the reformulated system

Δx	Lax-Friedrichs		MacCormack	
	L^2-error	Order	L^2-error	Order
0.569662	0.052002	–	0.020162	–
0.284831	0.040773	0.35	0.003910	2.37
0.142416	0.024022	0.76	0.000767	2.35
0.071208	0.012777	0.91	0.000161	2.25
0.035604	0.006621	0.95	0.000037	2.12

Table: Error for incoming soliton (ζ enforced)

Δx	Lax-Friedrichs		MacCormack	
	L^2-error	Order	L^2-error	Order
0.284831	0.024246	–	0.001088	–
0.142416	0.014574	0.73	0.000314	1.79
0.071208	0.008471	0.78	0.000101	1.64
0.035604	0.004751	0.83	0.000029	1.80
0.017802	0.002559	0.89	0.000008	1.86

Table: Error for outgoing soliton (ζ enforced)

Boussinesq-Peregrine system with varying bottom

Account for varying bottom with Boussinesq-Peregrine in (ζ, q)-coordinates

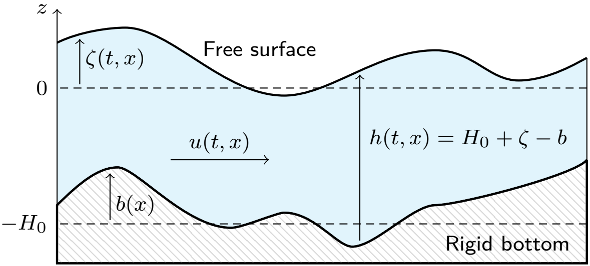
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 + h_b \mathcal{T}_b)\partial_t q + \partial_x f_{\mathrm{NSW}} = -gh\partial_x b \end{cases} \quad \text{in } (0, \ell)\ , \tag{BP}$$

under generating boundary conditions

$$\zeta(t, 0) = g_0(t), \qquad \zeta(t, \ell) = g_\ell(t),$$

with $h_b = H_0 - b$ (depth at rest) and

$$\mathcal{T}_b(\cdot) = -\frac{1}{3h_b}\partial_x\left(h_b^3 \partial_x \frac{(\cdot)}{h_b}\right) + \frac{(\cdot)}{2}\partial_x^2 b, \tag{6}$$



Note R_b^0 the inverse of $(1 + h_b\mathcal{T}_b)$ with **homogeneous Dirichlet** conditions. Then

$$\partial_t q = -R_b^0 \partial_x f_{\mathrm{NSW}} - g R_b^0 (h \partial_x b) + \mathfrak{s}_{(b,0)} \dot{q}_{|_0} + \mathfrak{s}_{(b,\ell)} \dot{q}_{|_\ell} \tag{7}$$

$$\text{where} \quad \begin{cases} (1 + h_b\mathcal{T}_b)\mathfrak{s}_{(b,0)} = 0 \\ \mathfrak{s}_{(b,0)}(0) = 1, \quad \mathfrak{s}_{(b,0)}(\ell) = 0 \end{cases} \quad \text{and} \quad \begin{cases} (1 + h_b\mathcal{T}_b)\mathfrak{s}_{(b,\ell)} = 0 \\ \mathfrak{s}_{(b,\ell)}(0) = 0, \quad \mathfrak{s}_{(b,\ell)}(\ell) = 1 \end{cases} .$$

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where $\begin{cases} (1 + h_b\mathcal{T}_b)\mathfrak{s}_{(b,0)} = 0 \\ \mathfrak{s}_{(b,0)}(0) = 1, \quad \mathfrak{s}_{(b,0)}(\ell) = 0 \end{cases}$ and $\begin{cases} (1 + h_b\mathcal{T}_b)\mathfrak{s}_{(b,\ell)} = 0 \\ \mathfrak{s}_{(b,\ell)}(0) = 0, \quad \mathfrak{s}_{(b,\ell)}(\ell) = 1 \end{cases}$.

Lemma 1 (generalization of $R^0\partial_x = \partial_x R^1$)

We can construct a nonlocal operator R_b^1 such that

$$R_b^0 \partial_x (\cdot) = \frac{1}{\alpha}(\partial_x + \phi)\Big[h_b^2 R_b^1\Big(\frac{(\cdot)}{h_b^2}\Big)\Big] - R_b^0\big((\cdot)\phi\big) \quad \textit{with} \quad \alpha = 1 + \frac{1}{4}(\partial_x b)^2 \quad \textit{and} \quad \phi = \frac{3}{2}\frac{\partial_x b}{h_b}$$

Definition 1 (Nonlocal flux and source terms)

$$\mathfrak{f} = h_b^2 R_b^1\Big(\frac{f_{\mathrm{NSW}}}{h_b^2}\Big), \qquad \mathfrak{S} = R_b^0(-gh\partial_x b) - \frac{\phi}{\alpha}\mathfrak{f} + R_b^0(\phi f_{\mathrm{NSW}})$$

Proposition 2 (Equivalent formulation with nonlocal flux)

Let (ζ, q) initially equal to $(\zeta^{\text{in}}, q^{\text{in}})$. The two assertions are equivalent:

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2. *The pair (ζ, q) satisfies*

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \dfrac{1}{\alpha}\partial_x \mathfrak{f}(U, x) = \mathfrak{S}(U, x) + \mathfrak{s}_{(b,0)}\dot{q}_0 + \mathfrak{s}_{(b,\ell)}\dot{q}_\ell \end{cases} \quad \text{in } (0, \ell) \tag{8}$$

and the trace equations

$$\begin{pmatrix} \mathfrak{s}'_{(b,0)}(0) & \mathfrak{s}'_{(b,\ell)}(0) \\ \mathfrak{s}'_{(b,0)}(\ell) & \mathfrak{s}'_{(b,\ell)}(\ell) \end{pmatrix} \begin{pmatrix} \dot{q}_0 \\ \dot{q}_\ell \end{pmatrix} = \Phi\Big(q_{|0,\ell}, \mathfrak{f}_{|0,\ell}, \partial_x R^0_{b|0,\ell}(\phi f_{\text{NSW}} - gh\partial_x b)\Big) - \begin{pmatrix} \ddot{g}_0 \\ \ddot{g}_\ell \end{pmatrix} \tag{9}$$

where $\Phi : \mathbb{R}^3 \to \mathbb{R}^2$ known.

Enforcing waves through boundary conditions

Question: starting from a wrong initial condition, can we recover the solution by enforcing appropriate boundary conditions?

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Setup:

- approximate U_{ref} solution in $(-\ell, 2\ell)$ with periodic conditions;
- extract $g_0(t^n) := \sigma(t^n)\xi^+[U_{\text{ref}}]_{|_0}(t^n)$ and $g_\ell(t^n) := \sigma(t^n)\xi^-[U_{\text{ref}}]_{|_\ell}(t^n)$;
- approximate new solution in $(0, \ell)$, initially at rest, with g_0, g_ℓ enforced at boundaries;

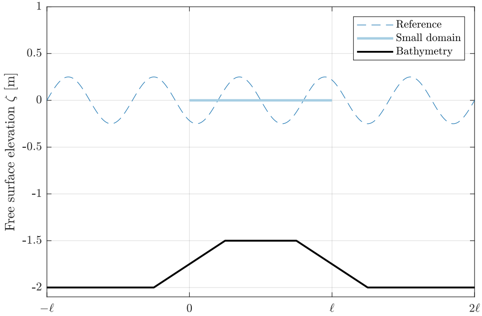
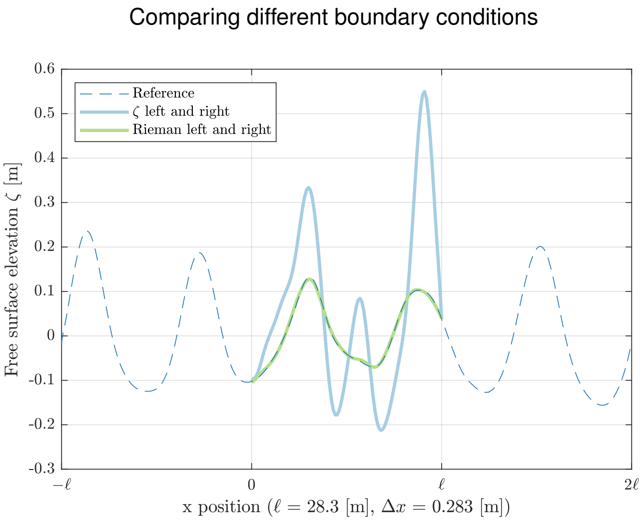
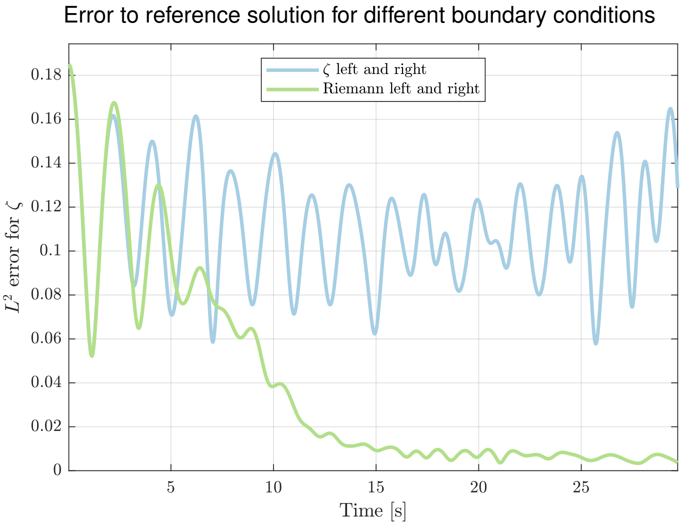


Figure: Initial condition ($(2\pi H_0)^2/\lambda^2 = 0.31$, $a_0/H_0 = 0.25$)

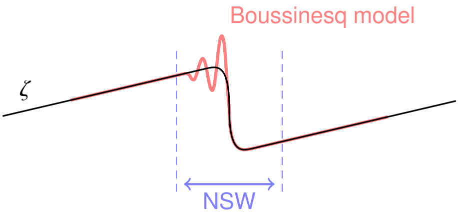
Enforcing waves through boundary conditions




Error to reference solution for different boundary conditions
ζ left and right
Riemann left and right
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
L^2 error for ζ
5
10
15
20
25
Time [s]

Coupling Boussinesq-Peregrine and shallow water models

Motivation: wave breaking with dispersive models → non physical oscillations.

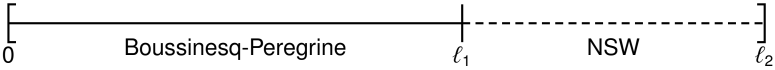


→ Cancel dispersive term near shock wave

$$\begin{cases} \partial_t \zeta_L + \partial_x q_L = 0 \\ \partial_t q_L + \frac{1}{\alpha}\partial_x \mathfrak{f}(U_L) = \mathfrak{S}(U_L) + \mathfrak{s}_{(b,0)}\dot{q}_{L|_0} + \mathfrak{s}_{(b,\ell_1)}\dot{q}_{L|_{\ell_1}} \end{cases} \quad \text{in } (0,\ell_1)$$

$$\begin{cases} \partial_t \zeta_R + \partial_x q_R = 0 \\ \partial_t q_R + \partial_x f_{NSW}(U_R) = -g h_R \partial_x b \end{cases} \quad \text{in } (\ell_1,\ell_2) \qquad (10)$$

Coupling conditions: $\mathcal{R}_+(U_R)_{|_{\ell_1}} = \mathcal{R}_+(U_L)_{|_{\ell_1}}, \quad \mathcal{R}_-(U_L)_{|_{\ell_1}} = \mathcal{R}_-(U_R)_{|_{\ell_1}}$



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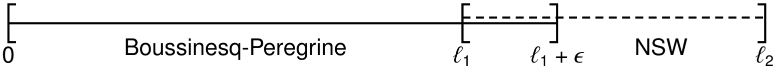
$$\begin{cases} \partial_t \zeta_R + \partial_x q_R = 0 \\ \partial_t q_R + \partial_x f_{NSW}(U_R) = -g h_R \partial_x b \end{cases} \text{in } (\ell_1, \ell_2) \qquad (10)$$

Coupling conditions: $\mathcal{R}_+(U_R)_{|_{\ell_1}} = \mathcal{R}_+(U_L)_{|_{\ell_1}}, \quad \mathcal{R}_-(U_L)_{|_{\ell_1}} = \mathcal{R}_-(U_R)_{|_{\ell_1}}$



In practice, **overlapping** helps to reduce oscillations/reflections

$$\begin{cases} \partial_t \zeta_{\mathrm{L}} + \partial_x q_{\mathrm{L}} = 0 \\ \partial_t q_{\mathrm{L}} + \frac{1}{\alpha}\partial_x \mathfrak{f}(U_{\mathrm{L}}) = \mathfrak{S}(U_{\mathrm{L}}) + \mathfrak{s}_{(b,0)}\dot{q}_{\mathrm{L}|_0} + \mathfrak{s}_{(b,\ell_1)}\dot{q}_{\mathrm{L}|_{\ell_1}} \end{cases} \text{in } (0,\ell_1)$$

$$\begin{cases} \partial_t \zeta_{\mathrm{R}} + \partial_x q_{\mathrm{R}} = 0 \\ \partial_t q_{\mathrm{R}} + \partial_x f_{\mathrm{NSW}}(U_{\mathrm{R}}) = -g h_{\mathrm{R}} \partial_x b \end{cases} \text{in } (\ell_1,\ell_2) \tag{10}$$

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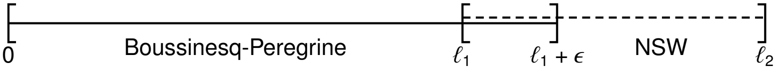
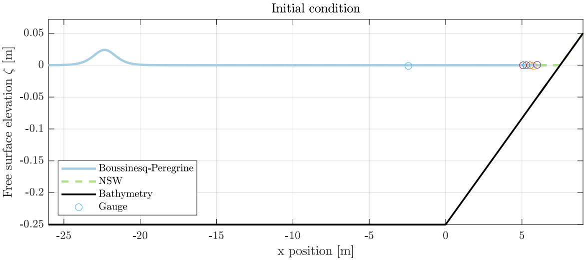
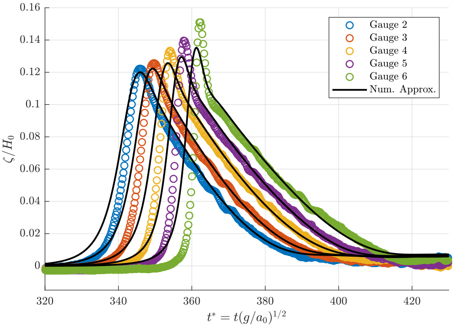


In practice, **overlapping** helps to reduce oscillations/reflections

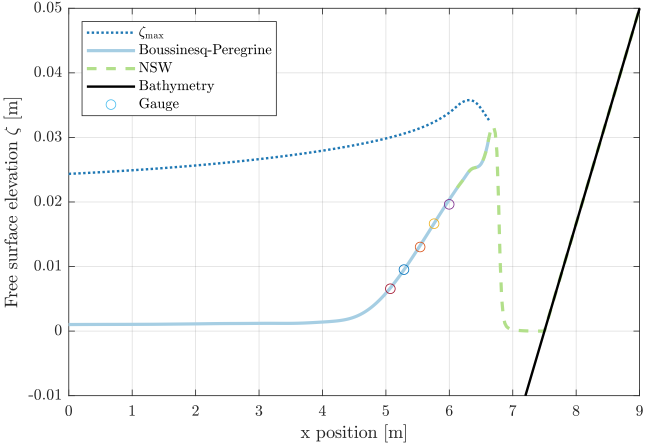
1. Approx. U_{R}^{n+1} with FV scheme + hydrostatic reconstruction; $\mathcal{R}_+(U_{\mathrm{R}})_{|_{\ell_1}} = \mathcal{R}_+(U_{\mathrm{L}})_{|_{\ell_1}}$
2. Approx. U_{L}^{n+1} with Lax-Friedrichs scheme + trace equations; $\mathcal{R}_-(U_{\mathrm{L}})_{|_{\ell_1+\epsilon}} = \mathcal{R}_-(U_{\mathrm{R}})_{|_{\ell_1+\epsilon}}$
3. Convex combination in overlapping area: $U_i^{n+1} = \rho(x_i)U_{\mathrm{L},i}^{n+1} + (1-\rho(x_i))U_{\mathrm{R},i}^{n+1}$.

Experimental testcase: LEGI





Experimental testcase: LEGI



Approximate transparent boundary conditions

Use coupling as a sponge layer to evacuate waves.

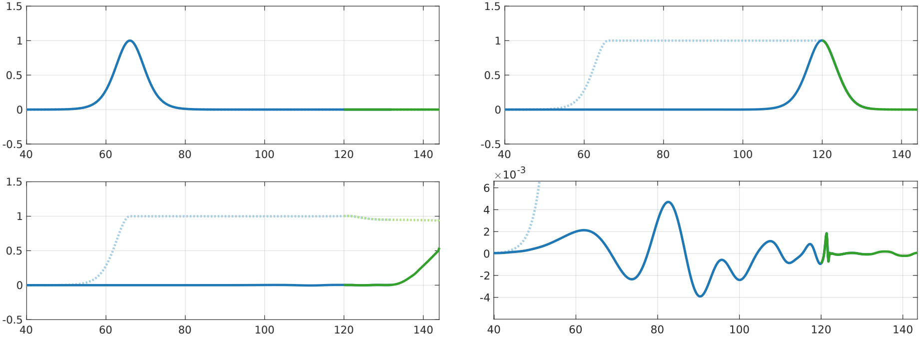


Figure: Outgoing soliton at times t = 0, 9.46, 14.19, and 23.16 [s]. Green domain corresponds to NSW.

Conclusion and perspectives

Over a flat bottom:

- Reformulation of Boussinesq-Abbott
- Generalized boundary conditions
- Efficient 1st and 2nd order schemes

Over a varying bottom:

- Approach extended to Boussinesq-Peregrine
- Coupling with shallow water model
- Implementation + validation (experimental data, various boundary conditions tested)

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Thank you for your attention!