General boundary conditions for the Boussinesq-Abbott model with varying bathymetry

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Séminaire CSM

Institut de Mathématiques de Bordeaux

Objectives

Until november 2023: postdoc supported by **IMPT** → **ANR Bourgeons**

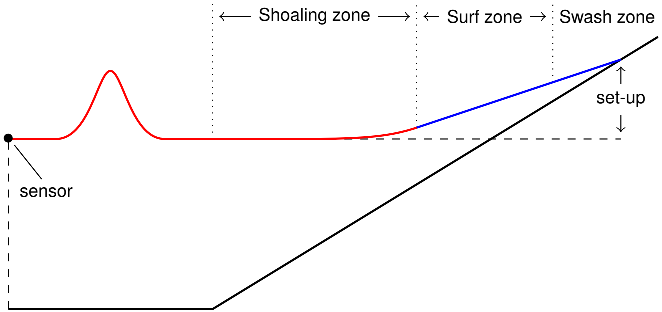
Supervision : David Lannes and Philippe Bonneton

Long term goal: study extreme waves in littoral area

Objectives

- Need accurate dispersive model: Boussinesq-type systems
- Boundary conditions are difficult to deal with

  Recently: Perfectly Matched Layer, source function method → costly



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- Need accurate dispersive model: Boussinesq-type systems
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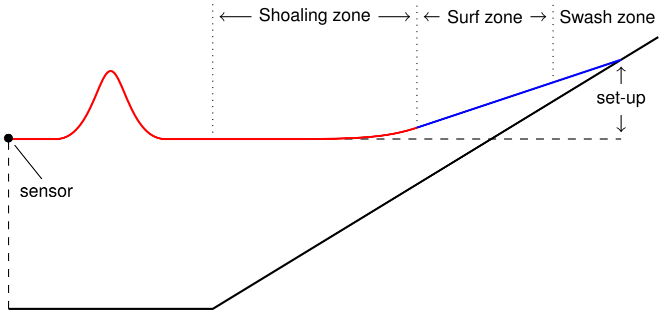
→ We propose a new and efficient method for boundary conditions.



Boussinesq-Abbott: dispersive model over flat bottom

Consider the Boussinesq-Abbott system

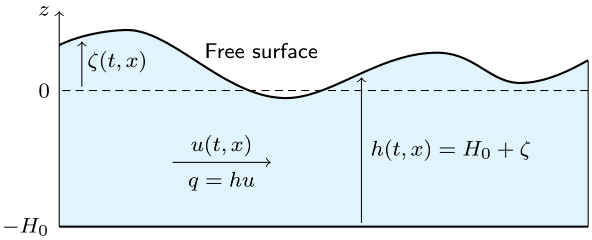
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_{xx}^2)\partial_t q + \partial_x f_{\mathrm{NSW}}(\zeta, q) = 0 \end{cases} \quad \text{in } (0, \ell) \tag{BA}$$

with *generating boundary conditions*

$$\zeta(t, 0) = g_0(t), \qquad \zeta(t, \ell) = g_\ell(t), \tag{1}$$

where $g_0, g_\ell \in C(0, T)$ and

$$\kappa^2 = H_0^2/3, \qquad f_{\mathrm{NSW}}(\zeta, q) = \frac{q^2}{h} + \frac{g}{2}h^2$$



Problematic

How to account for boundary conditions? How to recover $q_{|x=0,\ell}$?

- Hyperbolic case ($\kappa = 0$) : Riemann invariants
- Dispersive case ($\kappa > 0$) : need to invert $(1 - \kappa^2 \partial_{xx}^2)$ $\rightarrow$ requires knowledge on $\partial_t q_{|x=0,\ell}$

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Lannes and Weynans 2020

Equivalent writing of Boussinesq-Abbott with flat bottom over $(0, \infty)$

- substitute $(1 - \kappa^2 \partial_{xx}^2)$ for nonlocal flux & dispersive boundary layer
- ODE on $q_{|x=0}$
- local existence and unicity
- 1st order scheme

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Lannes and Beck 2022

Extension to wave-structure interactions

- transmission problem $\rightarrow$ ODE
- 2nd order scheme
- flat bottom and unbounded domain

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Outline of the talk

Reformulation of the model (flat bottom)

Recall the discharge equation:

$$(1 - \kappa^2 \partial_{xx}^2)\partial_t q + \partial_x f_{\mathrm{NSW}}(\zeta, q) = 0 \qquad \text{in } (0, \ell)$$

Fix $0 \leq t \leq T$, then $y(x) = \partial_t q(t, x)$ satisfies an ODE of the form

$$\begin{cases} y - \kappa^2 y'' = \phi(x) \\ y(0) = \dot{q}_{|_0}, \quad y(\ell) = \dot{q}_{|_\ell} \end{cases}$$

Equivalently: $y = y_h + y_b$ with $\begin{cases} y_h - \kappa^2 y_h'' = 0 \\ y_h(0) = \dot{q}_{|_0}, \quad y_h(\ell) = \dot{q}_{|_\ell} \end{cases}$ and $\begin{cases} y_b - \kappa^2 y_b'' = \phi(x) \\ y_b(0) = y_b(\ell) = 0 \end{cases}$

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Define R^0 as the inverse of $(1 - \kappa^2 \partial_{xx}^2)$ with **homogeneous Dirichlet** conditions at $x = 0, \ell$

$$\Rightarrow \quad \partial_t q = \underbrace{-R^0 \partial_x f_{\mathrm{NSW}}}_{y_b} + \underbrace{\mathfrak{s}_{(0)} \dot{q}_{|_0} + \mathfrak{s}_{(\ell)} \dot{q}_{|_\ell}}_{y_h}$$

$$\text{where} \quad \begin{cases} (1 - \kappa^2 \partial_{xx}^2)\mathfrak{s}_{(0)} = 0 \\ \mathfrak{s}_{(0)}(0) = 1, \quad \mathfrak{s}_{(0)}(\ell) = 0 \end{cases} \quad \text{and} \quad \begin{cases} (1 - \kappa^2 \partial_{xx}^2)\mathfrak{s}_{(\ell)} = 0 \\ \mathfrak{s}_{(\ell)}(0) = 0, \quad \mathfrak{s}_{(\ell)}(\ell) = 1 \end{cases} . \qquad (2)$$

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Proposition 1 (D. Lannes, R.)

Let (ζ, q) *initially equal to* $(\zeta^{\text{in}}, q^{\text{in}})$. *The two assertions are equivalent:*

1. *The pair* (ζ, q) *satisfies* (BA) *with generating conditions* $\zeta(\cdot, 0) = g_0$ *and* $\zeta(\cdot, \ell) = g_\ell$
2. *The pair* (ζ, q) *satisfies*

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x (R^1 f_{\text{NSW}}) = \mathfrak{s}_{(0)} \dot{q}_{|_0} + \mathfrak{s}_{(\ell)} \dot{q}_{|_\ell} \end{cases} \quad \text{in } (0, \ell), \tag{3}$$

with the trace ODEs

$$\begin{pmatrix} \mathfrak{s}'_{(0)}(0) & \mathfrak{s}'_{(\ell)}(0) \\ \mathfrak{s}'_{(0)}(\ell) & \mathfrak{s}'_{(\ell)}(\ell) \end{pmatrix} \begin{pmatrix} \dot{q}_{|_0} \\ \dot{q}_{|_\ell} \end{pmatrix} = \frac{1}{\kappa^2} \begin{pmatrix} (R^1 - \text{id})_{|_0} f_{\text{NSW}} \\ (R^1 - \text{id})_{|_\ell} f_{\text{NSW}} \end{pmatrix} - \begin{pmatrix} \ddot{g}_0 \\ \ddot{g}_\ell \end{pmatrix} \tag{4}$$

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Sketch of the proof:

- To get (3), check that $R^0 \partial_x = \partial_x R^1$.
- Apply ∂_x to the discharge eq. from (3); take the trace at $x = 0, \ell$ to get (4).

$$\underbrace{\partial_{xt}^2 q}_{-\partial_{tt}^2 \zeta} + \underbrace{(\partial_{xx} R^1 f_{\text{NSW}})}_{\frac{1}{\kappa^2}(R^1 - \text{id}) f_{\text{NSW}}} = \mathfrak{s}'_{(0)} \dot{q}_0 + \mathfrak{s}'_{(\ell)} \dot{q}_\ell$$

Reformulation of the model (flat bottom)

Advantages of the reformulated model

- Furnishes evolution equation on $(q_{|_0}, q_{|_\ell})$
- Under some assumptions† and denoting $\mathcal{S} = H^n(0,\ell) \times H^{n+1}(0,\ell)$ for $n \geq 1$:
  - If $(\zeta^{\text{in}}, q^{\text{in}}) \in \mathcal{S}$, reformulated model can be seen as an ODE on $(\zeta, q, q_{|_0}, q_{|_\ell})(t) \in \mathcal{S} \times \mathbb{R}^2$
  - Local existence and uniqueness by Cauchy-Lipschitz theorem

† Assumptions for well-posedness:

$$h_{|_{t=0}} > 0; \qquad g_0, g_\ell \in C^\infty(\mathbb{R}_+); \qquad \begin{cases} \zeta_{|_{t=0}}(0) = g_0(0), \\ \dot{g}_0(0) + \partial_x q^{\text{in}}(0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} \zeta_{|_{t=0}}(\ell) = g_\ell(0), \\ \dot{g}_\ell(0) + \partial_x q^{\text{in}}(\ell) = 0 \end{cases}$$

More general boundary conditions

Possibility to enforce general boundary conditions

$$\xi_0^+(\zeta_{|_0}, q_{|_0})(t) = g_0(t), \qquad \xi_\ell^-(\zeta_{|_\ell}, q_{|_\ell})(t) = g_\ell(t). \tag{5}$$

For instance, $\xi^\pm$ given by q or Saint-Venant Riemann invariants

$$\mathcal{R}_\pm(U) = u \pm 2\sqrt{gh}$$

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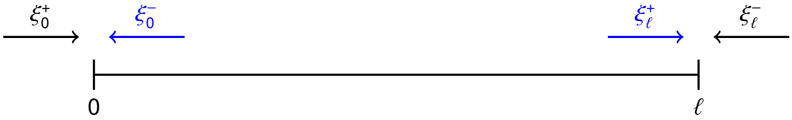
Adapt trace ODE in terms of missing data (outgoing information ξ_0^- and ξ_ℓ^+)

$$\begin{pmatrix} \mathfrak{s}'_{(0)}(0) & \mathfrak{s}'_{(\ell)}(0) \\ \mathfrak{s}'_{(0)}(\ell) & \mathfrak{s}'_{(\ell)}(\ell) \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} q(\xi_0^+, \xi_0^-) \\ q(\xi_\ell^+, \xi_\ell^-) \end{pmatrix} = \frac{1}{\kappa^2} \begin{pmatrix} (R^1 - \mathrm{id})_{|_0} f_{\mathrm{NSW}} \\ (R^1 - \mathrm{id})_{|_\ell} f_{\mathrm{NSW}} \end{pmatrix} - \frac{\mathrm{d}^2}{\mathrm{d}t^2} \begin{pmatrix} \zeta(\xi_0^+, \xi_0^-) \\ \zeta(\xi_\ell^+, \xi_\ell^-) \end{pmatrix}$$



More general boundary conditions

Introduce smooth reconstruction maps $\mathcal{H}_0, \mathcal{H}_\ell$: $\begin{cases} \mathcal{H}_0\Big(\xi_0^+(\zeta,q), \xi_0^-(\zeta,q)\Big) = (\zeta,q) \\ \mathcal{H}_\ell\Big(\xi_\ell^+(\zeta,q), \xi_\ell^-(\zeta,q)\Big) = (\zeta,q) \end{cases}$

Use chain rule to rewrite trace ODEs in term of (ξ_0^-, ξ_ℓ^+):

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q_{|_0} \\ q_{|_\ell} \end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \mathcal{H}_{0,2}(g_0, \xi_0^-) \\ \mathcal{H}_{\ell,2}(\xi_\ell^+, g_\ell) \end{pmatrix}, \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix} \zeta_{|_0} \\ \zeta_{|_\ell} \end{pmatrix} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix} \mathcal{H}_{0,1}(g_0, \xi_0^-) \\ \mathcal{H}_{\ell,1}(\xi_\ell^+, g_\ell) \end{pmatrix}$$

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Three cases considered here:

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$$\xi_0^+(\zeta,q) = \xi_\ell^-(\zeta,q) = \zeta, \qquad \xi_0^-(\zeta,q) = \xi_\ell^+(\zeta,q) = q.$$

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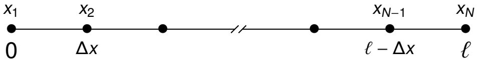
3. Incoming Riemann invariant enforced

$$\xi_0^+(\zeta,q) = \xi_\ell^+(\zeta,q) = u + 2\sqrt{gh}, \qquad \xi_0^-(\zeta,q) = \xi_\ell^-(\zeta,q) = u - 2\sqrt{gh}.$$

Discretize $(0, \ell)$ as follows:



Numerical schemes for the reformulated system

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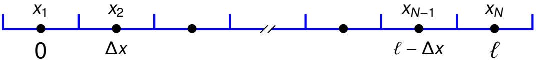

Note $U_i^n = (\zeta_i^n, q_i^n)^T$ the approximation of $\dfrac{1}{\Delta x}\displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} \begin{pmatrix}\zeta\\ q\end{pmatrix}(t^n, s)\,\mathrm{d}s$.

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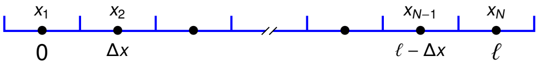

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Time stepping procedure

Step 1: Define $\underline{R}^1 f_{\mathrm{NSW}}^n$ as the vector $v \in \mathbb{R}^N$ satisfying

$$\begin{cases} v_i - \kappa^2 \dfrac{v_{i+1} - 2v_i + v_{i-1}}{\Delta x^2} = f_{\mathrm{NSW}}(U_i^n) \ \text{ for } \ 2 \le i \le N-1 \\ \dfrac{v_2 - v_1}{\Delta x} = \dfrac{v_N - v_{N-1}}{\Delta x} = 0 \end{cases}.$$

Similar definition for boundary layer functions $\underline{\mathfrak{s}}_{(0)}$ and $\underline{\mathfrak{s}}_{(\ell)}$.

Time stepping procedure

Step 2: Approx. trace ODEs using FD scheme to update output functions $(\xi_0^-)^{n+1}, (\xi_\ell^+)^{n+1}$

Numerical schemes for the reformulated system

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Step 3: Recover border elevation and discharge from reconstruction maps

$$(\zeta_1^{n+1}, q_1^{n+1}) = \mathcal{H}_0\Big(g_0^{n+1}, (\xi_0^-)^{n+1}\Big); \qquad (\zeta_N^{n+1}, q_N^{n+1}) = \mathcal{H}_\ell\Big((\xi_\ell^+)^{n+1}, g_\ell^{n+1}\Big)$$

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Step 4: For $2 \leq i \leq N$, finite volumes update with Lax-Friedrichs numerical flux

$$\begin{cases} \dfrac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} + \dfrac{1}{\Delta x}\Big(q_{i+1/2}^n - q_{i-1/2}^n\Big) = 0 \\ \dfrac{q_i^{n+1} - q_i^n}{\Delta t} + \dfrac{1}{\Delta x}\Big((\underline{R}^1 f_{\mathrm{NSW}}^n)_{i+1/2} - (\underline{R}^1 f_{\mathrm{NSW}}^n)_{i-1/2}\Big) = (\mathfrak{s}_{(0)})_i \delta_t q_1^n + (\mathfrak{s}_{(\ell)})_i \delta_t q_N^n \end{cases}.$$

Numerical schemes for the reformulated system

Second order extension: MacCormack prediction-correction method

Beck, Lannes, Weynans (preprint 2023)

- left upwinding during prediction
- right upwinding during correction
- final stage: average prediction and correction (similar to Heun)

Adapt to finite domain $(0, \ell)$ + general boundary conditions

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Solitary wave testcase: wave travelling without deforming

$$\zeta(t,x) = \widetilde{\zeta}(x - ct), \quad q(t,x) = \widetilde{q}(x - ct)$$

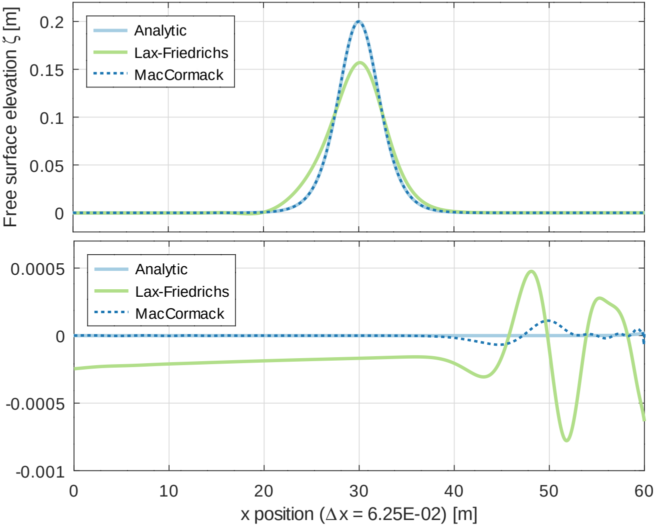
Compare different boundary conditions in two settings:

1. Generation of an incoming solitary wave
2. Evacuation of a solitary wave initially in the domain

→ no sponge layer shall be used

Solitary wave testcase

Figure: Top: incoming solitary wave, bottom: outgoing solitary wave



Solitary wave testcase

Δx	ζ enforced		q enforced		$R^{\pm}$ enforced	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
6.25E-02	3.263E-03	–	3.484E-03	–	3.312E-03	–
4.42E-02	2.387E-03	0.90	2.539E-03	0.91	2.420E-03	0.91
3.12E-02	1.727E-03	0.93	1.833E-03	0.94	1.750E-03	0.93
2.21E-02	1.244E-03	0.95	1.318E-03	0.95	1.260E-03	0.95
1.56E-02	8.906E-04	0.97	9.411E-04	0.97	9.015E-04	0.97

Table: Lax-Friedrichs scheme for the incoming solitary wave

Solitary wave testcase

Δx	ζ enforced		q enforced		$R^{\pm}$ enforced	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
6.25E-02	3.263E-03	–	3.484E-03	–	3.312E-03	–
4.42E-02	2.387E-03	0.90	2.539E-03	0.91	2.420E-03	0.91
3.12E-02	1.727E-03	0.93	1.833E-03	0.94	1.750E-03	0.93
2.21E-02	1.244E-03	0.95	1.318E-03	0.95	1.260E-03	0.95
1.56E-02	8.906E-04	0.97	9.411E-04	0.97	9.015E-04	0.97

Table: Lax-Friedrichs scheme for the incoming solitary wave

Δx	ζ enforced		q enforced		$R^{\pm}$ enforced	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
4.42E-02	1.785E-03	–	2.031E-03	–	1.070E-04	–
3.12E-02	1.322E-03	0.87	1.514E-03	0.85	8.401E-05	0.70
2.21E-02	9.725E-04	0.89	1.119E-03	0.87	6.503E-05	0.74
1.56E-02	7.053E-04	0.93	8.145E-04	0.92	4.939E-05	0.79
1.11E-02	5.098E-04	0.94	5.903E-04	0.93	3.711E-05	0.83

Table: Lax-Friedrichs scheme for the outgoing solitary wave.

Solitary wave testcase

Δx	ζ enforced		q enforced		$R^{\pm}$ enforced	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
5.00E-01	2.473E-03	–	3.789E-03	–	2.783E-03	–
2.50E-01	6.670E-04	1.89	1.026E-03	1.89	7.199E-04	1.95
1.25E-01	1.696E-04	1.98	2.662E-04	1.95	1.832E-04	1.97
6.25E-02	4.312E-05	1.98	6.804E-05	1.97	4.624E-05	1.99
3.12E-02	1.107E-05	1.96	1.731E-05	1.97	1.162E-05	1.99

Table: MacCormack scheme for the incoming solitary wave.

Solitary wave testcase

Δx	ζ enforced		q enforced		$R^{\pm}$ enforced	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
5.00E-01	2.473E-03	–	3.789E-03	–	2.783E-03	–
2.50E-01	6.670E-04	1.89	1.026E-03	1.89	7.199E-04	1.95
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Table: MacCormack scheme for the incoming solitary wave.

Δx	ζ enforced		q enforced		$R^{\pm}$ enforced	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
5.00E-01	1.885E-03	–	4.437E-03	–	1.993E-03	–
2.50E-01	5.568E-04	1.76	1.257E-03	1.82	5.607E-04	1.83
1.25E-01	1.459E-04	1.93	3.252E-04	1.95	1.318E-04	2.09
6.25E-02	3.674E-05	1.99	8.223E-05	1.98	2.826E-05	2.22
3.12E-02	9.162E-06	2.00	2.067E-05	1.99	7.085E-06	2.00

Table: MacCormack scheme for the outgoing solitary wave.

Boussinesq-Abbott system with varying bottom

The Boussinesq-Abbott system now reads

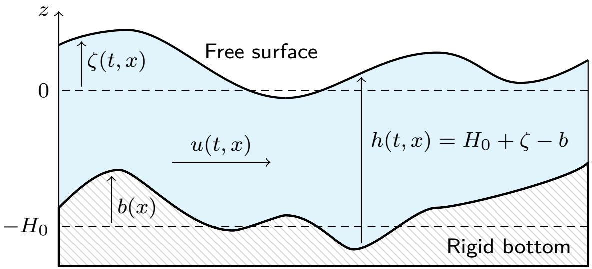
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 + h_b \mathcal{T}_b)\partial_t q + \partial_x f_{\mathrm{NSW}} = -gh\partial_x b \end{cases} \quad \text{in } (0, \ell)\ , \tag{BA}$$

under generating boundary conditions

$$\zeta(t, 0) = g_0(t), \qquad \zeta(t, \ell) = g_\ell(t),$$

with $h_b = H_0 - b$ (depth at rest) and

$$\mathcal{T}_b(\cdot) = -\frac{1}{3h_b}\partial_x\left(h_b^3 \partial_x \frac{(\cdot)}{h_b}\right) + \frac{(\cdot)}{2}\partial_{xx}^2 b, \tag{6}$$



Boussinesq-Abbott system with varying bottom

Recall of the main problematic

To invert $(1 + h_b \mathcal{T}_b)$, we need knowledge on $(\dot{q}_{|_0}, \dot{q}_{|_\ell})$

- Extend nonlocal reformulation to varying bottoms
- Obtain trace ODEs for missing data $(q_{|_0}, q_{|_\ell})$

Boussinesq-Abbott system with varying bottom

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- Extend nonlocal reformulation to varying bottoms
- Obtain trace ODEs for missing data $(q_{|_0}, q_{|_\ell})$

Version adapted to well-balancedness

Lake at rest $(\zeta, q) = (0, 0)$ is a steady state $\quad \Rightarrow \quad \partial_x f_{\mathrm{NSW}}(0,0) = -gh_b\partial_x b$

$$
\text{(BA)} \;\Leftrightarrow\; \begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 + h_b\mathcal{T}_b)\partial_t q + \partial_x\Big(\underbrace{f_{\mathrm{NSW}}(\zeta, q) - f_{\mathrm{NSW}}(0,0)}_{\widetilde{f}_{\mathrm{NSW}}(\zeta, q) = \frac{q^2}{h} + \frac{g}{2}(\zeta^2 + 2H_0\zeta)}\Big) = -g\underbrace{(h - h_b)}_{\zeta}\partial_x b \end{cases} \quad \text{in } (0, \ell)\,,
$$

- we have $\widetilde{f}_{\mathrm{NSW}}(\zeta, q) = -g\zeta\partial_x b = 0$ if $(\zeta, q) = (0, 0)$
- easy to preserve at discrete level → scheme naturally well-balanced
- advantage: can be generalized to other steady states

Nonlocal reformulation

Note R_b^0 the inverse of $(1 + h_b\mathcal{T}_b)$ with **homogeneous Dirichlet** conditions at $x = 0, \ell$

$$\Rightarrow \quad \partial_t q = -R_b^0 \partial_x \widetilde{f}_{\mathrm{NSW}} + R_b^0(-g\zeta\partial_x b) + \mathfrak{s}_{(b,0)} \dot{q}_{|_0} + \mathfrak{s}_{(b,\ell)} \dot{q}_{|_\ell} \tag{7}$$

$$\text{where} \quad \begin{cases} (1 + h_b\mathcal{T}_b)\mathfrak{s}_{(b,0)} = 0 \\ \mathfrak{s}_{(b,0)}(0) = 1, \quad \mathfrak{s}_{(b,0)}(\ell) = 0 \end{cases} \quad \text{and} \quad \begin{cases} (1 + h_b\mathcal{T}_b)\mathfrak{s}_{(b,\ell)} = 0 \\ \mathfrak{s}_{(b,\ell)}(0) = 0, \quad \mathfrak{s}_{(b,\ell)}(\ell) = 1 \end{cases} .$$

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where $\begin{cases} (1 + h_b\mathcal{T}_b)\mathfrak{s}_{(b,0)} = 0 \\ \mathfrak{s}_{(b,0)}(0) = 1, \quad \mathfrak{s}_{(b,0)}(\ell) = 0 \end{cases}$ and $\begin{cases} (1 + h_b\mathcal{T}_b)\mathfrak{s}_{(b,\ell)} = 0 \\ \mathfrak{s}_{(b,\ell)}(0) = 0, \quad \mathfrak{s}_{(b,\ell)}(\ell) = 1 \end{cases}$.

Lemma 1 (generalization of $R^0\partial_x = \partial_x R^1$)

We can construct a nonlocal operator R_b^1 such that

$$R_b^0 \partial_x(\cdot) = \left(\partial_x + \beta + \frac{\partial_x \alpha}{\alpha}\right)\left[\frac{h_b^2}{\alpha} R_b^1\left(\frac{(\cdot)}{h_b^2}\right)\right] - R_b^0\Big((\cdot)\beta\Big) \quad \textit{with} \quad \begin{cases} h_b = H_0 - b \\ \alpha = 1 + \frac{1}{4}(\partial_x b)^2 \\ \beta = \frac{3}{2} h_b^{-1} \partial_x b \end{cases}$$

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where $\begin{cases} (1 + h_b\mathcal{T}_b)\mathfrak{s}_{(b,0)} = 0 \\ \mathfrak{s}_{(b,0)}(0) = 1, \quad \mathfrak{s}_{(b,0)}(\ell) = 0 \end{cases}$ and $\begin{cases} (1 + h_b\mathcal{T}_b)\mathfrak{s}_{(b,\ell)} = 0 \\ \mathfrak{s}_{(b,\ell)}(0) = 0, \quad \mathfrak{s}_{(b,\ell)}(\ell) = 1 \end{cases}$.

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Definition 1 (Nonlocal flux and source terms)

$$\mathfrak{f} = \frac{h_b^2}{\alpha}R_b^1\Big(\frac{\widetilde{\mathfrak{f}}_{\mathrm{NSW}}}{h_b^2}\Big), \qquad \mathfrak{B} = R_b^0\Big(-gh\partial_x b + \beta\widetilde{\mathfrak{f}}_{\mathrm{NSW}}\Big) - \Big(\beta + \frac{\partial_x\alpha}{\alpha}\Big)\mathfrak{f}$$

Nonlocal reformulation

Proposition 2 (D. Lannes, R.)

Let (ζ, q) *initially equal to* $(\zeta^{\mathrm{in}}, q^{\mathrm{in}}) \in H^n(0,\ell) \times H^{n+1}(0,\ell)$. *The two assertions are equivalent:*

1. *The pair* (ζ, q) *satisfies* (BA) *with generating conditions* $\zeta(\cdot, 0) = g_0$ *and* $\zeta(\cdot, \ell) = g_\ell$
2. *The pair* (ζ, q) *satisfies*

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \mathfrak{f}(\zeta, q) = \mathfrak{B}(\zeta, q) + \mathfrak{s}_{(b,0)} \dot{q}_{|0} + \mathfrak{s}_{(b,\ell)} \dot{q}_{|\ell} \end{cases} \quad \textit{in } (0,\ell) \tag{8}$$

and the trace equations

$$\begin{pmatrix} \mathfrak{s}'_{(b,0)}(0) & \mathfrak{s}'_{(b,\ell)}(0) \\ \mathfrak{s}'_{(b,0)}(\ell) & \mathfrak{s}'_{(b,\ell)}(\ell) \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} q_{|0} \\ q_{|\ell} \end{pmatrix} = \begin{pmatrix} \mathcal{F}_0(\zeta, q, \zeta_{|0}, q_{|0}) \\ \mathcal{F}_\ell(\zeta, q, \zeta_{|\ell}, q_{|\ell}) \end{pmatrix} - \frac{\mathrm{d}^2}{\mathrm{d}t^2} \begin{pmatrix} \zeta_{|0} \\ \zeta_{|\ell} \end{pmatrix} \tag{9}$$

where $\mathcal{F}_0, \mathcal{F}_\ell : H^n(0,\ell) \times H^{n+1}(0,\ell) \times \mathbb{R}^2 \to \mathbb{R}$ *are known.*

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where $\mathcal{F}_0, \mathcal{F}_\ell : H^n(0,\ell) \times H^{n+1}(0,\ell) \times \mathbb{R}^2 \to \mathbb{R}$ *are known.*

- Can be adapted to general boundary conditions $(\xi_0^+, \xi_\ell^-) = (g_0, g_\ell)$
- Local existence and unicity by Cauchy-Lipschitz
- Numerical discretization same as before

Setup:

- Reference solution on $(-\ell, 2\ell)$ with periodic conditions (very fine mesh)
- Generate boundary conditions for small domain $(0, \ell)$
- Compare simulations in $(0, \ell)$ with reference solution

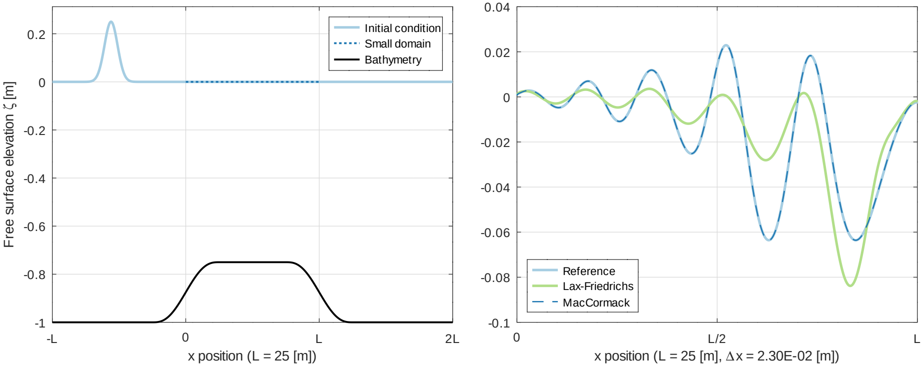

Figure: Gaussian over bump (left: initial time, right: t = 15 [s])

Question: starting from a wrong initial condition, can we recover the reference solution by enforcing appropriate boundary conditions?

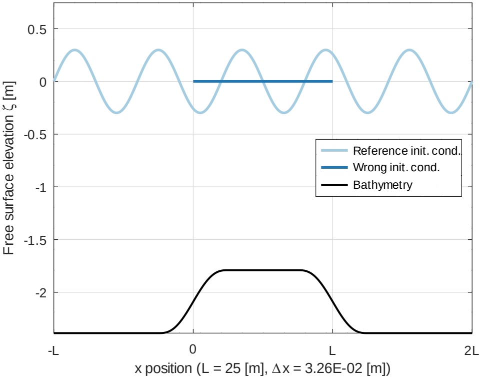

Figure: Sine over bump (shallowness $(2\pi H_0)^2/\lambda^2 = 1$)

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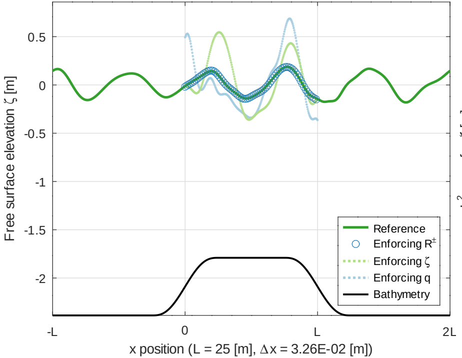

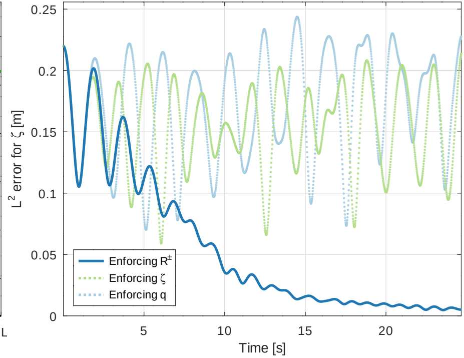

Figure: Sine over bump (shallowness $(2\pi H_0)^2/\lambda^2 = 1$)

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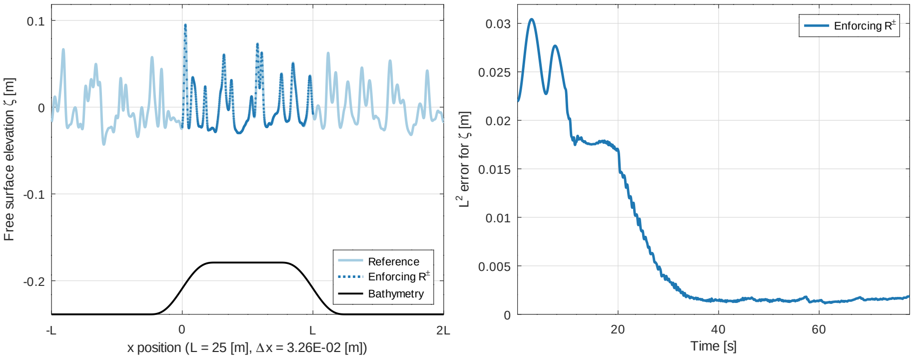
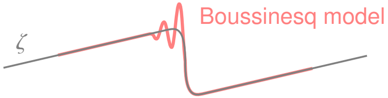
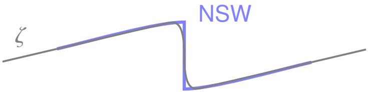


Figure: Sine over bump (shallowness $(2\pi H_0)^2/\lambda^2 = 10^{-2}$)

Perspectives: coupling with the shallow water model

Motivation: wave breaking with dispersive models → non physical oscillations.





Structure-preserving schemes (entropy stable, well-balanced)

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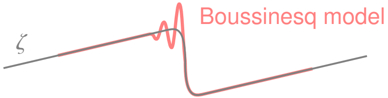
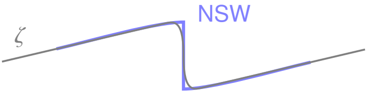




Structure-preserving schemes (entropy stable, well-balanced)

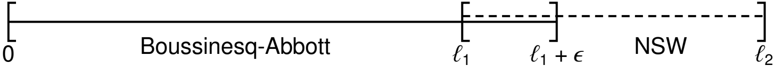
$$\begin{cases} \partial_t \zeta_L + \partial_x q_L = 0 & \\ \partial_t q_L + \partial_x \mathfrak{f}(\zeta_L, q_L) = \mathfrak{B}(\zeta_L, q_L) + \mathfrak{s}_{(0)} \dot{q}_{L|_{x=0}} + \mathfrak{s}_{(\ell_1)} \dot{q}_{L|_{x=\ell_1}} & \text{in } (0, \ell_1) \\ \partial_t \zeta_R + \partial_x q_R = 0 & \\ \partial_t q_R + \partial_x f_{NSW}(U_R) = -g h_R \partial_x b & \text{in } (\ell_1, \ell_2) \end{cases}$$

Coupling conditions: $\xi^+_{\ell_1}(U_{R|_{\ell_1}}) = \xi^+_{\ell_1}(U_{L|_{\ell_1}}), \quad \xi^-_{\ell_1}(U_{L|_{\ell_1}}) = \xi^-_{\ell_1}(U_{R|_{\ell_1}})$

Perspectives: coupling with the shallow water model

Preliminary observations and ideas

- Weird artifacts near coupling interface
- Much improved with a spatial overlapping...
- ... but difficult to interpret at continuous level



Perspectives: coupling with the shallow water model

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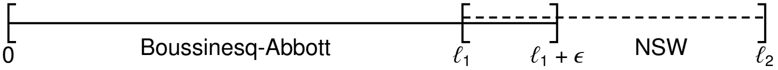
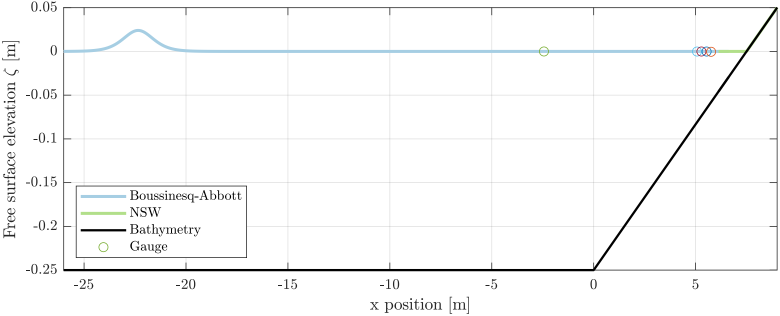


1. Approx. U_R^{n+1} with FV scheme + hydrostatic reconstruction; $\mathcal{R}_+(U_\mathrm{R})_{|\ell_1} = \mathcal{R}_+(U_\mathrm{L})_{|\ell_1}$
2. Approx. U_L^{n+1} with Lax-Friedrichs scheme + trace equations; $\mathcal{R}_-(U_\mathrm{L})_{|\ell_1+\epsilon} = \mathcal{R}_-(U_\mathrm{R})_{|\ell_1+\epsilon}$
3. Convex combination in overlapping area: $U_i^{n+1} = \rho(x_i)U_{\mathrm{L},i}^{n+1} + (1-\rho(x_i))U_{\mathrm{R},i}^{n+1}$.

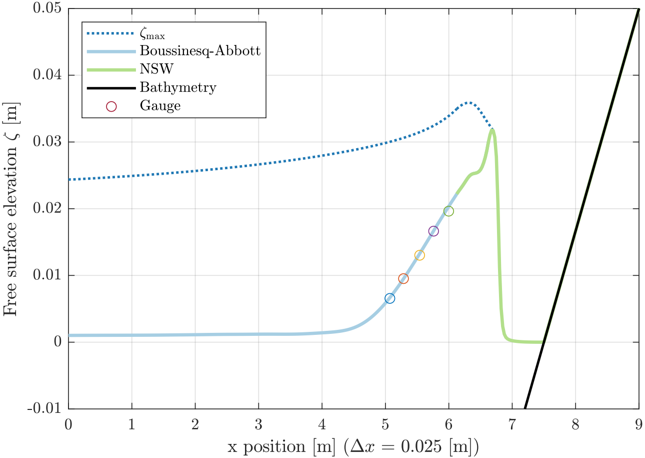
Experimental testcase: LEGI

Figure: Initial condition



Experimental testcase: LEGI

Figure: Time t = 18.5 [s]



Filippini, Bellec, Colin, Ricchiuto 2014

Boussinesq models can be written in (ζ, q) or (ζ, v) form

- in the 1st case, shoaling is **underestimated**;
- in the 2nd case, shoaling is **overestimated**;

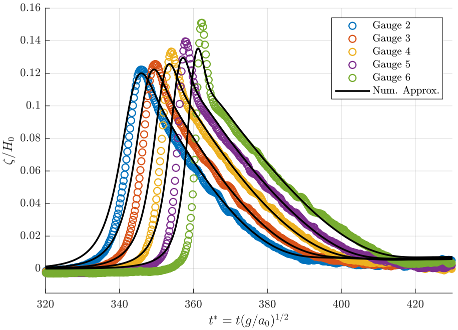


Figure: Predicted elevation for LEGI experiment

Issue: risk of bias when predicting extreme waves

Possible fix: try mixing the (ζ, q) and (ζ, v) formulations

Conclusion

Over a flat bottom:

- Reformulation of Boussinesq-Abbott in bounded domain
- Generalized boundary conditions
- Efficient 1st and 2nd order schemes

Over a varying bottom:

- Similar reformulation
- Numerical validation & asymptotic stability

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- Coupling with shallow water model
- Improve shoaling description
- Statistics of extreme waves: impact of bathymetry

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Thank you for your attention!

Constructing nonlocal operator R_b^1

Write dispersive operator $(1 + h_b \mathcal{T}_b)$ as $\alpha_b + h_b S^*(h_b S(\cdot))$, where

$$\alpha_b = 1 + \frac{1}{4}(\partial_x b)^2, \qquad S(\cdot) = -\frac{h_b}{\sqrt{3}}\partial_x\Big(\frac{\cdot}{h_b}\Big) + \frac{\sqrt{3}}{2}\frac{\partial_x b}{h_b}$$

Definition 2

$$R_b^1 : f \in L^2(0,\ell) \longmapsto u \in H^2(0,\ell) \ \text{ s.t. } \begin{cases} \Big[1 + S\Big(\frac{h_b}{\alpha_b}(h_b S)^*\Big)\Big]u = f \\ (h_b S)^* u(0) = (h_b S)^* u(\ell) = 0 \end{cases}$$

Lemma 3 (Commutation)

$$\forall f \in L^2(0,\ell), \ R_b^0(h_b S^* f) = \frac{h_b}{\alpha_b}(h_b S)^*\Big(R_b^1\Big(\frac{f}{h_b}\Big)\Big)$$

Approximate transparent boundary conditions

Use coupling as a sponge layer to evacuate waves.

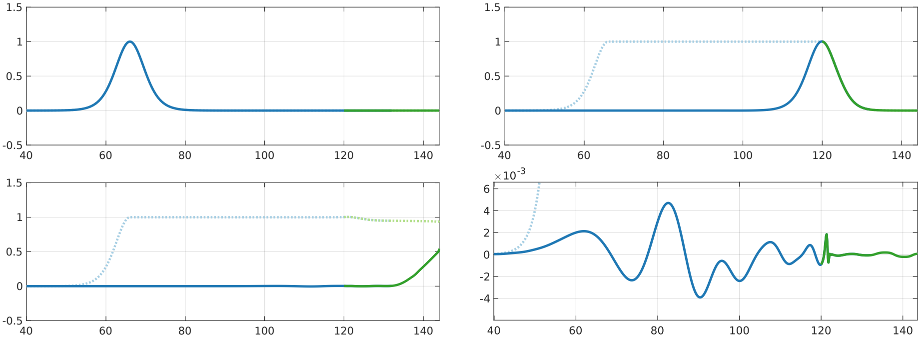

Figure: Outgoing soliton at times $t = 0, 9.46, 14.19$, and $23.16\,[s]$. Green domain corresponds to NSW.

Perspectives: statistics of extreme waves

Goal: study impact of bathymetry on extreme waves formation.

- Complex waves: different scales (swell/infragravity waves), two-ways propagation
- Need to randomly generate input data (ξ_0^+, ξ_ℓ^-)
- Probability distribution for incoming waves: Fuhrman, Klahn and Zhai 2023

$$p(\zeta) = \frac{1}{\sqrt{2\pi}} e^{-\zeta^2/2}\Big(1 + \frac{1}{6}\mathcal{S}(\zeta^3 - 3\zeta)\Big) + O(\epsilon^2), \qquad \begin{cases} \mathcal{S} = \text{skewness parameter} \\ \epsilon \ = \text{wave steepness} \end{cases}$$

- Efficient code required: implement new methods in UHAINA