Fully implicit kinetic scheme for the 1D Saint-Venant system

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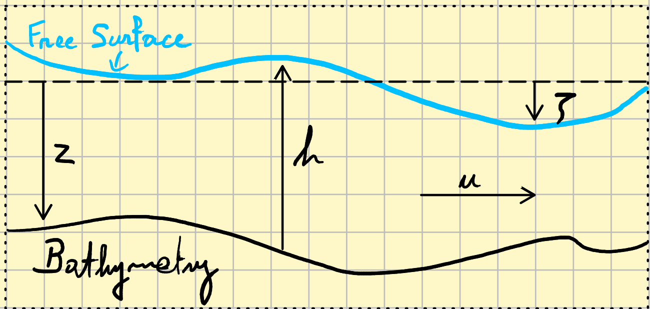
The 1D Saint-Venant system : notations & properties



$h(t,x) \longrightarrow$ Water height

$z(x) \longrightarrow$ Bathymetry

$\zeta = h + z \longrightarrow$ Free surface

$u(t,x) \longrightarrow$ Horizontal velocity

Hyperbolic system of conservation laws :

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + \frac{g}{2} h^2\right) = -gh\,\partial_x z \end{cases} \quad (SV) \longrightarrow \boxed{\partial_t U + \partial_x F(U) = S(U,z)}$$

↳ Eigenvalues of DF given by $\lambda^{\pm} = u \pm \sqrt{2}c$, $c = \sqrt{g\frac{h}{2}}$ sound speed

Positivity: $h \geq 0$, stationary state: $h + z = K$, $u = 0$

Energy $E(t,x) = \frac{hu^2}{2} + \frac{gh^2}{2} + ghz$ satisfies $\partial_t E + \partial_x\left[\left(E + \frac{gh^2}{2}\right)u\right] \leq 0 \quad (1)$

Kinetic representation for the 1D Saint-Venant system

Idea: $f(t,x,\xi)\geq 0$ particle distribution, new variable $\xi \to$ velocity

↳ Recover macroscopic quantities by integrating : $U_f(t,x)=\int_{\mathbb{R}}\begin{pmatrix}1\\ \xi\end{pmatrix} f(t,x,\xi)\,d\xi$

Boltzmann kinetic equation : $\underbrace{\partial_t f + \xi\,\partial_x f}_{\text{linear transport}} = \frac{1}{\varepsilon}\underbrace{Q[f](t,x,\xi)}_{\text{Collision operator with } \int_{\mathbb{R}}\binom{1}{\xi} Q\,d\xi = 0}$ (2)

Gibbs equilibrium : $f \in \operatorname{Ker} Q \iff f = M(U_f,\xi) := \frac{1}{g\pi}\sqrt{\left(2gh_f-(\xi-u_f)^2\right)_+}$

Maxwellian $M(U,\xi)$ satisfies the moment relations:

$$\forall U \in \mathbb{R}_+\times\mathbb{R},\quad \int_{\mathbb{R}}\begin{pmatrix}1\\ \xi\end{pmatrix} M(U,\xi)\,d\xi = U,\quad \int_{\mathbb{R}}\xi\begin{pmatrix}1\\ \xi\end{pmatrix} M(U,\xi)\,d\xi = F(U) \qquad (3)$$

Lemma 1 : $U=(h,hu)$ is a weak solution of (SV) iff $M(U,\xi)$ satisfies

$$\partial_t M + \xi\,\partial_x M - g(\partial_x z)\,\partial_\xi M = \mu(t,x,\xi) \qquad (KR)$$

with $\int_{\mathbb{R}}\begin{pmatrix}1\\ \xi\end{pmatrix}\mu(t,x,\xi)\,d\xi = 0$ for a.e. (t,x).

Explicit kinetic scheme with a flat bottom

Kinetic representation (KR) obtained as the limit of (2) when $\varepsilon \to 0$.

BGK operator $Q[f] = M(u_f, \xi) - f$ $\longrightarrow$ replace (KR) with BGK-splitting:

$$\begin{cases} \partial_t f = \frac{1}{\varepsilon}\left(M(u_f, \xi) - f\right) \\ \partial_t f + \xi \partial_x f = 0 \end{cases} \xrightarrow{\varepsilon \to 0}$$

Solve $\partial_t f + \xi \partial_x f = 0$ with initial data $f^0(x, \xi) = M(u_f^0(x), \xi)$.

Discretize: $t^n = n\Delta t$, $x_j = j\Delta x$, $C_j = (x_{j-1/2}, x_{j+1/2})$, $f_j^n(\xi) = \frac{1}{\Delta x}\int_{C_j} f(t^n, x, \xi)\, dx$

1st order upwind scheme: $$\frac{f_j^{n+1} - f_j^n}{\Delta t} + \frac{\xi}{\Delta x}\left(\mathbb{1}_{\xi<0}\left(f_{j+1}^n - f_j^n\right) + \mathbb{1}_{\xi>0}\left(f_j^n - f_{j-1}^n\right)\right) = 0 \quad (4)$$
where $f_j^n = M(u_j^n, \xi)$.

Integrate (4) against $\binom{1}{\xi}$: $$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{\Delta x}\left(F(u_j^n, u_{j+1}^n) - F(u_{j-1}^n, u_j^n)\right) = 0$$

Macroscopic kinetic flux $$F(u_L, u_R) = \int_{\xi<0} \xi \binom{1}{\xi} M(u_R, \xi)\, d\xi + \int_{\xi>0} \xi \binom{1}{\xi} M(u_L, \xi)\, d\xi$$

Implicit version of the kinetic scheme with flat bottom

Consider the scheme $\dfrac{f_j^{n+1}-f_j^n}{\Delta t}+\dfrac{\xi}{\Delta x}\left(\mathbb{1}_{\xi<0}\left(f_{j+1}^{n+1}-f_j^{n+1}\right)+\mathbb{1}_{\xi>0}\left(f_j^{n+1}-f_{j-1}^{n+1}\right)\right)=0$ (5)

N = number of cells, define $f^n=(f_1^n,f_2^n,\dots,f_N^n)^T\in(\mathbb{R}_+)^N$

(5) becomes $(I+\sigma L)\,f^{n+1}=f^n+\sigma B^{n+1}$ with $\sigma=\dfrac{\Delta t}{\Delta x}$, B^{n+1} accounts for limit conditions

↳ Ghost cells

$$L=\begin{pmatrix} |\xi| & \xi\mathbb{1}_{\xi<0} & & & 0 \\ -\xi\mathbb{1}_{\xi>0} & |\xi| & \xi\mathbb{1}_{\xi<0} & & \\ & \ddots & \ddots & \ddots & \\ & & -\xi\mathbb{1}_{\xi>0} & |\xi| & \xi\mathbb{1}_{\xi<0} \\ & 0 & & -\xi\mathbb{1}_{\xi>0} & |\xi| \end{pmatrix},\quad B^{n+1}=\begin{pmatrix} \xi M_0^{n+1}\mathbb{1}_{\xi>0} \\ 0 \\ \vdots \\ 0 \\ -\xi M_{N+1}^{n+1}\mathbb{1}_{\xi<0} \end{pmatrix}\in\mathbb{R}^N$$

__Remark 2:__ Matrix $(I+\sigma L)^{-1}$ can be computed by hand, and we have analytic expressions for $\int_{\mathbb{R}}(I+\sigma L)^{-1}M\,d\xi$ and $\int_{\mathbb{R}}\xi(I+\sigma L)^{-1}M\,d\xi$.

Since B^{n+1} unknown, proceed in two steps:

$$\begin{cases} 1/\ (h,hu)^{n+1/2}=\displaystyle\int_{\mathbb{R}}\binom{1}{\xi}\otimes(I+\sigma L)^{-1}M^n\,d\xi \\ 2/\ (h,hu)^{n+1}=\displaystyle\int_{\mathbb{R}}\binom{1}{\xi}\otimes(I-\sigma L)^{-1}B^{n+1/2}\,d\xi \end{cases}$$

Discreet entropy inequality and positivity

Energy inequality $\partial_t E + \partial_x\left[\left(E + g\frac{h^2}{2}\right)u\right] \leq 0$ $\longrightarrow$ discreet counterpart ?

Consider kinetic entropy $H(f,\xi) = \frac{\xi^2}{2} f + \frac{g^2\pi^2}{6} f^3 + gzf$ as an energy distribution

Lemma 3: $\forall\, \mathcal{U} \in \mathbb{R}_+ \times \mathbb{R}$, $\int_{\mathbb{R}} H(M(\mathcal{U},\xi),\xi)\,d\xi = E$, $\int_{\mathbb{R}} \xi H(M(\mathcal{U},\xi),\xi)\,d\xi = \left(E + g\frac{h^2}{2}\right)u$.

$M(\mathcal{U},\cdot)$ minimizes $f \mapsto \int_{\mathbb{R}} H(f(\xi),\xi)\,d\xi$ under the constraint $\int_{\mathbb{R}} \binom{1}{\xi} f(\xi)\,d\xi = \mathcal{U}$

Prop 4: Under CFL $\sigma|\xi| \leq 1$, explicit scheme (4) satisfies $h_j^{n+1} \geq 0$ and $E_j^{n+1} \leq E_j^n - \sigma\left[G_{j+1/2}^n - G_{j-1/2}^n\right]$

Proof: $f_j^{n+1} = (1-\sigma|\xi|)M_j^n + \sigma|\xi| M_{j\pm1}^n \xrightarrow{CFL}$ Convex combination $\Rightarrow f_j^{n+1} \geq 0$

Using lemma 3: $E_j^{n+1} = \int_{\mathbb{R}} H(M_j^{n+1}(\xi),\xi)\,d\xi \leq \int_{\mathbb{R}} H(f_j^{n+1}(\xi),\xi)\,d\xi \leq \int_{\mathbb{R}} (1-\sigma|\xi|)H_j^n + \sigma|\xi| H_{j\pm1}^n\,d\xi$

$\hookrightarrow$ Convexity of H

Prop 5: $\forall \Delta t > 0$, implicit scheme (5) satisfies $h_j^{n+1} \geq 0$ and $E_j^{n+1} \leq E_j^n - \sigma\left[\tilde{G}_{j+1/2}^{n+1} - \tilde{G}_{j-1/2}^{n+1}\right] + D_j$, $D_j \leq 0$

Proof: $I + \sigma L$ is monotone: $(I+\sigma L)f \geq 0 \Rightarrow f \geq 0$.

Define D_j such that $H(f_j^{n+1}) = H(M_j^n) \underbrace{- \sigma\, \partial_f H(f_j^{n+1})\left[f_j^{n+1} - M_j^n\right]}_{\text{Bounded from above by conservative term.}} + D_j \longrightarrow$ we find $D_j \leq 0$.

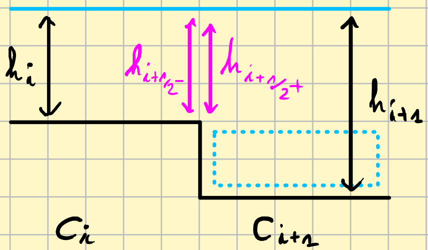
Accounting for nonflat bottom : hydrostatic reconstruction

Discretize source term $S(U,z)$ in (SV) : $\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + \frac{g}{2}h^2\right) = -gh\,\partial_x z \end{cases}$

Constraint : preserve lakes at rest $h+z=K$, $u=0$.

Problem : 1/ Upwinding introduces diffusion on h $\Rightarrow F(U_i, U_{i+1}) - F(U_{i-1}, U_i) \neq 0$

2/ Need to balance pressure variation with source term : $\partial_x\left(\frac{g}{2}h^2\right) = -gh\,\partial_x z$



Hydrostatic reconstruction : $z_{i+1/2} = \max(z_i, z_{i+1})$

$$h_{i+1/2+} = \left(h_{i+1} + z_{i+1} - z_{i+1/2}\right)_+ , \quad h_{i+1/2-} = \left(h_i + z_i - z_{i+1/2}\right)_+$$

$$\tilde{U}_{i+1/2+} = \begin{pmatrix} h_{i+1/2+} \\ h_{i+1/2+}\, u_{i+1} \end{pmatrix}, \quad \tilde{U}_{i+1/2-} = \begin{pmatrix} h_{i+1/2-} \\ h_{i+1/2-}\, u_i \end{pmatrix}$$

Modify numerical flux F as $\begin{cases} \mathcal{F}_{i+1/2-} = F(\tilde{U}_{i+1/2-}, \tilde{U}_{i+1/2+}) + \frac{g}{2}\left(h_i^2 - h_{i+1/2-}^2\right)\begin{pmatrix}0\\1\end{pmatrix} \\ \mathcal{F}_{i-1/2+} = F(\tilde{U}_{i-1/2-}, \tilde{U}_{i-1/2+}) + \frac{g}{2}\left(h_i^2 - h_{i-1/2+}^2\right)\begin{pmatrix}0\\1\end{pmatrix} \end{cases}$

Over lakes at rest, $U_{i+1/2-} = U_{i+1/2+} \Rightarrow$ No more diffusion on h.

Kinetic interpretation of the hydrostatic reconstruction

We remark that :

$$\int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} (\xi - u_i)\,(M_i - M_{i+1/2-})\, d\xi = \begin{pmatrix} 0 \\ \frac{g}{2}\left(h_i^2 - h_{i+1/2-}^2\right) \end{pmatrix} , \quad M_{i+1/2-} = M(\tilde{U}_{i+1/2-}, \xi)$$

$$\int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} (\xi - u_i)\,(M_i - M_{i-1/2+})\, d\xi = \begin{pmatrix} 0 \\ \frac{g}{2}\left(h_i^2 - h_{i-1/2+}^2\right) \end{pmatrix} , \quad M_{i-1/2+} = M(\tilde{U}_{i-1/2+}, \xi)$$

Define the interfacial Maxwellian $M_{i+1/2} = M(\tilde{U}_{i+1/2+}, \xi)\, \mathbb{1}_{\xi<0} + M(\tilde{U}_{i+1/2-}, \xi)\, \mathbb{1}_{\xi>0}$.

The previous explicit kinetic scheme (4) with hydrostatic reconstruction writes :

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + \frac{\xi}{\Delta x}\left(M^n_{i+1/2} - M^n_{i-1/2}\right) + \sigma\,(\xi - u_i^n)\left[M^n_{i-1/2+} - M^n_{i+1/2-}\right] = 0 \qquad (6)$$

Prop 6: Scheme (6) admits a discreet entropy inequality with an error term :

$$E_i^{n+1} \leqslant E_i^n - \sigma\left[\tilde{G}_{i+1/2} - \tilde{G}_{i-1/2}\right] + D_i , \qquad D_i \geqslant 0 \quad \longrightarrow \text{Total energy can increase.}$$

See E. Audusse, M.-O. Bristeau, B. Perthame, Kinetic entropy inequality and hydrostatic reconstruction scheme for the Saint-Venant system, 2000

Implicit kinetic scheme with hydrostatic reconstruction

Replace explicit kinetic scheme with hydrostatic reconstruction (6) by:

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + \frac{\xi}{\Delta x}\left(M_{i+1/2}^{n+1} - M_{i-1/2}^{n+1}\right) + \sigma\left(\xi - u_i^{n+1}\right)\left[M_{i-1/2+}^{n+1} - M_{i+1/2-}^{n+1}\right] = 0 \qquad (7)$$

↳ Nonlinear system

We propose an iterative approximation for (7):

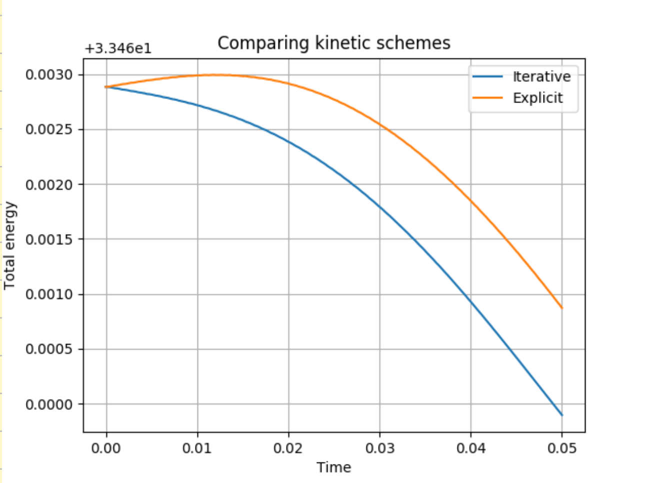
$$(1+\alpha) f_i^{n+1,k+1} = f_i^n + \alpha f_i^{n+1,k} - \sigma\xi\left(M_{i+1/2}^{n+1,k} - M_{i-1/2}^{n+1,k}\right) + \sigma\left(\xi - u_i^{n+1,k}\right)\left[M_{i+1/2-}^{n+1,k} - M_{i-1/2+}^{n+1,k}\right] \qquad (8)$$

$\alpha > 0$ is a relaxation parameter.

<u>Prop 7:</u> Let δ, K_1, K_2 be non-negative parameters. Assume that each iteration (8) keeps $U_i^{n+1,k} = \int_{\mathbb{R}} \binom{1}{\xi} f_i^{n+1,k} d\xi$ in the set $\Theta = \{(h, hu) \mid \delta \leq h \leq K_1,\ |u| \leq K_2\}$ for all $k \in \mathbb{N}$.

Then under CFL condition $\sigma \leq C(K_1, K_2, 1/\delta)$, sequence $\left(f_i^{n+1,k}\right)_k$ converges to f_i^{n+1} satisfying (7).

Comparing explicit and iterative kinetic schemes with hydrostatic reconstruction



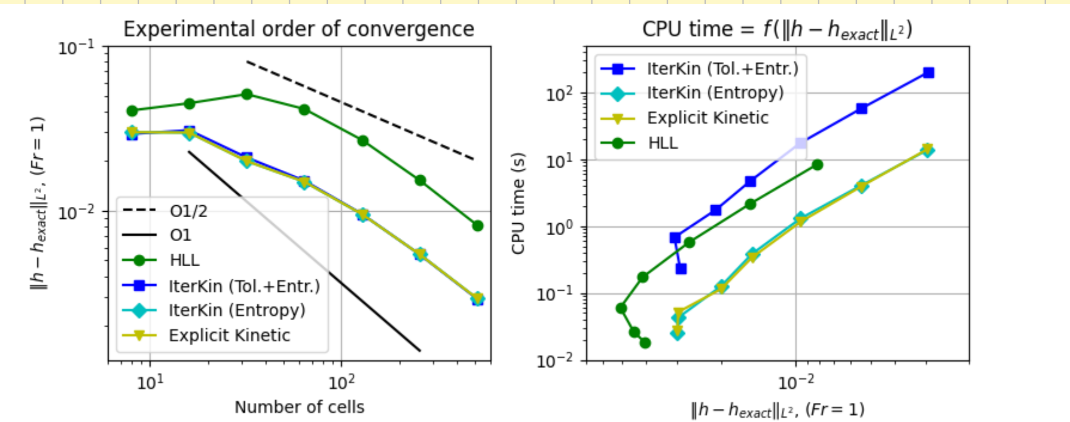
Domain : $[0,1]$ discretized with 100 cells

Boundary conditions : periodic

Initial data : $h+z = K$, $u = K' > 0$

We plot the total energy

$$E_{tot}(t) = \int_{[0,1]} E(t,x)\,dx$$



Perspectives:

- Proof of energy dissipation in the iterative case
- Weaker assumption to get the convergence of iterative scheme
- 2D over cartesian mesh ?

References:

Hydrostatic reconstruction:

* E. Audusse, F. Bouchut, M.-O. Bristeau, R. Klein, B. Perthame, A fast and stable well-balanced scheme with hydrostatic reconstruction for Shallow Water flows, 2004

1D Explicit Kinetic schemes

* E. Audusse, M.-O. Bristeau, B. Perthame, Kinetic entropy inequality and hydrostatic reconstruction scheme for the Saint-Venant system, 2000

* B. Perthame, C. Simeoni, A kinetic scheme with a source term, 2001

2D Explicit Kinetic schemes

* E. Audusse, M.-O. Bristeau, B. Perthame, Kinetic scheme for Saint-Venant equations with source terms on unstructured grids, 2000