Kinetic schemes and wave splitting for the Shallow Water system in low Froude regime

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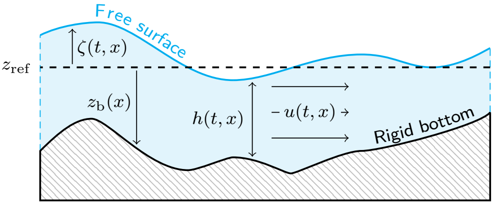

- water height h
- horizontal fluid velocity u
- horizontal discharge $q = hu$
- fixed rigid bottom z_b
- free surface elevation $\zeta = h + z_b$

Hyperbolic system of conservation laws: $\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{s}(\boldsymbol{U}, x)$ (SW)

$$\boldsymbol{U}(t,x) = \begin{pmatrix} h \\ q \end{pmatrix}, \quad \boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} q \\ q^2/h + gh^2/2 \end{pmatrix} \quad \boldsymbol{s}(\boldsymbol{U}, x) = \begin{pmatrix} 0 \\ -ghz_b'(x) \end{pmatrix}$$

Eigenvalues of the flux Jacobian: $\lambda_\pm(\boldsymbol{U}) = u \pm \sqrt{gh}$ ($\sqrt{gh}$ = gravity waves velocity)

Remarkable properties:

- $h(t,x) \geq 0$ (positivity)
- Water height conservation
- Steady state at rest: $u = 0, \quad h + z_b = cst$ (lake at rest)
- Single entropy inequality: $\partial_t \eta(\boldsymbol{U}) + \partial_x G(\boldsymbol{U}) \leq \nabla \eta(\boldsymbol{U})^T \boldsymbol{s}(\boldsymbol{U}, x)$

Goal: derive FV schemes with nice properties, efficient in **low Froude regime** ($|u| \ll \sqrt{gh}$)

Wave splitting and kinetic representations

Issue: CFL condition over restrictive with $\dfrac{\Delta x}{\Delta t^n} \geq \max\limits_{1\leq i\leq N}\left(|u_i \pm \sqrt{gh_i}|\right) \gg \max\limits_{1\leq i\leq N}(|u_i|)$

Solution: separate *convection* waves from *gravity* waves : $\boldsymbol{F} = \boldsymbol{F}_c + \boldsymbol{F}_g$

Choose $\boldsymbol{F}_g(\boldsymbol{U}, x) = \begin{pmatrix} 0 & 1 \\ -gz_b & 0 \end{pmatrix}\begin{pmatrix} h \\ q \end{pmatrix} + \begin{pmatrix} 0 \\ -gz_b^2/2 \end{pmatrix} \Rightarrow \lambda_c^0(\boldsymbol{U}) = 0,\ \lambda_c^1(\boldsymbol{U}) = 2u,\ \lambda_g^{\pm}(x) = \pm\sqrt{-gz_b}$

Consider the splitting $\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}_c(\boldsymbol{U}, x) = 0 & \text{explicit approx.} \\ \partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}_g(\boldsymbol{U}, x) = \boldsymbol{s}(\boldsymbol{U}, x) & \text{implicit approx.} \end{cases}$

Strategy: **kinetic relaxation** with BGK collision operator

$$\partial_t \boldsymbol{f} + v(\xi)\,\partial_x \boldsymbol{f} = [\boldsymbol{M}_c(\boldsymbol{U}_{\boldsymbol{f}}, x, \xi) - \boldsymbol{f}]/\varepsilon \quad \forall \xi \in \Xi_c \qquad \boldsymbol{U}_{\boldsymbol{f}}(t,x) \overset{\text{def}}{=} \int_{\Xi_c} \boldsymbol{f}(t,x,\xi)\,\mathrm{d}\xi$$

$$\partial_t \boldsymbol{k} + w(\xi)\,\partial_x \boldsymbol{k} + \begin{pmatrix} 0 & 0 \\ gz_b' & 0 \end{pmatrix} \boldsymbol{k} = [\boldsymbol{M}_g(\boldsymbol{U}_{\boldsymbol{k}}, x, \xi) - \boldsymbol{k}]/\varepsilon \quad \forall \xi \in \Xi_g \qquad \boldsymbol{U}_{\boldsymbol{k}}(t,x) \overset{\text{def}}{=} \int_{\Xi_g} \boldsymbol{k}(t,x,\xi)\,\mathrm{d}\xi$$

Moment relations: $\forall (\boldsymbol{U}, x) \in \mathbb{R}^2 \times \mathbb{R}$,

$$\int_{\Xi} \boldsymbol{M}(\boldsymbol{U}, x, \xi)\,\mathrm{d}\xi = \boldsymbol{U}, \quad \int_{\Xi_c} v(\xi)\boldsymbol{M}_c(\boldsymbol{U}, x, \xi)\,\mathrm{d}\xi = \boldsymbol{F}_c(\boldsymbol{U}, x), \quad \int_{\Xi_g} w(\xi)\boldsymbol{M}_g(\boldsymbol{U}, x, \xi)\,\mathrm{d}\xi = \boldsymbol{F}_g(\boldsymbol{U}, x)$$

Explicit scheme for the convection

In our case, $\Xi_c = \{-1, 1\}$, $\boldsymbol{M}_c(\boldsymbol{U}, x, \xi) = \boldsymbol{U} + v(\xi)^{-1}\boldsymbol{F}_c(\boldsymbol{U}, x)$, $v(\xi) = cst \cdot \xi$

Update based on a BGK splitting:
$$\begin{cases} \partial_t \boldsymbol{f} = (\boldsymbol{M}_c(\boldsymbol{U}_{\boldsymbol{f}}, x, \xi) - \boldsymbol{f})/\varepsilon & (1a) \\ \partial_t \boldsymbol{f} + v(\xi)\partial_x \boldsymbol{f} = 0 & (1b) \end{cases}$$

Step 1: relaxation
Start from $\boldsymbol{U}(t^n, \cdot)$, perform all the collisions at once ($\varepsilon \to 0$) $\Rightarrow \boldsymbol{f}(t^n, x, \xi) = \boldsymbol{M}_c(\boldsymbol{U}(t^n, x), x, \xi)$

Step 2: transport
Find $\boldsymbol{f}(t^{n+1-}, x, \xi)$ sol. at time Δt^n of (1b) with initial condition $\boldsymbol{M}_c(\boldsymbol{U}(t^n, x), x, \xi)$

Assume $\boldsymbol{U}(t^n, \cdot)$ piece-wise constant and approximate step 2 using an upwind FV scheme:

$$\boldsymbol{f}_i^{n+1-}(\xi) = \boldsymbol{M}^n_{c,i}(\xi) - \frac{\Delta t^n}{\Delta x}[\widetilde{\boldsymbol{F}}^n_{i+1/2} - \widetilde{\boldsymbol{F}}^n_{i-1/2}], \quad \widetilde{\boldsymbol{F}}^n_{i+1/2} = \begin{cases} v(\xi)\boldsymbol{M}^n_{c,i+1}(\xi) & \text{if } v(\xi) < 0 \\ v(\xi)\boldsymbol{M}^n_{c,i}(\xi) & \text{if } v(\xi) > 0 \end{cases}$$

where $\boldsymbol{M}^n_{c,i}(\xi) \stackrel{\text{def}}{=} \boldsymbol{M}_c(\boldsymbol{U}^n_i, x_i, \xi)$. At the macroscopic level:

$$\boldsymbol{U}^{n+1}_{c,i} \stackrel{\text{def}}{=} \int_{\Xi_c} \boldsymbol{f}_i^{n+1-}(\xi)\,\mathrm{d}\xi = \boldsymbol{U}^n_i - \frac{\Delta t^n}{\Delta x}[\boldsymbol{F}_{\text{kinetic}}(\boldsymbol{U}^n_i, \boldsymbol{U}^n_{i+1} x_i, x_{i+1}) - \boldsymbol{F}_{\text{kinetic}}(\boldsymbol{U}^n_{i-1}, \boldsymbol{U}^n_i, x_{i-1}, x_i)]$$

with $\boldsymbol{F}_{\text{kinetic}}(\boldsymbol{U}^n_i, \boldsymbol{U}^n_{i+1} x_i, x_{i+1}) = \int_{v(\xi)<0} v(\xi)\boldsymbol{M}_c(\boldsymbol{U}^n_{i+1}, x_{i+1}, \xi)\,\mathrm{d}\xi + \int_{v(\xi)>0} v(\xi)\boldsymbol{M}_c(\boldsymbol{U}^n_i, x_i, \xi)\,\mathrm{d}\xi.$

Implicit scheme for the gravity waves

Similarly, $\Xi_g = \{-1, 1\}$, $\boldsymbol{M}_g(\boldsymbol{U}, x, \xi) = \boldsymbol{U} + w(\xi)^{-1}\boldsymbol{F}_g(\boldsymbol{U}, x)$, $w(\xi) = cst' \cdot \xi$

$$\frac{\boldsymbol{k}_i^{n+1} - \boldsymbol{k}_i^{n+1-}}{\Delta t^n} + w(\xi)\frac{\boldsymbol{k}_{i+1/2}^{n+1} - \boldsymbol{k}_{i-1/2}^{n+1}}{\Delta x} + P_b \boldsymbol{k}_i^{n+1} = \frac{\boldsymbol{M}_{g,i}^{n+1}[\boldsymbol{k}](\xi) - \boldsymbol{k}_i^{n+1}}{\varepsilon}, \quad \boldsymbol{k}_i^{n+1-} \stackrel{\text{def}}{=} \boldsymbol{M}_g(\boldsymbol{U}_{c,i}^{n+1}, x_i, \xi)$$

$\Rightarrow$ system of the form $A\boldsymbol{x}_\varepsilon^{n+1} = \boldsymbol{b}^n + (2\varepsilon)^{-1}R\boldsymbol{x}_\varepsilon^{n+1}$ with:

The vector of $4N$ unknowns	$(\boldsymbol{x}_{\varepsilon,j}^{n+1})_{4i-3\le j\le 4i} = (k_{h,i}^{n+1}(-1),\ k_{h,i}^{n+1}(1),\ k_{q,i}^{n+1}(-1),\ k_{q,i}^{n+1}(1))$
The relaxation matrix $(4N \times 4N)$	$R = PDP^{-1}$ with $D = \mathrm{diag}(-2, \cdots, -2, 0, \cdots, 0)$

Discrete equivalent of $\boldsymbol{k}(t, x, \xi) \xrightarrow{\varepsilon\to 0} \boldsymbol{M}_g(\boldsymbol{U}_{\text{eq}}, x, \xi)$: $\overline{\boldsymbol{x}}^{n+1} \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} \boldsymbol{x}_\varepsilon^{n+1} \in \ker R$

Basis change: $\boldsymbol{y}_\varepsilon^{n+1} = P^{-1}\boldsymbol{x}_\varepsilon^{n+1}$, $\widetilde{A} = P^{-1}AP$, $\widetilde{\boldsymbol{b}}^n = P^{-1}\boldsymbol{b}^n \Rightarrow \widetilde{A}\boldsymbol{y}_\varepsilon^{n+1} = \widetilde{\boldsymbol{b}}^n + (2\varepsilon)^{-1}D\boldsymbol{y}_\varepsilon^{n+1}$

Introduce $\mathcal{I} = \{i \in [\![1, 4N]\!],\ D_{ii} \neq 0\}$ and $\mathcal{J} = [\![1, 4N]\!] \setminus \mathcal{I}$. In the limit:

$$\overline{\boldsymbol{y}}^{n+1} \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} \boldsymbol{y}_\varepsilon^{n+1} \in \ker D \quad \Rightarrow \quad \begin{cases} \overline{\boldsymbol{y}}_{\mathcal{I}}^{n+1} = 0 \\ \widetilde{A}_{\mathcal{J}\times\mathcal{J}}\overline{\boldsymbol{y}}_{\mathcal{J}}^{n+1} = \widetilde{\boldsymbol{b}}_{\mathcal{J}}^n \end{cases}$$

Results and perspectives