Designing accurate and efficient schemes for the low Froude regime

Mathieu Rigal

Supervisors: Nina Aguillon, Nathalie Ayi

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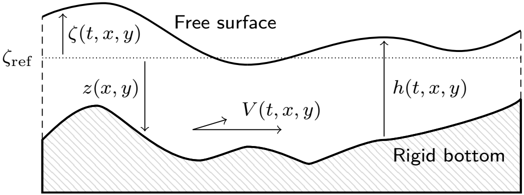
The 2D Shallow Water system



- water height h
- horizontal velocity $V \in \mathbb{R}^2$
- horizontal discharge hV
- fixed rigid bottom z

Low Froude regime: $\mathrm{Fr} \overset{\mathrm{def}}{=} \dfrac{\text{particles velocity}}{\text{surface waves velocity}} \ll 1$

Shallow Water system: $\partial_t U + \nabla \cdot F(U) = S(U,z)$ (SW)

$$U(t,x,y) = \begin{pmatrix} h \\ hV \end{pmatrix}_{3\times 1} \quad F(U) = \begin{pmatrix} hV^T \\ hV \otimes V + \frac{h^2}{2\mathrm{Fr}^2}\mathbf{I} \end{pmatrix}_{3\times 3} \quad S(U,z) = \begin{pmatrix} 0 \\ -\frac{h}{\mathrm{Fr}^2}\nabla z \end{pmatrix}_{3\times 1}$$

Wave splitting and time integration

Explicit FV schemes: stability $\Rightarrow$ prohibitively small timesteps

$$\Delta t \leq \frac{\mathrm{Fr}}{2} \min\left(\frac{\mathrm{mes}(\Gamma)}{\mathrm{Fr}\,|V \cdot n| + \sqrt{h}}\right) = \mathcal{O}(\mathrm{Fr}\Delta x,\ \mathrm{Fr}\Delta y)$$

Instead: try to split the system in two spatial operators:

$$\nabla \cdot F(U) - S(U,z) = \nabla \cdot K(U,z) + \mathcal{G}(U,z)$$

$K(U,z) \to$ convective flux (slow dynamics)

- its eigenvalues must remain bounded as $\mathrm{Fr} \to 0$;
- it can be nonlinear;

$\mathcal{G}(U,z) \to$ gravity waves diff. operator (fast dynamics)

- its eigenvalues can be unbounded as $\mathrm{Fr} \to 0$;
- it must be linear;

Wave splitting and time integration

We will consider IMplicit-EXplicit time integrators:

$$\frac{U^{n+1} - U^n}{\Delta t} + \nabla \cdot K(U^n, z) + \mathcal{G}(U^{n+1}, z) = 0$$

Space discretization on a **cartesian mesh**:

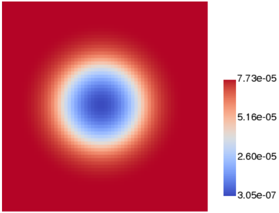
- $\nabla \cdot K(U, z)$ discretized using Godunov-type fluxes;
- $\mathcal{G}(U, z)$ discretized using centered differences;

Define scheme $\mathcal{S}^1_{\mathrm{Fr}}$ with properties:

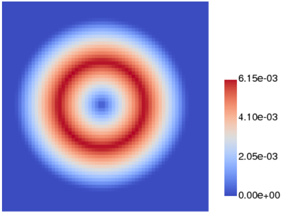
- Second order accuracy;
- Stable for a scale-independent timestep ($\Delta t = \mathcal{O}(\Delta x,\ \Delta y)$);
- Well balanced (lakes at rest are preserved);
- Asymptotic preserving ($\mathcal{S}^1_0$ consistent with limiting eq.);

A first IMEX scheme

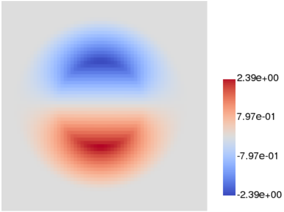
Gresho vortex (steady state) at time $t = 1/2$ with $\mathrm{Fr} = 10^{-2}$



Free surface



Local Froude number



X-discharge

A first IMEX scheme

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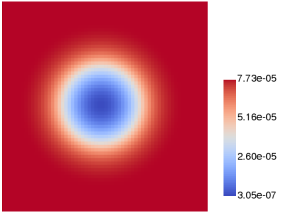

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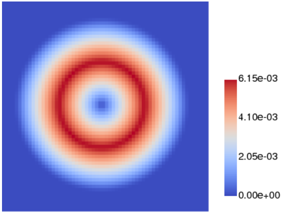

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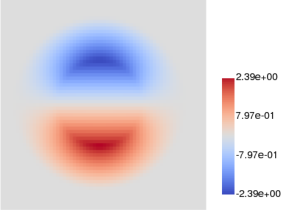

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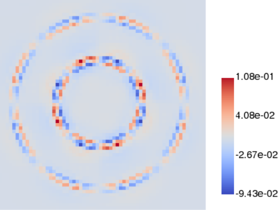

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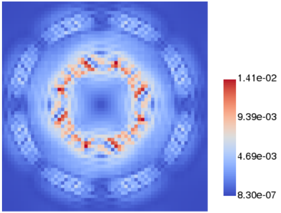

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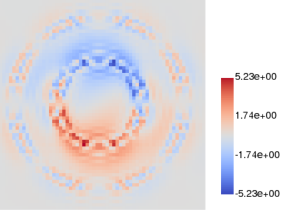


X-discharge

Preserving nearly incompressible states

How should solutions behave when $\mathrm{Fr} \ll 1$?

Limiting system:

$$\begin{cases} \overline{U}(t > 0, \cdot) \in \ker \mathcal{G}(\cdot, z) \overset{\text{def}}{=} \left\{ \begin{pmatrix} h \\ hV \end{pmatrix}, \ \nabla(h + z) = 0, \ \nabla \cdot (hV) = 0 \right\} \\ \partial_t \overline{V} + (\overline{V} \cdot \nabla)\overline{V} + \nabla \Pi = 0 \end{cases} \qquad (\mathrm{SW}_0)$$

From now on, take $z \equiv \mathrm{Cst} \Rightarrow \ker \mathcal{G}$ is the set of incompressible states

Theorem 1 (Schochet, 1994)

Solutions of (SW) remain "close" to $\ker \mathcal{G}$ if the initial condition is "close".

Study modified equation associated with $\mathcal{S}^1_{\mathrm{Fr}}(K = 0)$: $\quad \partial_t U + \mathcal{F} U = 0$

→ Does it preserve nearly incompressible states?

Preserving nearly incompressible states

Theorem 2 (Dellacherie)

Assume $(\partial_t + \mathcal{F})U = 0$ *leaves* $\ker \mathcal{G}$ *invariant. Sol.* U *of this eq. satisfy*

$$\inf_{W \in \ker \mathcal{G}} \|U(t=0,\cdot) - W\| = \mathcal{O}(\mathrm{Fr}) \Rightarrow \inf_{W \in \ker \mathcal{G}} \|U(t>0,\cdot) - W\| = \mathcal{O}(\mathrm{Fr})$$

Proposition 1

The modified equation of $\mathcal{S}^1_{\mathrm{Fr}}(K=0)$ *doesn't leave* $\ker \mathcal{G}$ *invariant.*

Intuition: Neglecting K, the modified equation of $\mathcal{S}^1_{\mathrm{Fr}}(K=0)$ reads

$$\left(\frac{\partial}{\partial t} + \mathcal{G}\right)U = [\mathcal{R}_{\Delta t} - \mathcal{R}_{\Delta x}]U$$

We have $\ker \mathcal{G} \subset \ker \mathcal{R}_{\Delta t}$, but $\ker \mathcal{G} \not\subset \ker \mathcal{R}_{\Delta x}$

Preserving nearly incompressible states

Solution: introduce $\widetilde{\mathcal{R}}_{\Delta x}$ such that $\ker \widetilde{\mathcal{R}}_{\Delta x} \supset \ker \mathcal{G}$

Proposition 2

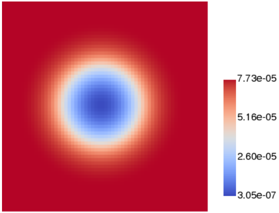
Define $\mathcal{S}^2_{\mathrm{Fr}}$ by substituting $\mathcal{G}$ with $\mathcal{G} + \widetilde{\mathcal{R}}_{\Delta x} - \mathcal{R}_{\Delta x}$ in $\mathcal{S}^1_{\mathrm{Fr}}$. Its modified eq. reads

$$\left(\frac{\partial}{\partial t} + \mathcal{G}\right) U = [\mathcal{R}_{\Delta t} - \widetilde{\mathcal{R}}_{\Delta x}] U \,,$$

hence it leaves $\ker \mathcal{G}$ invariant.

Modified scheme $\mathcal{S}^2_{\mathrm{Fr}}$

Gresho vortex (steady state) at time $t = 1/2$ with $\mathrm{Fr} = 10^{-2}$



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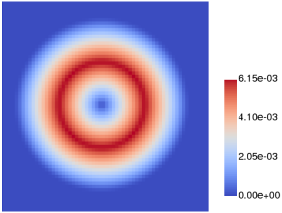

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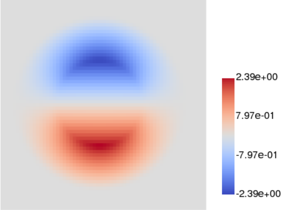

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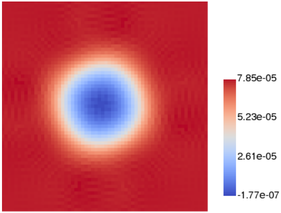

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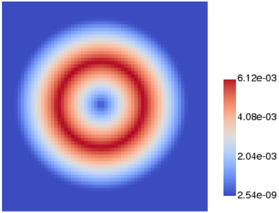

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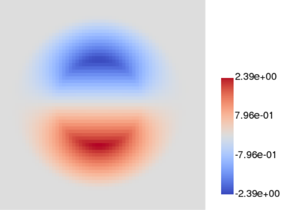

X-discharge

Experimental order of convergence and efficiency

Gresho vortex (steady state) at time $t = 1/2$ with $\mathrm{Fr} = 10^{-2}$