ON POLYNOMIALLY BOUNDED OPERATORS ACTING ON A BANACH SPACE

MOHAMED ZARRABI

Abstract. By the Von Neumann inequality every contraction on a Hilbert space is polynomially bounded. A simple example shows that this result does not extend to Banach space contractions. In this paper we give general conditions under which an arbitrary Banach space contraction is polynomially bounded. These conditions concern the thinness of the spectrum and the behaviour of the resolvent or the sequence of negative powers. To do this we use techniques from harmonic analysis, in particular, results concerning thin sets such as Helson sets, Kronecker sets and sets that satisfy spectral synthesis.

1. Introduction

We denote by $\mathbb{D}$ the open unit disk of the complex plane $\mathbb{C}$ and by $\mathbb{T}$ its boundary, $\mathbb{T} = \{z,\ |z| = 1\}$. Let $\mathcal{L}(X)$ denote the algebra of all bounded linear operators on the complex Banach space X. For $T \in \mathcal{L}(X)$, we denote by $\mathrm{Sp}(T)$ the spectrum of T. An operator $T \in \mathcal{L}(X)$ is said to be *polynomially bounded* if there exists a constant $C > 0$ such that for all polynomials P we have

$$\|P(T)\| \leq C \sup_{|z|\leq 1} |P(z)|.$$

The Von Neumann inequality asserts that every contraction T on a Hilbert space is polynomially bounded. This result doesn't extend to contractions acting on a Banach space. To see this it suffices to consider the operator of multiplication by z on some non uniform Banach algebras of functions defined on $\mathbb{D}$.

In this paper we give sharp conditions on the spectrum of a contraction T, on the growth of the resolvent $(\lambda - T)^{-1}$, and on the growth of the sequence of negative powers $(T^{-n})_{n\geq 0}$, which imply that T is polynomially bounded.

To describe the results let us introduce some definitions and notations. If E is a closed subset of $\mathbb{T}$, we denote by $\mathcal{C}(E)$ the space of all continuous functions on E. The classical Wiener algebra $A(\mathbb{T})$ is defined by

$$A(\mathbb{T}) = \left\{ f \in \mathcal{C}(\mathbb{T}) \ :\ \|f\|_1 = \sum_{n=-\infty}^{+\infty} |\hat{f}(n)| < +\infty \right\},$$

2000 *Mathematics Subject Classification.* 47A30; 43A46.

Key words and phrases. polynomially bounded operators; Helson and Kronecker sets; spectral synthesis.

$$A(\mathbb{T}) = \left\{ f \in \mathcal{C}(\mathbb{T}) \ : \ \|f\|_1 = \sum_{n=-\infty}^{+\infty} |\hat{f}(n)| < +\infty \right\},$$

where $\hat{f}(n)$ is the n^{th} Fourier coefficient of f. We set

$$A^+(\mathbb{T}) = \left\{ f \in A(\mathbb{T}) \ : \ \hat{f}(n) = 0, \ n < 0 \right\}.$$

Let E be a closed subset of $\mathbb{T}$. We recall that E is called a Helson set if every continuous function on E can be represented as an absolutely convergent Fourier series. We say that E satisfies spectral synthesis (for $A(\mathbb{T})$) if for every function $f \in A(\mathbb{T})$ vanishing on E, there exists a sequence of functions vanishing on a neighborhood of E, which converges to f for the norm $\|.\|_1$.

In Section 4 we show (Theorem 4.1) that if T is an invertible isometry such that $\mathrm{Sp}(T)$ is a Helson set and satisfies spectral synthesis then T is polynomially bounded. Conversely if E is a closed subset of T that is not a Helson set or that does not satisfy spectral synthesis then there exists an isometry with spectrum E that is not polynomially bounded.

We are also interested in operators with countable spectra. We prove (Theorem 4.2) that if T is a contraction such that $\mathrm{Sp}(T)$ is a countable Helson set and if

$$\limsup_{|z|\to 1-} (1 - |z|) \log^+ \|(z - T)^{-1}\| = 0.$$

or equivalently

$$\limsup_{n\to+\infty} \frac{\log \|T^{-n}\|}{\sqrt{n}} = 0.$$

then T is polynomially bounded.

Notice that in this result the assumption that $\mathrm{Sp}(T)$ is a Helson set is essential (Theorem 4.1). Also we show by examples that the hypotheses about countability and the growth of $\|(z-T)^{-1}\|$ or $\|T^{-n}\|$ in this result are best possible. Indeed for E any uncountable closed subset of $\mathbb{T}$, we construct a non polynomially bounded contraction T such that $\mathrm{Sp}(T) \subset E$, $\mathrm{Sp}(T)$ is a Helson set and T satisfies the condition

$$\limsup_{|z|\to 1-} (1 - |z|) \log^+ \|(z - T)^{-1}\| = 0.$$

Also for every $\epsilon > 0$ we get a non polynomially bounded contraction T such that $\mathrm{Sp}(T) = \{1\}$ and

$$\limsup_{|z|\to 1-} (1 - |z|) \log^+ \|(z - T)^{-1}\| \leq \epsilon.$$

To obtain these results we first study contractions of the form

$$\begin{aligned} T \ : A^+(\mathbb{T})/I &\longrightarrow A^+(\mathbb{T})/I \\ f + I &\longrightarrow \alpha f + I \end{aligned}$$

where I is a closed ideal of $A^+(\mathbb{T})$ and $\alpha : z \to z$ is the identity map. We show (Theorem 3.1) that T is polynomially bounded if and only if I has the form $I = \Theta I^+(E)$, where E is a closed Helson subset of $\mathbb{T}$, Θ is a nonzero constant or a finite Blaschke product and $I^+(E) = \{f \in A^+(\mathbb{T}),\ f_{|E} = 0\}$.

In Section 5 we are interested in contractions for which the spectrum is a Kronecker set. We recall that a closed subset E of $\mathbb{T}$ is called a Kronecker set if the set of functions $\{\alpha^n,\ n \in \mathbb{Z}\}$ is dense in $\{f \in \mathcal{C}(E),\ |f(z)| = 1,\ (z \in E)\}$, for the supremum norm on E. We observe that if T is an isometry such that $\mathrm{Sp}(T)$ is a Kronecker set, then T is polynomially bounded. This follows from Theorem 4.1 and a result of Varopolous ([21]), which says that a Kronecher set is a Helson set and satisfies spectral synthesis. On the other hand we show in Theorem 5.4 that if $(\beta_n)_{n\geq 0}$ is a sequence of real numbers such that $\lim_{n\to+\infty} \beta_n = +\infty$ and $\beta_n > 1,\ n \geq 1$, then there exists a non polynomially bounded contraction T such that $\mathrm{Sp}(T)$ is a Kronecker set and $\|T^{-n}\| \leq \beta_n,\ n \geq 0$.
Notice finally that the study and results concerning thin sets considered here can be found in the books [12], [7], [16] and [17] .

2. An interpolation theorem

Let Θ be an inner function in the unit disk. It is well known that Θ admits a factorisation of the form $\Theta = \lambda_\Theta B_\Theta S_\Theta$, where λ_Θ is a complex number of modulus 1, B_Θ is the Blaschke product associated to the zero set of Θ and S_Θ is a singular inner function.

For $\lambda \in \mathbb{D}$ we set $b_\lambda(z) = \frac{|\lambda|}{\lambda}\frac{\lambda - z}{1-\bar{\lambda}z}$, $z \in \mathbb{D}$, with the understanding that $b_0(z) = z$. We have $B_\Theta = \prod_{\lambda\in\Lambda_\Theta} b_\lambda^{k_\lambda}$, where Λ_Θ is the zero set of Θ, k_λ the multiplicity of λ and

$$S_\Theta(z) = \exp\left(\int_\pi \frac{z+\zeta}{z-\zeta} d\mu_\Theta(\zeta)\right),$$

where μ_Θ is a non–negative singular measure. Following [18], pp. 62-63, we set

$$\begin{aligned}\sigma(\Theta) &= \{\lambda \in \overline{\mathbb{D}}\ :\ \liminf_{\zeta\in\mathbb{D},\zeta\to\lambda} \Theta(\lambda) = 0\}\\ &= \overline{\Lambda_\Theta} \cup \mathrm{supp}(\mu_\Theta),\end{aligned}$$

where $\mathrm{supp}(\mu_\Theta)$ is the support of the measure μ_Θ.

Let $\mathfrak{A}(\mathbb{D})$ be the disk algebra, that is the set of all functions which are continuous on $\overline{\mathbb{D}}$ and holomorphic in $\mathbb{D}$. It will be equipped with the norm $\|f\|_\infty = \sup_{z\in\mathbb{T}} |f(z)|$. We denote by H^∞ the set of all holomorphic and bounded functions on $\mathbb{D}$. For $f, g \in H^\infty$, we say that f divides g if $g/f \in H^\infty$.

If $\mathfrak{I}$ is a non empty subset of $\mathfrak{A}(\mathbb{D})$ we set

$$h(\mathfrak{I}) = \{z \in \overline{\mathbb{D}}\ :\ f(z) = 0\ (f \in \mathfrak{I})\},$$

and we denote by $\Theta_\mathfrak{I}$ the greatest common divisor of the inner factors of all nonzero functions in $\mathfrak{I}$ (see [19], Lemma 2 or [8], p. 85). If E is a closed subset of $\mathbb{T}$ we set

$$\mathfrak{I}(E) = \{f \in \mathfrak{A}(\mathbb{D}) \ : \ f_{|E} = 0\}.$$

It is shown in [19], Theorem 1, that if $\mathfrak{I}$ is a closed ideal of $\mathfrak{A}(\mathbb{D})$ then $\mathfrak{I} = \Theta_{\mathfrak{I}}\mathfrak{I}(h(\mathfrak{I})\cap\mathbb{T})$ (see also [8], p. 85). Notice also the following result which we will use below : if Θ is an inner function, $h \in H^\infty$ and if $\Theta h \in \mathfrak{A}(\mathbb{D})$ then $h \in \mathfrak{A}(\mathbb{D})$ and h vanishes on $\sigma(\Theta)\cap\mathbb{T}$ ([19], Lemma 6 and Lemma 7).

We recall that

$$A(\mathbb{T}) = \{f \in \mathcal{C}(\mathbb{T}) \ : \ \|f\|_1 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)| \ < +\infty\}$$

and

$$A^+(\mathbb{T}) = \{f \in A \ : \ \ \hat{f}(n) = 0, \ n < 0\}.$$

Every function $f = \sum_{n\geq 0} \hat{f}(n)z^n$ in $A^+(\mathbb{T})$, extended to $\overline{\mathbb{D}}$, defines an element of $\mathfrak{A}(\mathbb{D})$. So $A^+(\mathbb{T})$ will also be regarded as a subalgebra of $\mathfrak{A}(\mathbb{D})$.

Let E be a closed subset of $\mathbb{T}$. We denote by $\mathfrak{A}(E)$ the algebra of all functions on E which are the restrictions to this set of functions in $\mathfrak{A}(\mathbb{D})$, that is

$$\mathfrak{A}(E) = \{f_{|E} \ : \ f \in \mathfrak{A}(\mathbb{D})\}.$$

Similarly we define $A(E)$ and $A^+(E)$. Thus E is a Helson set if $\mathcal{C}(E) = A(E)$. Notice also that E is called a Carleson set if $\mathcal{C}(E) = A^+(E)$. Obviously a Carleson set is a Helson set. Wik showed in [23] that the converse of this result is true. These sets are studied in many papers and books ([7], [9], [12], [16], [23]).

Let $\mathcal{M}(\mathbb{T})$ denote the space of all finite Borel measures on $\mathbb{T}$ and let

$$H_0^1 = \{f \in L^1(\mathbb{T}), \ \hat{f}(n) = 0, \ n \leq 0\}.$$

The distance of a measure $\mu \in \mathcal{M}(\mathbb{T})$ to H_0^1 is

$$\mathrm{dist}(\mu, H_0^1) = \inf_{f\in H_0^1} \|\mu - f dm\|_{\mathcal{M}(\mathbb{T})},$$

where m is the Lebesgue measure on $\mathbb{T}$.

If $\mathfrak{I}$ is a closed subspace of $\mathfrak{A}(\mathbb{D})$, we set

$$\mathfrak{I}^\perp = \left\{\mu \in \mathcal{M}(\mathbb{T}), \ \int_{\mathbb{T}} f(\zeta)d\mu(\zeta) = 0, \ (f \in \mathfrak{I})\right\}.$$

The following theorem is the main result of this section.

Theorem 2.1. *Let $\mathfrak{I}$ be a closed ideal of $\mathfrak{A}(\mathbb{D})$. Then the following are equivalent.*
i) $\mathfrak{A}(\mathbb{D}) = A^+(\mathbb{T}) + \mathfrak{I}$.
ii) $h(\mathfrak{I})\cap\mathbb{T}$ is a Helson set and $\Theta_{\mathfrak{I}}$ is a constant or a finite Blaschke product.
iii) There exists a positive constant C such that for every $\mu \in \mathfrak{I}^\perp$, we have

$$dist(\mu, H_0^1) \leq C \sup_{n\leq 0} |\hat{\mu}(n)|.$$

The proof of this result requires some lemmas. If $\mathfrak{I}$ is a nonempty subset of $\mathfrak{A}(\mathbb{D})$ and $g \in \mathfrak{A}(\mathbb{D})$, we set

$$\mathfrak{I}_g = \{f \in \mathfrak{A}(\mathbb{D}) : \; gf \in \mathfrak{I}\}.$$

Lemma 2.2. *Let $\mathfrak{I}$ be a closed ideal of $\mathfrak{A}(\mathbb{D})$. Assume that B is a nonzero constant or a finite Blaschke product that divides $\Theta_{\mathfrak{I}}$. The following are equivalent.*
i) $\mathfrak{A}(\mathbb{D}) = A^+(\mathbb{T}) + \mathfrak{I}$.
ii) $\mathfrak{A}(\mathbb{D}) = A^+(\mathbb{T}) + \mathfrak{I}_B$.

Proof. The implication $i) \Rightarrow ii)$ is obvious since $\mathfrak{I} \subset \mathfrak{I}_B$. Assume now that the equality $\mathfrak{A}(\mathbb{D}) = A^+(\mathbb{T}) + \mathfrak{I}_B$ holds. Let $f \in \mathfrak{A}(\mathbb{D})$. Since B is a nonzero constant or a finite Blaschke product it is easily seen that there exist a polynomial g and a function $h \in \mathfrak{A}(\mathbb{D})$ such that $f = g + Bh$. Now, by hypothesis, there exist $g_1 \in A^+(\mathbb{T})$ and $h_1 \in \mathfrak{I}_B$ such that $h = g_1 + h_1$. So $f = g + Bg_1 + Bh_1$. We have clearly $g + Bg_1 \in A^+(\mathbb{T})$ and $Bh_1 \in \mathfrak{I}$. Thus $\mathfrak{A}(\mathbb{D}) = A^+(\mathbb{T}) + \mathfrak{I}$, which finishes the proof. $\square$

Lemma 2.3. *Let Θ be an inner function. Then the following are equivalent.*
i) Θ is a constant or a finite Blaschke product.
ii) $H^\infty = \mathfrak{A}(\mathbb{D}) + \Theta H^\infty$.

Proof. The implication $i) \Rightarrow ii)$ is straightforward. To prove the converse assume that the equality $H^\infty = \mathfrak{A}(\mathbb{D}) + \Theta H^\infty$ holds. Assume also that $\Theta = \Theta_1\Theta_2$ where Θ_1, Θ_2 are inner functions. Hence there exists $f \in \mathfrak{A}(\mathbb{D})$ and $h \in H^\infty$ such that $\Theta_1 = f + \Theta h$. We have $\Theta_1(1-\Theta_2 h) = f \in \mathfrak{A}(\mathbb{D})$. As we have observed in beginning of this section, the function $1-\Theta_2 h$ belongs to $\mathfrak{A}(\mathbb{D})$ and vanishes on $\sigma(\Theta_1)\cap\mathbb{T}$. It follows that $\Theta_2 h \in \mathfrak{A}(\mathbb{D})$ so $h \in \mathfrak{A}(\mathbb{D})$ and vanishes on $\sigma(\Theta_2)\cap\mathbb{T}$. Thus the function $1-\Theta_2 h$ vanishes on $\sigma(\Theta_1)\cap\mathbb{T}$ and equals 1 on $\sigma(\Theta_2)\cap\mathbb{T}$, which implies that $\sigma(\Theta_1)\cap\sigma(\Theta_2)\cap\mathbb{T} = \emptyset$. Now, to finishes the proof, it suffices to show that if Θ is neither a constant nor a finite Blaschke product then there exist two inner functions Θ_1, Θ_2 such that $\Theta = \Theta_1\Theta_2$ and $\sigma(\Theta_1)\cap\sigma(\Theta_2)\cap\mathbb{T} \neq \emptyset$. Indeed if the singular inner factor S_Θ of Θ is not constant, we take $\Theta_1 = S_\Theta^{1/2}$ and $\Theta_2 = \Theta/\Theta_1$, where

$$S_\Theta^{1/2}(z) = \exp\left(\frac{1}{2}\int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta} d\mu_\Theta(\zeta)\right), \quad z \in \mathbb{D}.$$

In this case the support of μ_Θ is contained in $\sigma(\Theta_1)\cap\sigma(\Theta_2)\cap\mathbb{T}$. Now if Θ has infinitely many zeros, take $(\lambda_n)_{n\geq 0}$ to be a sequence of zeros of Θ which converges to some $\zeta \in \mathbb{T}$. We set $\Theta_1 = \Pi_{k\geq 0} b_{\lambda_{2k}}$ and $\Theta_2 = \Theta/\Theta_1$. We have $\zeta \in \sigma(\Theta_1)\cap\sigma(\Theta_2)\cap\mathbb{T}$. This finishes the proof. $\square$

Lemma 2.4. *Let $\mathfrak{I}$ be a closed ideal of $\mathfrak{A}(\mathbb{D})$. If*

$$\mathfrak{A}(\mathbb{D}) = A^+(\mathbb{T}) + \mathfrak{I}$$

then

$$H^\infty = A^+(\mathbb{T}) + \Theta_{\mathfrak{I}} H^\infty.$$

Proof. Assume that $\mathfrak{A}(\mathbb{D}) = A^+(\mathbb{T}) + \mathfrak{I}$. Notice that $\mathfrak{I} = \Theta_{\mathfrak{I}}\mathfrak{I}(E)$, where $E = h(\mathfrak{I}) \cap \mathbb{T}$. By the open mapping theorem there exists a constant $c > 0$ such that for every $f \in \mathfrak{A}(\mathbb{D})$ there exist $g \in A^+(\mathbb{T})$ and $h \in \mathfrak{I}(E)$ with $f = g + \Theta_{\mathfrak{I}}h$ and $\|g\|_1 \leq c\|f\|_\infty$. We have

$$\|h\|_\infty = \|\Theta_{\mathfrak{I}}h\|_\infty = \|f - g\|_\infty \leq (1 + c)\|f\|_\infty.$$

Let $f \in H^\infty$. For $0 \leq r < 1$, we set $f_r(z) = f(rz),\ z \in \overline{\mathbb{D}}$. The functions f_r are clearly in $\mathfrak{A}(\mathbb{D})$. By the observation above there exist $g_r \in A^+(\mathbb{T})$ and $h_r \in \mathfrak{I}(E)$ such that $f_r = g_r + \Theta_{\mathfrak{I}}h_r$, $\|g_r\|_1 \leq c\|f\|_\infty$ and $\|h_r\|_\infty \leq (1 + c)\|f\|_\infty$. The mapping $f \to (\hat{f}(n))_{n\geq 0}$ is an isometric isomorphism of the Banach space $A^+(\mathbb{T})$ onto ℓ^1, the Banach space of all complex sequences $(x_n)_{n\geq 0}$ such that $\sum_{n\geq 0} |x_n| < +\infty$. We may then identify $A^+(\mathbb{T})$ with the dual of c_0, the Banach space of all complex sequences that converge to 0. This induces a w*-topology on $A^+(\mathbb{T})$. For this topology, the closed bounded subsets of $A^+(\mathbb{T})$ are compact. Since the family $\{g_r,\ 0 \leq r < 1\}$ is bounded in $A^+(\mathbb{T})$, there exist $g \in A^+(\mathbb{T})$ and a sequence $r_k \to 1$ such that $(g_{r_k})_k$ converges to g for the w*-topology. This implies in particular that $(g_{r_k})_k$ converges to g uniformly on every compact subset of $\mathbb{D}$. Now the sequence $(h_{r_k})_k$ is bounded for the supremum norm on $\mathbb{D}$. It follows from the Montel theorem that there exists a subsequence of $(h_{r_k})_k$ which converges uniformly on every compact subset of $\mathbb{D}$ to some function $h \in H^\infty$. Now it is easily seen that $f = g + \Theta_{\mathfrak{I}}h$, which finishes the proof. $\square$

Remark 2.5. Let Θ be an inner function. Notice that equality $H^\infty = A^+(\mathbb{T}) + \Theta H^\infty$ implies that $\mathfrak{A}(\mathbb{D}) = A^+(\mathbb{T}) + \mathfrak{I}$, where $\mathfrak{I} = \Theta\mathfrak{I}(\sigma(\Theta) \cap \mathbb{T})$.

Proof of Theorem 2.1 : *i)* $\Rightarrow$ *ii)* Suppose that $\mathfrak{A}(\mathbb{D}) = A^+(\mathbb{T}) + \mathfrak{I}$. Since every function in $\mathfrak{I}$ vanishes on $h(\mathfrak{I}) \cap \mathbb{T}$, we have $\mathfrak{A}(h(\mathfrak{I}) \cap \mathbb{T}) = A^+(h(\mathfrak{I}) \cap \mathbb{T})$. Notice that $\mathfrak{I} \neq \{0\}$ and hence $h(\mathfrak{I}) \cap \mathbb{T}$ is of Lebesgue measure zero. Thus we have $\mathcal{C}(h(\mathfrak{I}) \cap \mathbb{T}) = \mathfrak{A}(h(\mathfrak{I}) \cap \mathbb{T})$ ([8], p. 81). It follows then that $h(\mathfrak{I}) \cap \mathbb{T}$ is a Helson set. On the other hand, combining Lemma 2.4 and Lemma 2.3 we see that $\Theta_{\mathfrak{I}}$ is constant or a finite Blaschke product.

ii) $\Rightarrow$ *i)* Notice that a Helson set is of Lebesgue measure zero. Using again a result from ([8], p. 81), we get that $\mathcal{C}(E) = \mathfrak{A}(E)$, where $E = h(\mathfrak{I}) \cap \mathbb{T}$. It follows from [23] that $\mathcal{C}(E) = A^+(E)$. Thus $\mathfrak{A}(E) = A^+(E)$ or equivalently $\mathfrak{A}(\mathbb{D}) = A^+(\mathbb{T}) + \mathfrak{I}(E)$. Since $\Theta_{\mathfrak{I}}$ is constant or a finite Blaschke product we have, clearly, $\mathfrak{I}_{\Theta_{\mathfrak{I}}} = \mathfrak{I}(E)$. So $\mathfrak{A}(\mathbb{D}) = A^+(\mathbb{T}) + \mathfrak{I}_{\Theta_{\mathfrak{I}}}$. It follows now from Lemma 2.2 that $\mathfrak{A}(\mathbb{D}) = A^+(\mathbb{T}) + \mathfrak{I}$.

i) $\Leftrightarrow$ *iii)* The Riesz representation theorem identifies the dual of $\mathcal{C}(\mathbb{T})$ with the space $\mathcal{M}(\mathbb{T})$, by the formula

$$< f, \mu >= \int_{\mathbb{T}} f(\zeta)d\mu(\zeta), \quad (f \in \mathcal{C}(\mathbb{T}), \mu \in \mathcal{M}(\mathbb{T})).$$

By the F. and M. Riesz theorem ([8], p. 47), we may identify the dual of $\mathfrak{A}(\mathbb{D})$ with the space $\mathcal{M}(\mathbb{T})/H^1_0$. So the norm of a measure $\mu \in \mathcal{M}(\mathbb{T})$, in the dual of $\mathfrak{A}(\mathbb{D})$, is equal to $\operatorname{dist}(\mu, H_0^1)$. Moreover the norm of μ in the dual of $A^+(\mathbb{T})$ is equal to $2\pi \sup_{n\leq 0} |\hat{\mu}(n)|$. On the other hand the equality $\mathfrak{A}(\mathbb{D}) = A^+(\mathbb{T}) + \mathfrak{I}$ is equivalent to the surjectivity of the map $f \to f + \mathfrak{I}$, from $A^+(\mathbb{T})$ into $\mathfrak{A}(\mathbb{D})/\mathfrak{I}$. So the equivalence between the assertions $i)$ and $iii)$ follows from [16], Lemma 2, p. 244. $\square$

3. Polynomially bounded operators

If X is a Banach space, we denote by $PB(X)$ the set of all polynomially bounded operators acting on X. If E is a closed ideal of $\mathbb{T}$, we set

$$I^+(E) = \{f \in A^+(\mathbb{T}),\ f = 0 \text{ on } E\}.$$

Let I be a closed ideal of $A^+(\mathbb{T})$. We denote by I^∞ the closed ideal of $\mathfrak{A}(\mathbb{D})$ generated by I. In fact I^∞ is the closure of I in $\mathfrak{A}(\mathbb{D})$ for the norm $\|.\|_\infty$. Notice that we have $h(I) = h(I^\infty)$ and $\Theta_I = \Theta_{I^\infty}$.

We will use the same symbol $\|.\|_1$ for the norm in the quotient algebra $A^+(\mathbb{T})/I$ as we use for the norm in $A^+(\mathbb{T})$: if $f \in A^+(\mathbb{T})$ we write $\|f + I\|_1 := \inf_{g\in I} \|f + g\|_1$. Similarly the norm in the quotient algebras of $\mathfrak{A}(\mathbb{D})/\mathfrak{I}$ will be denoted by $\|.\|_\infty$.

Theorem 3.1. *Let I be a closed ideal of $A^+(\mathbb{T})$ and let*

$$\begin{aligned} T_I\ :\ A^+(\mathbb{T})/I &\longrightarrow A^+(\mathbb{T})/I \\ f + I &\longrightarrow \alpha f + I \end{aligned}$$

where $\alpha : z \to z$ is the identity map. Then the following are equivalent.
i) $T_I \in PB(A^+(\mathbb{T})/I)$.
ii) $h(I) \cap \mathbb{T}$ is a Helson set, Θ_I is a constant or a finite Blaschke product and $I = \Theta_I I^+(h(I) \cap \mathbb{T})$.

Proof. $i) \Rightarrow ii)$ Suppose that T_I is polynomially bounded. By definition there exists a constant $C > 0$ such that, for all polynomials P, we have $\|P(T_I)\| \leq C\|P\|_\infty$. Since the set of polynomials is dense in $\mathfrak{A}(\mathbb{D})$ we can define the functionnal calculus $g(T_I)$, $g \in \mathfrak{A}(\mathbb{D})$, and the inequality $\|g(T_I)\| \leq C\|g\|_\infty$ remains true. For $g \in A^+(\mathbb{T})$, $g(T_I)$ is the operator $f + I \longrightarrow gf + I$, $f \in A^+(\mathbb{T})/I$, and $\|g(T_I)\| = \|g + I\|_1$. Hence if $g \in A^+(\mathbb{T})$ and $h \in I$, we have

$$\|g + I\|_1 = \|g + h + I\|_1 = \|(g + h)(T_I)\| \leq C\|g + h\|_\infty.$$

Since I is dense in I^∞, we obtain

$$\|g + I\|_1 \leq C\|g + I^\infty\|_\infty. \tag{3.1}$$

This shows that the range of the canonical map

$$\begin{aligned} i\ :\ A^+(\mathbb{T})/I &\longrightarrow \mathfrak{A}(\mathbb{D})/I^\infty \\ f + I &\longrightarrow f + I^\infty \end{aligned}$$

is closed. Since it is dense in $\mathfrak{A}(\mathbb{D})/I^\infty$, it follows that i is surjective. This means that $\mathfrak{A}(\mathbb{D}) = A^+(\mathbb{T}) + I^\infty$. Now Theorem 2.1 implies that $h(I) \cap \mathbb{T}$ is

a Helson set, Θ_I is a constant or a finite Blaschke product.
Notice that the imbeding $I \subset I^\infty \cap A^+(\mathbb{T})$ holds always, while the converse $I^\infty \cap A^+(\mathbb{T}) \subset I$ follows immediately from inequality (3.1). So we have $I = I^\infty \cap A^+(\mathbb{T})$. Now since $I^\infty = \Theta_I \mathfrak{I}(h(I)\cap\mathbb{T})$ and since Θ_I is a constant or a finite Blaschke product, we get easily that $I = \Theta_I I^+(h(I)\cap\mathbb{T})$.

ii) $\Rightarrow$ *i)* By Lemma 2.1 we have $\mathfrak{A}(\mathbb{D}) = A^+(\mathbb{T}) + I^\infty$, which means that the canonical map $i \ : A^+(\mathbb{T})/I \longrightarrow \mathfrak{A}(\mathbb{D})/I^\infty$ is surjective. In fact i is invertible since $I = \Theta_I I^+(h(I)\cap\mathbb{T})$. Consider now the operator U defined by

$$\begin{aligned} U \ &: \mathfrak{A}(\mathbb{D})/I^\infty \longrightarrow \mathfrak{A}(\mathbb{D})/I^\infty \\ &f + I^\infty \longrightarrow \alpha f + I^\infty \end{aligned}$$

Clearly U is a bounded linear operator on $\mathfrak{A}(\mathbb{D})/I^\infty$, and for every polynomial P we have

$$\|P(U)\| = \|P + I^\infty\|_\infty \le \|P\|_\infty.$$

On the other hand for every polynomial P we have $P(T_I) = i^{-1}P(U)i$. So $\|P(T_I)\| \le \|i^{-1}\|\|P\|_\infty$ and $T_I \in PB(A^+/I)$, which finishes the proof. $\square$

Let T be an operator on a Banach space X such that $\sup_{n\ge 0}\|T^n\| < +\infty$. We associate T with the following continuous morphism

$$\begin{aligned} \Phi_T \ &: A^+(\mathbb{T}) \longrightarrow \mathcal{L}(X) \\ &f \longrightarrow f(T) = \sum_{n\ge 0} \hat{f}(n)T^n. \end{aligned} \tag{3.2}$$

Notice that the kernel $\ker(\Phi_T)$ is a closed ideal of $A^+(\mathbb{T})$. The following result gives in particular a criterion in terms of $\ker(\Phi_T)$ which implies that $T \in PB(X)$.

Theorem 3.2. *Let I be a closed ideal of $A^+(\mathbb{T})$. Then the following are equivalent.*
i) Every operator T such that $\sup_{n\ge 0}\|T^n\| < +\infty$ and $I \subset \ker(\Phi_T)$ is polynomially bounded.
ii) $h(I)\cap\mathbb{T}$ is a Helson set, Θ_I is a constant or a finite Blaschke product and $I = \Theta_I I^+(h(I)\cap\mathbb{T})$.

Proof. *i)* $\Rightarrow$ *ii)* Consider the contraction defined in Theorem 3.1

$$\begin{aligned} T_I \ &: A^+(\mathbb{T})/I \longrightarrow A^+(\mathbb{T})/I \\ &f + I \longrightarrow \alpha f + I \end{aligned}$$

We have clearly $\ker(\Phi_{T_I}) = I$ so that $T_I \in PB(A^+(\mathbb{T})/I)$. It follows from Theorem 3.1 that *ii)* holds.

ii) $\Rightarrow$ *i)* Let T be an operator on a Banach space X such that $\sup_{n\ge 0}\|T^n\| < +\infty$ and $I \subset \ker(\Phi_T)$. There exists a continuous morphism

$$\widetilde{\Phi_T} \ : \ A^+(\mathbb{T})/I \to \mathcal{L}(X)$$

such that $\Phi_T = \widetilde{\Phi_T} \circ \pi$, where $\pi \ : \ A^+(\mathbb{T}) \to A^+(\mathbb{T})/I$ is the canonical surjection.

Let $f \in A^+(\mathbb{T})$. We have

$$\|f(T)\| = \|\Phi_T(f)\| = \|\widetilde{\Phi_T} \circ \pi(f)\| \leq \|\widetilde{\Phi_T}\| \|\pi(f)\|_1 \leq \|\Phi_T\| \|f(T_I)\|.$$

It follows from Theorem 3.1 that $\|f(T_I)\| \leq C\|f\|_\infty$, where C is a positive constant independent of f. So for every function $f \in A^+(\mathbb{T})$, we have $\|f(T)\| \leq C\|\widetilde{\Phi_T}\| \|f\|_\infty$, which proves that $T \in PB(X)$. $\square$

We deduce the following well-known result (see for example [24]).

Corollary 3.3. *Let T be an operator such that $\sup_{n\geq 0} \|T^n\| < +\infty$. Assume that there exists a nonzero polynomial P such that $P(T) = 0$. Then T is polynomially bounded.*

Proof. We set $Z = \{\lambda \in \overline{\mathbb{D}},\ P(\lambda) = 0\}$. Notice that Z is not empty since it contains $\mathrm{Sp}(T)$. Let I be the closed ideal of $A^+(\mathbb{T})$ generated by P. Since $h(I) = Z$ is a finite set it follows from the principal result of [10] that $I = I^\infty \cap A^+(\mathbb{T})$ (see also [4] and [6]). It is easily seen that $\Theta_I = \prod_{\lambda\in\Lambda} b_\lambda^{k_\lambda}$, where $\Lambda = Z \cap \mathbb{D}$ and k_λ is the mutiplicity of λ. So we have $I = \Theta_I I^+(Z \cap \mathbb{T})$. Now the corollary follows from the inclusion $I \subset \ker(\Phi_T)$ and Theorem 3.2. $\square$

4. Isometries and contractions with countable spectra

If M is a subset of $\mathcal{C}(\mathbb{T})$, we set

$$\widetilde{h}(M) = \{z \in \mathbb{T},\ f(z) = 0,\ (f \in M)\},$$

and if $f \in \mathcal{C}(\mathbb{T})$, we recall that $\|f\|_\infty = \sup_{z\in\mathbb{T}} |f(z)|$.

Let $\omega = (\omega_n)_{n\geq 1}$ be a sequence of real numbers with $\omega_n \geq 1$ and $\omega_{n+m} \leq \omega_n\omega_m$, for all $n, m \in \mathbb{Z}$. We say then that ω is a weight. The Beurling algebra $A_\omega(\mathbb{T})$ defined by the weight ω is the set of functions

$$f(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\theta} \quad \text{with} \quad \|f\|_\omega := \sum_{n\in\mathbb{Z}} |\hat{f}(n)|\omega_n < +\infty.$$

Let E be a closed subset of $\mathbb{T}$. We set

$$I_\omega(E) = \{f \in A_\omega(\mathbb{T}),\ f_{|E} = 0\}$$

and we denote by $J_\omega(E)$ the closure in $A_\omega(\mathbb{T})$ of the set of functions in $A_\omega(\mathbb{T})$ which vanish on a neighborhood of E. Clearly $I_\omega(E)$ and $J_\omega(E)$ are closed ideals of $A_\omega(\mathbb{T})$ and we have $J_\omega(E) \subset I_\omega(E)$. Suppose that $\sum_{n\in\mathbb{Z}} \frac{\log \omega_n}{1+n^2} < +\infty$, so that the algebra $A_\omega(\mathbb{T})$ is regular ([14], p. 118, Ex. 7). Then we have $\widetilde{h}(J_\omega(E)) = \widetilde{h}(I_\omega(E)) = E$. Moreover if I is a closed ideal of $A_\omega(\mathbb{T})$ such that $\widetilde{h}(I) = E$ then we have $J_\omega(E) \subset I \subset I_\omega(E)$. We say that E satisfies spectral synthesis for $A_\omega(\mathbb{T})$ if $J_\omega(E) = I_\omega(E)$, which is equivalent to the existence of a unique closed ideal I of $A_\omega(\mathbb{T})$ such that $\widetilde{h}(I) = E$. When $\omega(n) = 1$ for every integer n, we will write $I(E)$ [resp.

$J(E)$] instead of $I_\omega(E)$ [resp. $J_\omega(E)$]. Notice that every closed countable subset of $\mathbb{T}$ satisfies spectral synthesis for $A_\omega(\mathbb{T})$ for all weights ω such that $\omega(n) = 1$ for $n \geq 0$ and $\lim_{n\to+\infty} \frac{\log \omega(-n)}{\sqrt{n}} = 0$ ([25]).

We have the following result, which gives a general criterion for an isometry to be polynomially bounded.

Theorem 4.1. *Let E be a closed subset of $\mathbb{T}$. Then the following are equivalent.*
i) Every isometry T such that $Sp(T) \subset E$ is polynomially bounded.
ii) E is a Helson set and satisfies spectral synthesis for $A(\mathbb{T})$.

Proof. *i)* $\Rightarrow$ *ii)* Let I be a closed ideal of $A(\mathbb{T})$ such that $\widetilde{h}(I) = E$ and consider the isometry

$$\begin{aligned} T\ &: A(\mathbb{T})/I \longrightarrow A(\mathbb{T})/I \\ &f + I \longrightarrow \alpha f + I \end{aligned}$$

We have $\mathrm{Sp}(T) = E$ and then $T \in PB(A(\mathbb{T})/I)$. Let $P = \sum_{|k|\leq n} a_k \alpha^k$ be a trigonometric polynomial. Since multiplication by α acts as an isometry on $A(\mathbb{T})$ and on $\mathcal{C}(\mathbb{T})$, and since $T \in PB(A(\mathbb{T})/I)$, we have

$$\|P + I\|_1 = \|\alpha^n P + I\|_1 = \|(\alpha^n P)(T)\| \leq C\|\alpha^n P\|_\infty \leq C\|P\|_\infty,$$

where C is a constant independent of P. It follows that for every $f \in A(\mathbb{T})$, we have

$$\|f + I\|_1 \leq C\|f + I\|_\infty.$$

Let M denote the closed ideal generated by I in $\mathcal{C}(\mathbb{T})$. Since the set of trigonometric polynomials is dense in $\mathcal{C}(\mathbb{T})$, it is easily seen that M is the closure of I in $\mathcal{C}(\mathbb{T})$. We have then, with the help of the above inequality, that

$$\|f + I\|_1 \leq C\|f + M\|_\infty, \quad (f \in A(\mathbb{T})). \tag{4.1}$$

Notice that $\widetilde{h}(M) = E$ and, according to the structure of closed ideals of $\mathcal{C}(\mathbb{T})$, we have $M = \{f \in \mathcal{C}(\mathbb{T}),\ f_{|E} = 0\}$. It follows from this and inequality (4.1) that $I = I(E)$, which shows that E satisfies spectral synthesis for $A(\mathbb{T})$. Inequality (4.1) shows also that the range of the canonical imbedding $i\ :\ A(\mathbb{T})/I \to \mathcal{C}(\mathbb{T})/M$ is closed. The range of i is dense in $\mathcal{C}(\mathbb{T})/M$ and so i is surjective. Therefore we have $\mathcal{C}(\mathbb{T}) = A(\mathbb{T}) + M$, which means that E is a Helson set.

ii) $\Rightarrow$ *i)* Let T be an isometry on X such that $\mathrm{Sp}(T) \subset E$. We set

$$\begin{aligned} \Psi_T\ &: A(\mathbb{T}) \longrightarrow \mathcal{L}(X) \\ &f \longrightarrow f(T) = \sum_{n\in\mathbb{Z}} \hat{f}(n)T^n. \end{aligned}$$

The kernel $\ker(\Psi_T)$ is a closed ideal of $A(\mathbb{T})$ such that $\widetilde{h}(\ker(\Psi_T)) = \mathrm{Sp}(T)$. Thus $\ker(\Psi_T) \supset J(\mathrm{Sp}(T)) \supset J(E)$. Since E satisfies spectral synthesis in $A(\mathbb{T})$ we have $J(E) = I(E)$ and so $\ker(\Psi_T) \supset I(E)$. We have $\Psi_T = \Phi_T$ on

$A^+(\mathbb{T})$, where Φ_T is defined in (3.2). This implies that $\ker(\Phi_T) = \ker(\Psi_T) \cap A^+(\mathbb{T}) \supset I^+(E)$. It follows then from Theorem 3.2 that $T \in PB(X)$. $\square$

Now we will turn to contractions with countable spectra. For such operators we obtain the following result.

Theorem 4.2. *Let $T \in \mathcal{L}(X)$ be a contraction on a Banach space X such that $Sp\ (T)$ is a countable Helson subset of $\mathbb{T}$ and*

$$\limsup_{|z|\to 1-} (1-|z|)\log^+ \|(z-T)^{-1}\| = 0. \tag{4.2}$$

or equivalently

$$\limsup_{n\to+\infty} \frac{\log \|T^{-n}\|}{\sqrt{n}} = 0. \tag{4.3}$$

Then T is polynomially bounded.

Proof. Notice that the equivalence between Conditions (4.2) and (4.3) follows from [2], Lemma 2. Assume now that T is a contraction such that $\mathrm{Sp}(T)$ is a countable Helson set and such that Condition (4.3) holds. We set $\omega(n) = \|T^n\|$, $(n \in \mathbb{Z})$. Clearly we have $\omega(n) = 1$ for $n \geq 0$ and $\lim_{n\to+\infty} \frac{\log \omega(-n)}{\sqrt{n}} = 0$. For $f \in A_\omega(\mathbb{T})$, we set $\Psi_T(f) = f(T) := \sum_{-\infty}^{\infty} \hat{f}(n)T^n$. The kernel $\ker(\Psi_T)$ is a closed ideal of $A_\omega(\mathbb{T})$ such that $\widetilde{h}(\ker(\Psi_T)) = \mathrm{Sp}(T)$. Since closed countable sets satisfy spectral synthesis in $A_\omega(\mathbb{T})$ ([25]), we obtain that $\ker(\Psi_T) = \{f \in A_\omega(\mathbb{T}),\ f = 0 \text{ on } Sp(T)\}$. The equality $\Psi_T = \Phi_T$ holds on $A^+(\mathbb{T})$ and therefore $\ker(\Phi_T) = I^+(\mathrm{Sp}(T))$. Now by Theorem 3.2 we get that $T \in PB(X)$. $\square$

Remark 4.3. 1. It is shown in [26] that if T is a contraction such that $\mathrm{Sp}(T)$ is a countable subset of $\mathbb{T}$ and satisfies Condition (4.2) or (4.3) then T is an isometry. So combining this result and Theorem 4.1 we obtain Theorem 4.2.

2. Notice that finite unions of independent countable closed subets of $\mathbb{T}$ are Helson sets. Also finite unions of Hadamard sets are Helson sets ([9], p. 54; see also [16], Proposition 6, p. 332). Moreover these sets satisfy spectral synthesis for $A(\mathbb{T})$. See also Section 5 and [16], Chapter X, for some complementary information about sets which are Helson sets and satisfy the spectral synthesis for $A(\mathbb{T})$.

Now we will be interested in the hypotheses of Theorem 4.2 and we will show that they are best possible. We know by Theorem 4.1 that if E is not a Helson set then there exists a non polynomially bounded isometry such that $\mathrm{Sp}(T) \subset E$. Obviously an invertible isometry satisfies a stronger conditions than (4.2) and (4.3). Now we study the condition related to the countability of the spectrum and conditions (4.2) and (4.3). To do this let us first make some observations. Let $I \neq \{0\}$ be a closed ideal of $A^+(\mathbb{T})$. We denote by T_I the operator defined on $A^+(\mathbb{T})/I$ by

$$\begin{aligned} T_I \ : A^+(\mathbb{T})/I &\longrightarrow A^+(\mathbb{T})/I \\ f + I &\longrightarrow \alpha f + I. \end{aligned}$$

Notice that T_I is a contraction such that $\mathrm{Sp}(T_I) = h(I)$. Let $f \in A^+(\mathbb{T})$. We denote by μ_I (resp. μ_f) the measure defining the singular inner factor of Θ_I (resp. f) and by ν_I (resp. ν_f) the discrete part of μ_I (resp. μ_f). For $\lambda \in \mathbb{D}$, $z \in \overline{\mathbb{D}}$, we set

$$\phi_\lambda(z) = \begin{cases} \frac{f(z)-f(\lambda)}{z-\lambda} & \text{if} \quad z \neq \lambda \\ f'(\lambda) & \text{if} \quad z = \lambda. \end{cases}$$

The function ϕ_λ is in $A^+(\mathbb{T})$ and we have the equality in the Banach algebra $A^+(\mathbb{T})/I$,

$$(\phi_\lambda + I) = (\alpha - \lambda + I)^{-1}(f - f(\lambda) + I), \ \lambda \in \mathbb{D} \setminus h(I).$$

Suppose now that $h(I) \subset \mathbb{T}$ and $f \in I$. We have

$$\begin{aligned} \|f(\lambda)(\alpha - \lambda + I)^{-1}\|_1 &= \|\phi_\lambda + I\|_1 \\ &\leq \|\phi_\lambda\|_1 \\ &\leq \frac{2\|f\|_1}{1 - |\lambda|}. \end{aligned}$$

It follows then from [1], part d) of Lemma 5, that for every $\epsilon > 0$, we have

$$\|(\lambda - T_I)^{-1}\| = O\left(\frac{1}{1-|\lambda|} \exp\left(\frac{\epsilon + 2\|\nu_f\|}{1-|\lambda|}\right)\right), \quad |\lambda| \to 1- \, .$$

It is shown in [6], Lemma 1.3, that there exists a sequence $\{f_n\}_{n\geq 1} \subset I$ such that $\lim_{n\to\infty} \|\mu_{f_n} - \mu_I\| = 0$, which implies in particular that $\lim_{n\to\infty} \|\nu_{f_n} - \nu_I\| = 0$. Thus we obtain

$$\limsup_{|\lambda|\to 1-} (1 - |\lambda|) \log^+ \|(\lambda - T_I)^{-1}\| \leq 2\|\nu_I\|. \tag{4.4}$$

The following result shows that Condition (4.2) (or (4.3)) in Theorem 4.2 is best possible.

Proposition 4.4. *Let $\epsilon > 0$. There exists a contraction T such that $Sp(T) = \{1\}$,*

$$\limsup_{|z|\to 1-} (1 - |z|) \log^+ \|(z - T)^{-1}\| \leq \epsilon,$$

and T is not polynomially bounded.

Proof. Let Θ be the inner function defined by

$$\Theta(z) = \exp\left(\frac{\epsilon}{2} \frac{z+1}{z-1}\right), \ |z| < 1,$$

and let $I = \Theta H^\infty \cap A^+(\mathbb{T})$. Using Theorem 3.1 and inequality (4.4) it is easily seen that $T = T_I$ is the desired contraction. $\square$

Now we will show that the hypothesis in Theorem 4.2 about countability of the spectum is best possible. First we recall that E is called a multiplicity set if there exists a nonzero complex sequence $(c_n)_{n\in\mathbb{Z}}$ such that $\lim_{n\to+\infty}\sum_{k=-n}^{n} c_n e^{ikt} = 0$ for every $e^{it} \in \mathbb{T}\setminus E$.

Proposition 4.5. *Let E be a closed uncountable subset of $\mathbb{T}$. There exists a contraction T such that $Sp(T)$ is a Helson set contained in E, T satisfies Condition (4.2) and T is not polynomially bounded.*
Moreover if we assume that E is a multiplicity set then T can be chosen to be an isometry.

Proof. Since E is closed and uncountable, by [1], Lemma 5.3, there exists a perfect closed set F contained in E and which satisfies the Carleson condition, that is

$$\int_0^{2\pi} \log\frac{1}{d(e^{it},F)}dt < +\infty, \tag{4.5}$$

where $d(e^{it},F)$ is the distance from e^{it} to F. Now by a well known result ([9], p. 92) there exists a perfect closed Kronecker set G contained in F (see also [16], p. 338). Since G is uncountable, a classical result of Lebesgue asserts that G supports a nonzero continuous positive measure μ. Let Θ be the singular inner function defined by μ and let

$$I = \Theta H^\infty \cap A^+(\mathbb{T}).$$

Since $G \subset F$, G satisfies also the Carleson condition (4.5). According to [20], Theorem 3.3, there exists a nonzero outer function h which is infinitely differentiable on $\overline{\mathbb{D}}$ and which vanishes together with all its derivatives exactly on G. It is easily seen that $\Theta f \in I$. Hence $h(I) \subset G$. We set $T = T_I$. Thus T is a contraction with $\mathrm{Sp}(T) = h(I) \subset G$. Notice that every Kronecker set is a Helson set ([10], p. 89; see also section 5 below). It follows that G and then $\mathrm{Sp}(T)$ are Helson sets. We get from inequality (4.4) that T satisfies Condition (4.2). Finally T is not polynomially bounded by Theorem 3.1. So T satisfies all the required conditions.

Assume now that E is a multiplicity set. By a theorem of Kaufman, [15], there exists a Helson set $F \subset E$ which is also a multiplicity set (see also [16], Theorem 9, p. 250). It follows from [12], Theorème III and IV, pp. 142-143, that F does not satisfy spectral synthesis in $A(\mathbb{T})$. By Theorem 3.2 there exists an isometry T which is not polynomially bounded and such that $\mathrm{Sp}(T) \subset F$. It is clear that $\mathrm{Sp}(T)$ is a Helson set, which finishes the proof. $\square$

5. Contractions with spectra contained in Kronecker sets

A holomorphic function φ on $\mathbb{C}\setminus\mathbb{T}$ such that $\lim_{|z|\to+\infty}\varphi(z)=0$ is called a hyperdistribution on $\mathbb{T}$ (or hyperfunction). We denote by $HD(\mathbb{T})$ the space of all hyperdistributions on $\mathbb{T}$. Let $\varphi\in HD(\mathbb{T})$. The Fourier coefficients $(\hat\varphi(n))_{n\in\mathbb{Z}}$ of $\varphi\in HD(\mathbb{T})$ are given by the formula

$$\varphi(z)=\begin{cases} \sum_{n\geq 0}\hat\varphi(n)z^n, & |z|<1\\ -\sum_{n<0}\hat\varphi(n)z^n, & |z|>1.\end{cases}$$

The support of $\varphi\in HD(\mathbb{T})$, which we denote by $\mathrm{supp}(\varphi)$, is the smallest closed subset F of $\mathbb{T}$ such that there exists a holomorphic function on $\mathbb{C}\setminus F$, which agrees with φ on $\mathbb{C}\setminus\mathbb{T}$.

A hyperdistribution $\varphi\in HD(\mathbb{T})$ is called a pseudofunction [resp. pseudomeasure] if $\lim_{|n|\to\infty}\hat\varphi(n)=0$ [resp. $\sup_{n\in\mathbb{Z}}|\hat\varphi(n)|<+\infty$]. We denote by $PM(E)$ [resp. $M(E)$] the set of the pseudomeasures [resp. measures] with support contained in E.

Let ω be a regular weight and E a closed subset of $\mathbb{T}$. We denote by $HD_\omega(E)$ the set of all hyperdistributions φ such that $\mathrm{supp}\,(\varphi)\subset E$ and $\|\varphi\|_\omega^\star:=\sup_{n\in\mathbb{Z}}\frac{|\hat\varphi(n)|}{\omega(-n)}<\infty$. We set

$$HD^0_\omega(E)=\{\varphi\in HD_\omega(E),\ \lim_{n\to-\infty}\hat\varphi(n)=0\}.$$

The space $(HD_\omega(\mathbb{T}),\|.\|_\omega)$ can be identified with the dual of $A_\omega(\mathbb{T})$, the duality being implemented by the formula

$$<f,\varphi>=\sum_{n\in\mathbb{Z}}\hat f(n)\hat\varphi(-n),\quad (f\in A_\omega(\mathbb{T}),\ \varphi\in HD_\omega(\mathbb{T})).$$

A closed subset E of $\mathbb{T}$ is said to be without true pseudomeasures (WTP) if $PM(E)=M(E)$. Notice that E is a set WTP if and only if E is a Helson set and satisfies spectral synthesis for $A(\mathbb{T})$ (see [16], Chap. X, p. 330). Moreover Varopoulos showed in [21] that every Kronecker set is WTP. So a Kronecker set is a Helson set and satisfies spectral synthesis for $A(\mathbb{T})$.

We will need the following two elementary lemmas to establish the main result of this section.

Lemma 5.1. *Let ω be a weight such that $\omega(n)=1,\ n\geq 0$ and let I be a closed ideal of $A_\omega(\mathbb{T})$. Consider the operator*

$$\begin{aligned} T\ &: A_\omega(\mathbb{T})/I\longrightarrow A_\omega(\mathbb{T})/I\\ &f+I\longrightarrow \alpha f+I\end{aligned}$$

If $T\in PB(A_\omega(\mathbb{T})/I)$ then $I^+=\Theta_{I^+}\mathfrak{I}(h(I^+)\cap\mathbb{T})\cap A^+(\mathbb{T})$, where $I^+=I\cap A^+(\mathbb{T})$.

Proof. Suppose that $T\in PB(A_\omega(\mathbb{T})/I)$. There exists a constant $C>0$ such that for every $f\in A^+(\mathbb{T})$,

$$\|f+I\|_\omega=\|f(T)\|\leq C\|f\|_\infty.$$

For $g \in I^+$, we have $\|f+I\|_\omega \leq C\|f+g\|_\infty$. Since I^+ is dense in $(I^+)^\infty$, the closed ideal generated by I in $\mathfrak{A}(\mathbb{D})$, we obtain $\|f+I\|_\omega \leq C\|f+(I^+)^\infty\|_\infty$. It follows from this inequality that if $f \in (I^+)^\infty \cap A^+(\mathbb{T})$ then we have $f \in I$. So $(I^+)^\infty \cap A^+(\mathbb{T}) \subset I^+$. Therefore $i)$ holds since the inclusion $I^+ \subset (I^+)^\infty \cap A^+(\mathbb{T})$ is obvious and $(I^+)^\infty = \Theta_{I^+}\mathfrak{I}(h(I^+)\cap\mathbb{T})$. □

Remark 5.2. Notice that if I^+ is a closed ideal of $A^+(\mathbb{T})$ such that $h(I^+)\cap \mathbb{T}$ is a finite or countable set, or contained in a Cantor set, then $I^+ = \Theta_{I^+}\mathfrak{I}(h(I^+)\cap\mathbb{T})\cap A^+(\mathbb{T})$ ([10], [4], [6]). In the opposite direction J. Esterle constructed in [5] a closed ideal in $A^+(\mathbb{T})$ such that $I^+ \neq \Theta_{I^+}\mathfrak{I}(h(I^+)\cap \mathbb{T})\cap A^+(\mathbb{T})$, which shows that the Bennett-Gilbert conjecture about the structure of the closed ideals of $A^+(\mathbb{T})$ fails.

Lemma 5.3. *Let $(\delta_n)_{n\geq 1}$ be a sequence of real numbers such that $\lim_{n\to+\infty}\delta_n = +\infty$ and $\delta_n > 1$ for every $n \geq 1$. Then there exists an unbounded non–decreasing sequence $(\gamma_n)_{n\geq1}$ such that*

$$1 \leq \gamma_n \leq \delta_n \quad \textit{and} \quad \gamma_{n+m} \leq \gamma_n\gamma_m, \quad n,m \geq 1. \tag{5.1}$$

Proof. Without loss of generality we can assume that $(\delta_n)_{n\geq1}$ is non–decreasing, since otherwise we could consider the sequence $(\delta'_n)_{n\geq1}$ defined by $\delta'_n = \inf_{k\geq n}\delta_k$.

For $n \geq 1$, we set

$$\gamma_n = \inf \delta_{n_1}\delta_{n_2}\cdots\delta_{n_k},$$

where the infimum is taken over all families of integers $k, n_1 \cdots n_k$, greater than or equal to 1 and such that $n_1+n_2+\cdots n_k = n$. A simple computation shows that $(\gamma_n)_{n\geq1}$ is non–decreasing and satisfies Condition (5.1). It remain to check that $(\gamma_n)_{n\geq1}$ is unbounded. For this consider integers $n\geq1$, $k\geq1$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ be such that $n_1+n_2+\cdots+n_k = n$ and $\gamma_n = \delta_{n_1}\delta_{n_2}\cdots\delta_{n_k}$. We have clearly $\gamma_n \geq \max(\delta_{n_1}^k, \delta_{n_k})$. Since $kn_k \geq n$, we have $k \geq \sqrt{n}$ or $n_k \geq \sqrt{n}$. So $\gamma_n \geq \min(\delta_1^{\sqrt{n}}, \delta_{[\sqrt{n}]})$, where $[\sqrt{n}]$ is the nonnegative integer such that $[\sqrt{n}] \leq \sqrt{n} < [\sqrt{n}]+1$. Now since $\delta_1 > 1$ and $\lim_{n\to+\infty}\delta_n = +\infty$ we have also $\lim_{n\to+\infty}\gamma_n = +\infty$. □

Before stating the main result of this section we will make an observation. Denote by $\mathcal{C}^1([0,1])$ [resp. $\mathcal{C}^1(\mathbb{T})$] the Banach algebra of all continuously differentiable functions on $[0,1]$ [resp. $\mathbb{T}$]. The structure of the closed ideals of $\mathcal{C}^1([0,1])$ is known (see [22] and [13]). Indeed I is a closed ideal of $\mathcal{C}^1([0,1])$ if and only if there exist two closed subsets $E_1 \subset E_0$ of $[0,1]$ such that

$$I = \{f \in \mathcal{C}^1([0,1]),\ f^{(k)} = 0 \text{ on } \ E_k\ (k=0,1)\}$$

Since $\mathcal{C}^1([0,1])$ is a regular Banach algebra it follows that a function which vanishes with its derivative on a closed set $E \subset [0,1]$, satisfies spectral synthesis for E. This means that there exists a sequence of functions $(f_n)_n \subset \mathcal{C}^1([0,1])$ vanishing on a neighbohood of E such that $\lim_{n\to\infty}\|f_n - f\|_{\mathcal{C}^1([0,1])} = 0$. Now it is easily seen that this result holds also in $\mathcal{C}^1(\mathbb{T})$.

Theorem 5.4. *i) If T is an invertible isometry such that $Sp(T)$ is a Kronecker set then T is polynomially bounded.*
ii) Let $(\beta_n)_{n\geq 1}$ be a sequence of real numbers such that $\lim_{n\to+\infty}\beta_n = +\infty$ and $\beta_n > 1$ for every $n \geq 1$. Then there exists a non polynomially bounded contraction T such that $Sp(T)$ is a Kronecker subset of $\mathbb{T}$ and $\|T^{-n}\| \leq \beta_n, \quad n \geq 1$.

Proof. As we have observed before, every Kronecker set is a Helson set and satisfies the spectral synthesis for $A(\mathbb{T})$. So the assertion $i)$ follows from Theorem 4.1.
Now we prove $ii)$. We set $\delta_n = \min(\beta_n, (1+n)^{1/3}),\ n \geq 1$. Then $\lim_{n\to+\infty}\delta_n = +\infty$ and $\delta_n > 1$ for every $n \geq 1$. By Lemme 5.3 there exists an unbounded non-decreasing sequence $(\gamma_n)_{n\geq 1}$ satisfying Condition (5.1). We set $\omega(n) = 1$ for $n \geq 0$ and $\omega(n) = \delta_{-n}$ for $n < 0$. Then ω is a regular weight.
Let E be a totally discontinous closed subset of $\mathbb{T}$ which is a multiplicity set. Assume moreover that E satisfies the Carleson condition (4.5). Take for example $E = E_\xi$, where E_ξ is the perfect symmetric set of constant ratio $\xi \in (0, 1/2)$ with $1/\xi$ not a Pisot number (see [12], Théorème IV, p. 74). There exists a nonzero pseudofunction with support contained in E ([12], Chapitre V, p. 54). It follows then from Esterle's result ([5], Theorem 4.4) that there exists a Kronecker set $F \subset E$ and a nonzero hyperdistribution $\varphi \in HD^0_\omega(F)$.
Notice that $A^+(\mathbb{T})$ can be regarded as the dual of c_0, the space of all complex sequences converging to zero. This induces a w*-topology on $A^+(\mathbb{T})$. As observed before, since F is a Kronecker set, F is also a Helson set and by the Wik result [23], F is a Carleson set, that is $\mathfrak{A}(E) = A^+(E)$. So F is an AA^+ set, which means that $A(E) = A^+(E)$. Thus we have $\sup_{n\geq 0}\|\pi(\alpha)^{-n}\|_1 < +\infty$, where $\pi :\ A^+(\mathbb{T}) \to A^+(\mathbb{T})/I^+(F)$ is the canonical surjection and $\alpha :\ z \to z$ is the identity map. It follows from [5], Theorem 3.1, that $I^+(F)$ is dense in $A^+(\mathbb{T})$ for the w*-topology. Thus $\varphi \notin I^+(F)^\perp$. On the other hand we have $\varphi \in J_\omega(F)^\perp$. Therefore $I^+(F) \neq J_\omega(F)^+$, where $J_\omega(F)^+ = J_\omega(F) \cap A^+(\mathbb{T})$.
Since $F \subset E$, F also satisfies the Carleson condition. According to [20], Theorem 3.3, there exists a non zero outer function h which is infinitely differentiable on $\overline{\mathbb{D}}$ and which vanishes together with all its derivatives, exactly on F. By the observation we made before the theorem, there exists a sequence of functions in $\mathcal{C}^1(\mathbb{T})$, vanishing in a neighborhood of F and converging to h for the norm $\|.\|_{\mathcal{C}^1(\mathbb{T})}$. Since we have $\omega_n \leq (1+|n|)^{1/3},\ n \in \mathbb{Z}$, the imbeding $\mathcal{C}^1(\mathbb{T}) \subset A_\omega(\mathbb{T})$ is continuous. We deduce that $h \in J_\omega(F)$. Hence $\Theta_{J_\omega(F)^+}$ is constant and $h(J_\omega(F)^+) = F$. Therefore $\Theta_{J_\omega(F)^+}\mathfrak{I}\left(h(J_\omega(F)^+) \cap \mathbb{T}\right) \cap A^+(\mathbb{T}) = I^+(F)$. Since $I^+(F) \neq J_\omega(F)^+$, it follows from Lemma 5.1 that the operator

$$\begin{aligned} T\ &: A_\omega(\mathbb{T})/J_\omega(F) \longrightarrow A_\omega(\mathbb{T})/J_\omega(F) \\ &f + J_\omega(F) \longrightarrow \alpha f + J_\omega(F) \end{aligned}$$

is not polynomially bounded. Notice that T is a contraction such that $\mathrm{Sp}\,(T) = F$ and $\|T^{-n}\| \le \omega_{-n} \le \beta_n,\ n \ge 1$, which finishes the proof. $\square$

Remark 5.5. 1. The condition "T is an invertible isometry" in part i) of Theorem 5.4 may be replaced by the conditon

$$\sup_{n\in\mathbb{Z}} \|T^n\| < +\infty. \tag{5.2}$$

Indeed, assume that $T \in \mathcal{L}(X)$ satisfies (5.2), where X is a Banach space with norm $\|.\|$. For $x \in X$, we set $|||x||| = \sup_{n\in\mathbb{Z}} \|T^n x\|$. Then the norm $|||.|||$ is equivalent to the norm $\|.\|$ and T is an isometry with respect to $|||.|||$.

2. The conditions on the sequence $(\beta_n)_{n\ge1}$ in part *ii*) of Theorem 5.4 are optimal. Indeed, let T be a contraction such that $\mathrm{Sp}(T)$ is a Kronecker set and such that $\|T^{-n}\| \le \beta_n$. Assume that $(\beta_n)_{n\ge1}$ does not converge to $+\infty$ or that $\beta_n = 1$ for some $n \ge 1$. Then it is easily seen that there exists a subsequence of $(T^{-n})_{n\ge1}$ which is bounded. Since $(\|T^{-n}\|)_{n\ge1}$ is non–decreasing we get that $\sup_{n\ge1} \|T^{-n}\| < +\infty$. It follows now from the above remark and part *i*) of Theorem 5.4 that T is polynomially bounded.

Acknowledgements. I thank Professor Thomas Ransford for the interesting discussions we had which motivated me to work with polynomially bounded operators.

References

[1] C. Agrafeuil, *Idéaux fermés de certaines algèbres de Beurling et application aux opérateurs à spectre dénombrable*, preprint.

[2] A. Atzmon, *Operators which are annihilated by analytic functions and invariant subspaces* , Acta Math. **144** (1980), 27-63.

[3] C. Badea and V. Müller, *Growth conditions and inverse producing extensions*, preprint.

[4] C. Bennett and J.E. Gilbert, *Homogenuous algebras on the circle: I-Ideals of analytic functions*, Ann. Inst. Fourier Grenoble (3) **22** (1972), 1-19.

[5] J. Esterle, *Distributions on Kronecker sets, strong form of uniqueness, and closed ideals of A^+*, J. Reine angew. Math. **450** (1994), 43-82.

[6] J. Esterle, E. Strouse and F. Zouakia, *Closed ideals of A^+ and the Cantor set*, J. Reine angew. Math. **449** (1994), 65-79.

[7] C.C. Graham and O.C. Mc Gehee, *Essays in Commutative Harmonic Analysis*, Springer-Verlag, Berlin-Heidelberg-New York 1979.

[8] K. Hoffman, *Banach spaces of analytic functions*, Prentice Hall, Englwood Cliffs, New-Jersey 1962.

[9] J. P. Kahane, *Séries de Fourier absolument convergentes*, Erg. Math. **50** Springer-Verlag, Berlin-Heidelberg-New York 1970.

[10] J. P. Kahane, *Idéaux fermés dans certaines algèbres de Banach de fonctions analytiques*, Actes table ronde internat. C.N.R.S., Montpellier, Lect. Notes 336, Springer Verlag, Berlin (1973), 5-14.

[11] J. P. Kahane and Y. Katznelson, *Sur les algèbres de restrictions des séries de Taylor absolument convergentes à un fermé du cercle*, J. Anal. Math. **23** (1970), 185-197.
[12] J. P. Kahane and R. Salem, *Ensembles parfaits et série trigonométriques*, Hermann, Paris 1994.
[13] L. G. Khanin, *The structure of closed ideals some algebras of smooth functions*, Amer. Math. Soc. Transl. **149** (1991), 97-113.
[14] Y. Katznelson, *An introduction to Harmonic Analysis*, Wiley, New york 1968.
[15] R. Kaufman, *M-sets and distributions*, Astérisque 5 (1973), 225-230.
[16] A. Kecheris and A. Louveau, *Descriptive set theory and the structure of sets of uniqueness*, London Math. Soc. Lect. Notes Math. **128** (1986), Cambridge University Press 1987.
[17] L.A. Lindahl and F. Poulsen (Editors), *Thin Sets in Harmonic Analysis*, Marcel Decker, New York, 1971.
[18] N.K. Nikolskii, *Treatise on the shift operator*, Springer Verlag, Heidelberg 1986.
[19] W. Rudin, *The closed ideals in an algebra of analytic functions*, Can. J. Math. **9** (1957),426-434.
[20] B.A. Taylor and D.L. Williams, *Ideals in rings of analytic functions with smooth boundary values*, Can. J. Math. **22** (1970), 1266-1283.
[21] N. Varopoulos, *Sur les ensembles parfaits et les séries trigonométriques*, C. R. Acad. Sci. Paris (A) **260** (1965), 3831-3834.
[22] H. Whitney, *On ideals of differentiable functions*, Amer. J. Math. **70** (1948), 635-658.
[23] I. Wik, *On linear independence in closed sets*, Ark. Mat. (4) **209** (1960), 209-218.
[24] P. Vitse, *Functional calculus under Kreiss type conditions*, preprint.
[25] M. Zarrabi, *Synthèse spectrale dans certaines algèbres de Beurling sur le cercle unité*, Bull. Soc. Math. France (2) **118** (1990), 241-249.
[26] M. Zarrabi, *Contractions à spectre dénombrable et propriétés d'unicité des fermés dénombrable du cercle*, Ann. Inst. Fourier (1) **43** (1993), 251-263.

LaBAG, CNRS-UMR 5467, Université Bordeaux I, 351 cours de la Libération, 33405 Talence (FRANCE).
E-mail address: Mohamed.Zarrabi@math.u-bordeaux1.fr