Abstract

We show that a periodic orbit of large period of a diffeomorphism or flow, either admits a dominated splitting of a prescribed strength, or can be turned into a sink or a source by a $C^1$-small perturbation along the orbit. As a consequence we show that the linear Poincaré flow of a $C^1$-vector field admits a dominated splitting over any robustly transitive set.

Contents

1 Introduction 2

2 Statement of the results 4
  2.1 Elementary definitions : linear cocycles 4
  2.2 Dominated decomposition 5
  2.3 Periodic cocycles 5
  2.4 Perturbations 7
  2.5 Statement of the results for linear cocycles 8
  2.6 Statement of the results for diffeomorphisms and flows 10

3 Proof of Theorem 2.1 12
  3.1 Perturbations on subbundles and quotient bundles 12
  3.2 Cocycle of dimension 2 13
  3.3 Proof of Theorem 2.1 and Scholium 2.16 14

4 Proof of Theorem 2.2 14
  4.1 Lyapunov diameter 14
  4.2 Splitting in subbundles strictly without domination 17
  4.3 Decreasing the Lyapunov spectrum: proof of Theorem 4.1 19

5 Appendix 24
  5.1 Elementary definitions for time-continuous systems 24
  5.2 Perturbation results for time-continuous systems 25
  5.3 Preliminary 2-dimensional analytic result 27
  5.4 Generalization to higher dimension 30
  5.5 Proof of Theorem 5.2 30
1 Introduction

In [Ma], R. Mañé proved that any robustly transitive diffeomorphism $f$ of a closed surface $S$ is an Anosov diffeomorphism. Let us give a sketch of his proof.

Mañé first establishes that the tangent bundle of the surface splits into a direct sum $TS = E \oplus F$ of two line bundles, and that this splitting is dominated: the vectors in $F$ are uniformly more expanded by $Df$ than those in $E$. In order to prove this, he shows that, if there were no dominated splitting, then it would be possible to perturb (in the $C^1$-topology) the differential of the diffeomorphism along some periodic orbits in order to create a complex eigenvalue, thus creating a sink or a source and breaking the transitivity. This is a purely linear-algebraic argument on periodic sequences of matrices in $GL(2, \mathbb{R})$. It remains to show that the vectors in $E$ are indeed uniformly contracted and the vectors in $F$ uniformly expanded. This requires deeper arguments, in particular the ergodic closing lemma.

The first argument of this proof has been adapted in higher dimension, first in [DPU] in dimension 3 and then in [BDP] in any dimension: any robustly transitive diffeomorphisms admits a dominated splitting. However, the arguments in these papers fail to be purely algebraic: both of them use the classical fact that, if $P$ and $Q$ are homoclinically related hyperbolic saddles, there are periodic saddles (with arbitrarily long periods) that remain arbitrarily close to the orbit of $P$ during an arbitrarily long time, then jump into a neighborhood of $Q$ in a bounded time, remain there during an arbitrarily long time, come back in a small neighborhood of $P$ in a bounded time, and so on. This dynamical argument allows in some sense to multiply large positive iterates of the derivative of $f$ corresponding to different periodic orbits. This "semi-group-like" property has been formalized in [BDP] through the notion of linear cocycles with transitions, whose archetype is the cocycle induced by $Df$, over the set of homoclinically related periodic orbits in some homoclinic class. [BDP] shows that, if a linear cocycle with transitions does not admit any dominated splitting, then a small perturbation of the diffeomorphism enables to turn the differential along a periodic orbit into a homothety, thus turning the periodic orbit into a sink or a source.

The notion of transition introduced in [BDP] is a very strong tool, but it turns out to be also heavy and not very flexible. For instance, it is not easy to adapt its definition to continuous time systems. Another problem was that [BDP] does not give any information on the existence of dominated splittings for sequences of periodic orbits with trivial homoclinic class.

However, transitions are needed to get the linear result in [BDP]. Given a linear cocycle without transition and without dominated splitting, it is indeed not always possible to turn the matrix at the period into an homothety by a small perturbation of the cocycle. We can nevertheless recover almost the same property: for any linear cocycle over periodic orbits (without transition hypothesis) without dominated splitting, there are arbitrarily small perturbations such that the matrix at the period (corresponding to some periodic orbit of the system) has all its eigenvalues real and with same modulus.

Let us just state a consequence of this result for the periodic orbits of a diffeomorphism:

\[ Df^i(x, n) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \forall i \in \{0, \ldots, p_n - 2\} \text{ and } Df^p(x) = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}. \]

Then these orbits admit no dominated splitting, but there is $\varepsilon > 0$ such that for any $\varepsilon$-perturbation $g$ of $f$ preserving the orbit of one $x_n$, the differential $Dg^{p_n}(x_n)$ is not an homothety.

\[1\text{Consider for example a sequence of periodic points } x_n \text{ with periods } p_n \to \infty \text{ of a diffeomorphism } f \text{ such that the differential along the orbit can be written (in local coordinates) as: } Df^i(x_n) = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \forall i \in \{0, \ldots, p_n - 2\} \text{ and } Df^p(x_n) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \text{ Then these orbits admit no dominated splitting, but there is } \varepsilon > 0 \text{ such that for any } \varepsilon\text{-perturbation } g \text{ of } f \text{ preserving the orbit of one } x_n, \text{ the differential } Dg^{p_n}(x_n) \text{ is not an homothety.} \]
Corollary 2.19 Let $f: M \to M$ be a diffeomorphism of a compact Riemannian manifold. Then for any $\varepsilon > 0$ there are two integers $\ell$ and $n$ such that, for any periodic point $x$ of period $p(x) \geq n$:

- either $f$ admits an $\ell$-dominated splitting along the orbit of $x$;
- or, for any neighborhood $U$ of the orbit of $x$, there exists an $\varepsilon$-perturbation $g$ of $f$ for the $C^1$-topology, coinciding with $f$ outside $U$ and on the orbit of $x$, and such that $x$ is a source or a sink of $g$ for which the differential $Dg^{p(x)}(x)$ has all eigenvalues real and with same modulus.

An analogous result for flows, see Corollary 2.22, has the following consequence:

Corollary 2.23 Let $X$ be a $C^1$-vector field on a compact manifold. Assume that $K$ is an invariant compact set that is robustly transitive. Then the linear Poincaré flow of $X$ on $K \setminus \text{Sing}(X)$ admits a dominated splitting.

(See Section 2.6 for the definition of robust transitivity and for the linear Poincaré flow).

Our results can be connected to a recent result by S. Gan in [Ga]²: given any $\varepsilon > 0$, there is $\ell > 0$ such that, for any periodic point $x$,

- either there is an $\ell$-dominated splitting $E \oplus F$ along the orbit of $x$ with $\dim E = i$,
- or there is an $\varepsilon$-small $C^1$-perturbation of $f$, that makes equal the moduli of the $i^{\text{th}}$ and the $(i+1)^{\text{th}}$ eigenvalues associated to the orbit of $x$.

Our results on periodic orbits of diffeomorphisms derive from analogous results obtained for linear cocycles (see subsection 2.1 for the definition and subsection 2.5 for the statement of the result) together with a lemma of Franks which enables to achieve any perturbation of the differential of a diffeomorphism as the differential of a perturbation of the diffeomorphism. These statements for linear cocycles also induce similar results for vector fields (see Corollary 2.22), thanks to a preliminary result for vector fields equivalent to Franks’ Lemma. Although an equivalent statement of Franks’ Lemma has already been used by several authors, as far as we know, no proof for this result has been yet published: we shall henceforth give precise statements and proofs in the Appendix.

We would like to thank Marie-Claude Arnaud and Sylvain Crovisier for many enlightening discussions, and the referee for pointing out the relation between Gan’s result and ours.

²It would be tempting to try to deduce our result from Gan’s result. If a periodic orbit has no $\ell$-dominated splitting, then one may apply Gan’s result to any pair of eigenvalues. So, one can hope to make equal, successively, every pair of eigenvalues. This idea relies on the fact that, what is done at a step does not destroy the effect of the previous perturbations; in other words, it needs the robustness of the proximity of some of the Lyapunov exponents, independently of the period of the periodic orbit. Unfortunately, one can easily build sequences of periodic orbits $\gamma_n$, whose period tends to infinity, having a pair of equal Lyapunov exponents that are separated by more than a constant (for $n$ large enough) by an arbitrarily small perturbation.
2 Statement of the results

2.1 Elementary definitions: linear cocycles

Definition 2.1. We shall call linear cocycle of dimension $d$ any 4-uple $A = (\Sigma, f, E, A)$ such that:

- $\Sigma$ is a set and $f : \Sigma \to \Sigma$ is a one-to-one map.
- $\pi : E \to \Sigma$ is a linear bundle of dimension $d$ over $\Sigma$, whose fibers are endowed with a Euclidean metric $\| \|$. The fiber over the point $x \in \Sigma$ will be denoted by $E_x$.
- $A : x \in \Sigma \mapsto A_x \in GL(E_x, E_{f(x)})$ is a map.

We shall say that a linear cocycle $A$ is bounded if there exists a constant $K > 0$ such that, for any point $x \in \Sigma$, we have $\|A_x\| < K$ and $\|A_x^{-1}\| < K$. We shall say that $A$ is a cocycle of matrices if the bundle $\pi : E \to \Sigma$ is trivial and if the metric on each fiber is the standard euclidean norm of $\mathbb{R}^d$. We shall then consider $A_x$ as an element of $GL(\mathbb{R}, d)$. Notice that any linear cocycle $A$ bounded by $K$ is, up to a choice of orthonormal basis over each fiber $E_x$, conjugated to a cocycle of matrices bounded by the same constant $K$.

Let $A$ be a linear cocycle. For any integer $n$, and any point $x \in \Sigma$, we shall denote by $A^n_x$ the product $A^n_x = A_{f^{n-1}(x)} \circ \ldots \circ A_{f(x)} \circ A_x$. Moreover, $A^n = (\Sigma, f^n, E, A^n)$ is a linear cocycle bounded by $K^n$.

We shall say that a subbundle $F \subset E$ is invariant if its fibers $F_x$ have constant dimension for any point $x \in \Sigma$ and if $F_{f(x)} = A_x(F_x)$. We denote by $F^\perp$ the (a priori non invariant) orthogonal subbundle of $F$, that is, for each point $x \in \Sigma$, $F_x^\perp$ is the orthogonal supplement of $F_x$ in $E_x$. These subbundles are both naturally endowed with the metric induced by the metric defined on $E$.

Definition 2.2. Let $A$ be a linear cocycle, $\Sigma'$ an $f$-invariant subset of $\Sigma$ and $F$ an invariant subbundle over $\Sigma$.

1. The restriction of $A$ to the subset $\Sigma'$ is a linear cocycle $(\Sigma', f, E, A)$ denoted by $A|_{\Sigma'}$.

2. The linear cocycle $A_F$ induced by $A$ on $F$ is the linear cocycle $(\Sigma, f, F, A)$ obtained by restricting $A$ to the subbundle $F$.

3. The quotient cocycle of $A$ by $F$, denoted by $A/F$, is the linear cocycle $(\Sigma, f, F^\perp, A/F)$, where $(A/F)_x : F^\perp_x \to F^\perp_{f(x)}$ is the projection on $F^\perp_{f(x)}$, parallel to $F_{f(x)}$, of the restriction $(A|_{F^\perp})_x : F^\perp_x \to E_{f(x)}$ of $A_x$.

Notice that if $A$ is bounded by $K$, then $A|_{\Sigma'}$, $A_F$ and $A/F$ are also bounded by $K$: this is obvious for the first two cocycles. This is true for $A/F$ because the invariance of $F$ by $A$ implies

$$(A/F)^{-1} = A^{-1}/F,$$

and because the use of an orthogonal projection decreases the norm of the application (in fact, for any integer $n$, $(A/F)^n = A^n/F$).

We shall use the natural notions of transverse subbundles, and of direct sum of transverse subbundles.
2.2 Dominated decomposition

**Definition 2.3.** Let $A$ be a linear cocycle bounded by $K$, and $F,G$ two invariant subbundles. We shall say that $G$ dominates $F$, denoted by $F \prec G$ or $F \prec_\ell G$, if there exists an integer $\ell$ such that, for any point $x \in \Sigma$ and any pair of vectors $(u,v) \in F_x \times G_x$, the following inequality holds:

$$\frac{\|A^\ell_x(u)\|}{\|u\|} \leq \frac{1}{2} \frac{\|A^\ell_x(v)\|}{\|v\|}.$$

**Remark 2.4.**

1. $\ell$-domination does not imply $(\ell + 1)$-domination;
2. For any constant $K > 0$, any integer $\ell$, there is an integer $L$ such that, for any linear cocycle bounded by $K$ and any $\ell$-dominated decomposition $F \prec_\ell G$, the following assertion holds:

$$\forall \ell' \geq L, F \prec_{\ell'} G.$$

We then use the following notation: $F \prec^L G$. We call characteristic time of domination the smallest $L$ such that $F \prec^L G$.

**Definition 2.5.** A bounded linear cocycle $A$ admits a dominated splitting (or dominated decomposition) if there exist two transverse invariant subbundles $F,G$ such that $E = F \oplus G$ and $F \prec G$.

More generally, for any $F_1,...,F_k$ invariant subbundles such that $E = F_1 \oplus \cdots \oplus F_k$, the decomposition is said dominated if, for any $i \in \{1,...,k-1\}$, $F_i \prec F_{i+1}$; indeed, [BDP] proves that, under these assumptions, the decomposition $E = \left( \bigoplus^i F_j \right) \oplus \left( \bigoplus^{k+1} F_j \right)$ is dominated for any $i \in \{1,...,k-1\}$.

We shall also use the following Lemma ([BDP, Lemma 4.4])

**Lemma 2.6.** Let $A = (\Sigma, f, E, A)$ be a bounded linear cocycle and assume that $E$ admits an invariant decomposition $E = F \oplus G \oplus H$. Consider the quotient cocycle $A/G$. The projection on $E/G$ of the subbundles $F$ and $H$ induces an invariant decomposition denoted by $E/G = F/G \oplus H/G$.

Assume that we have the dominations $F/G \prec H/G$ and $G \prec H$, then we have $(F \oplus G) \prec H$.

Symmetrically, we get:

$$(F \prec G \text{ and } F/G \prec H/G) \implies F \prec (G \oplus H).$$

2.3 Periodic cocycles

We shall consider linear cocycles over a system $(\Sigma, f)$ satisfying the following three properties:

**P1** $\Sigma$ is infinite,

**P2** any point $x \in \Sigma$ is periodic, and we shall denote its period by $p(x)$,

**P3** for any integer $k > 0$, the set of points $x \in \Sigma$ such that $p(x) \leq k$ is finite.
A system verifying (P2) is a periodic system. A system $(\Sigma, f)$ verifying (P1), (P2) and (P3) is a large periods system. In particular, for any large periods system $(\Sigma, f)$ the set $\Sigma$ is countable, and up to an indexation of the orbits, $\Sigma$ can be regarded as a sequence of periodic orbits whose periods tend to infinity. The restriction of $(\Sigma, f)$ to any infinite invariant subset of $\Sigma$ is another large periods system.

Let $(\Sigma, f)$ be a periodic system. For any linear cocycle $A = (\Sigma, f, E, A)$ and any $x \in \Sigma$ we define:

$$M_{x,A} = A^{p(x)} : E_x \to E_x.$$ 

We shall use the abridged notation $M_x$ when there is no ambiguity on the considered cocycle.

**Definition 2.7.** Let $A = (\Sigma, f, E, A)$ be a linear bounded cocycle over an infinite periodic system.

1. We shall say that the cocycle $A$ is strictly without dominated decomposition (or equivalently strictly without domination) if the only invariant subsets $\Sigma'$, in restriction to which the cocycle admits a dominated splitting, are finite.

2. An invariant splitting $E = F \oplus G$ shall be said strictly not dominated if the only invariant subsets $\Sigma'$ in restriction to which the splitting $F \oplus G$ is dominated are finite.

**Remark 2.8.** If $A$ is strictly without domination and if $\Sigma' \subset \Sigma$ is an infinite $f$-invariant subset, then the restriction $A\mid_{\Sigma'}$ is also strictly without domination.

**Lemma 2.9.** Let $A = (\Sigma, f, E, A)$ be a linear bounded cocycle over an infinite periodic system.

1. Let $E = F \oplus G$ be an invariant splitting, then:

   - either there exists an infinite invariant subset $\Sigma' \subset \Sigma$ such that the splitting is strictly not dominated in restriction to $\Sigma'$;
   - or there exists a finite invariant subset $\Sigma_0 \subset \Sigma$ such that the splitting is dominated in restriction to $\Sigma \setminus \Sigma_0$.

2. In any case, $A$ verifies one of the following properties:

   - either there exists an infinite invariant subset $\Sigma' \subset \Sigma$ such that the cocycle is strictly without dominated decomposition in restriction to $\Sigma'$;
   - or there exists a partition $\Sigma = \Sigma_0 \cup \Sigma_1 \cdots \cup \Sigma_k$, with $k < d$, such that, for any $i \in \{0, ..., k\}$ the subsets $\Sigma_i$ are $f$-invariant, $\Sigma_0$ is a finite set and the cocycle admits a dominated decomposition in restriction to each $\Sigma_i$, $i \geq 1$.

**Proof:**

1. Let $E = F \oplus G$ be an invariant splitting. For any integer $L \geq 1$, let us consider the set $\Sigma_L$ of points $x \in \Sigma$ such that the restriction of the cocycle over the orbit of $x$ verifies $F \prec^L G$. If there exists an integer $L$ such that the $\Sigma \setminus \Sigma_L$ is finite, then the second assertion holds. Assume that for any integer $L$, $\Sigma \setminus \Sigma_L$ is an infinite set, then we can construct an infinite sequence of points $x_n$ with disjoints orbits such that, for any $n$, $x_n$ belongs to $\Sigma - \Sigma_n$. The union $\Sigma'$ of the orbits of the points $x_n$ satisfies the first assertion.
2. Assume $\mathcal{A}$ admits no dominated splitting and consider, for any pair of integers $(L \geq 0, i \in \{1, ..., d - 1\})$, the set $\Sigma_{(L,i)}$ of points such that the restriction of the cocycle over the orbit of $x$ admits a splitting $E = F \oplus G$ verifying $F \prec^L G$ and $\dim(F) = i$. If there exists an integer $L$ such that the set $\Sigma \setminus \bigcup_{i=1}^{d-1} \Sigma_{(L,i)}$ is finite, then the second assertion holds. Assume that for any integer $L$, $\Sigma \setminus \bigcup_{i=1}^{d-1} \Sigma_{(L,i)}$ is an infinite set, then we can construct an infinite sequence of points $x_n$ with disjoint orbits such that, for any $n$, $x_n$ belongs to $\Sigma - \bigcup_i \Sigma_{(n,i)}$. The union $\Sigma'$ of the orbits of the points $x_n$ satisfies the first assertion.

\[\square\]

2.4 Perturbations

Definition 2.10.

1. Let $\mathcal{A} = (\Sigma, f, E, A)$ be a bounded linear cocycle. For any given $\varepsilon$, any bounded linear cocycle $\mathcal{B} = (\Sigma, f, E, B)$ such that $\|A_x - B_x\| \leq \varepsilon$ and $\|A_x^{-1} - B_x^{-1}\| \leq \varepsilon$ for any $x \in \Sigma$ is an $\varepsilon$-perturbation of $\mathcal{A}$.

2. Let $\mathcal{A} = (\Sigma, f, E, A)$ be a bounded linear cocycle over an infinite periodic system. A linear cocycle $\mathcal{B} = (\Sigma, f, E, B)$ is a perturbation of $\mathcal{A}$ if, for any $\varepsilon > 0$, the set $\{x \in \Sigma, \|A_x - B_x\| \geq \varepsilon\}$ is finite.

Remark 2.11.

1. Any linear cocycle $\mathcal{A}$ over a finite (hence periodic) system $(\Sigma, f)$ admits a dominated decomposition if and only if there exist an integer $i \in \{1, ..., d - 1\}$ and, for any point $x \in \Sigma$ two spaces $F_x$ and $G_x$ invariant by $M_x$ with dimension $i$, (respectively $d - i$,) such that the modulus of any eigenvalue of the restriction $M_x|F_x$ is strictly smaller than the modulus of any eigenvalue of $M_x|G_x$.

2. As a consequence, any linear cocycle over a finite system $(\Sigma, f)$ with dimension greater than 3 admits arbitrarily small perturbations such that there exists a decomposition of the set $\Sigma = \Sigma_1 \cup \Sigma_2$ verifying the two following assertions: the restriction of the linear cocycle to $\Sigma_1$ admits a dominated splitting $E = F_1 \oplus G_1$ where $\dim(F_1) = 1$, and the restriction of the linear cocycle to $\Sigma_2$ admits a dominated splitting $E = F_2 \oplus G_2$ where $\dim(F_2) = 2$.

Both these remarks explain why we shall neglect finite invariant subsets of the system in our further study.

Remark 2.12. Let $(\Sigma, f)$ be an infinite periodic system, and $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be three bounded linear cocycles of dimension $d$ over $(\Sigma, f)$. Then :

- if $B$ is a perturbation of $A$, then $A$ is a perturbation of $B$;
- if $B$ is a perturbation of $A$ and if $C$ is a perturbation of $B$, then $C$ is a perturbation of $A$;

In other words, "to be a perturbation of" defines an equivalence relation on the set of bounded cocycles over $(\Sigma, f)$.
Remark 2.13. Let $\mathcal{A} = (\Sigma, f, E, A)$ be a bounded linear cocycle over an infinite periodic system and let $\mathcal{B}$ be a perturbation of $\mathcal{A}$. Then there is a scalar function $\alpha : \Sigma \to \mathbb{R}$ with the following property:

- There is a finite subset $\Sigma_0 \subset \Sigma$ such that, for all $x \in \Sigma \setminus \Sigma_0$ the determinant of the matrix $\alpha(x).B_x$ is equal to the determinant of $A_x$;
- $\alpha(x)$ converges to 1 when $x \to \infty$, that is, for all $\varepsilon > 0$ the set $\{x \in \Sigma \mid |\alpha(x) - 1| > \varepsilon\}$ is finite.

In particular, the cocycle $C$ defined by $C_x = \alpha(x)B_x$ is a perturbation of the cocycle $\mathcal{A}$.

Given a bounded linear cocycle admitting a dominated splitting, [BDP] proves that any small enough perturbation of the cocycle admits a dominated splitting (with same dimension of the subbundles). More precisely, [BDP] proves that:

**Lemma 2.14.** Given any dimension $d$, any positive constant $K$ and any integer $\ell > 0$, there is an $\varepsilon > 0$ such that, for any linear cocycle $\mathcal{A} = (\Sigma, f, E, A)$ bounded by $K$, of dimension $d$, admitting an $\ell$-dominated splitting $F \prec_\ell G$, one has that:

\[ \text{any } \varepsilon\text{-perturbation } \mathcal{B} \text{ of } \mathcal{A} \text{ admits a dominated splitting } F' \prec G' \text{ with } \dim(F') = \dim(F). \]

**Corollary 2.15.** Let $\mathcal{A} = (\Sigma, f, E, A)$ be a bounded linear cocycle over an infinite periodic system. Assume $\mathcal{A}$ is strictly without dominated splitting. Then any perturbation of $\mathcal{A}$ is strictly without dominated splitting.

**Proof:** We argue by contradiction: assume that there exists a perturbation $\mathcal{B}$ of $\mathcal{A}$ whose restriction to an infinite invariant subset $\Sigma'$ admits a dominated splitting $E = F \oplus G$. Since $\mathcal{A}$ is bounded, the linear cocycle $\mathcal{B}$ is bounded by some positive constant $K$. Fix $\ell > 0$ such that $F \prec_\ell G$. By Lemma 2.14, there exists an $\varepsilon > 0$ such that any $\varepsilon$-perturbation of $\mathcal{B}$ admits a dominated splitting. By definition of a perturbation, there exists a finite invariant subset $\Sigma_0$ such that the restriction of the linear cocycle $\mathcal{A}$ to the infinite set $\Sigma' \setminus \Sigma_0$ is an $\varepsilon$-perturbation of $\mathcal{B}$, hence admits a dominated splitting. This contradicts the assumption that $\mathcal{A}$ is strictly without dominated splitting. \qed

### 2.5 Statement of the results for linear cocycles

**Theorem 2.1.** Any bounded linear cocycle $\mathcal{A}$ over a large periods system admits a perturbation $\mathcal{B}$ such that for any point $x \in \Sigma$, all eigenvalues of $M_{x,B}$ are real, with multiplicity 1 and different moduli.

**Scholium 2.16.** For any $x \in \Sigma$, $\Lambda(x, \mathcal{A})$ denotes the $d$-uple $(\sigma_1, \ldots, \sigma_d)$ of the Lyapunov exponents of $x$, considered with multiplicity and in increasing order. That is, each $\sigma_i$ is of the form

$$\sigma_i = \frac{\log(|\lambda_i|)}{p(x)},$$

where $\lambda_i$ is an eigenvalue of the matrix $M_{x,A}$.

The proof of Theorem 2.1 will show that, in the statement of Theorem 2.1, we can require that $\Lambda(x, \mathcal{B}) - \Lambda(x, \mathcal{A}) \in \mathbb{R}^d$ converges to 0 in $\mathbb{R}^d$ when $p(x)$ goes to infinity.
The main result in this section is the following:

**Theorem 2.2.** Let $A$ be a bounded linear cocycle over a large periods system. Assume that $A$ is strictly without dominated decomposition. Then there exists a perturbation $B$ and an infinite invariant subset $\Sigma'$ such that, for any point $x \in \Sigma'$, all eigenvalues of $M_{x,B}$ are real, with same modulus.

This result can be restated in the following stronger version:

**Corollary 2.17.** Let $A$ be a bounded linear cocycle over a large periods system. Assume that $A$ is strictly without dominated decomposition. Then there exists a perturbation $B$ such that, for any point $x \in \Sigma$, all eigenvalues of $M_{x,B}$ are real, with same modulus.

Let us prove Corollary 2.17 using Theorem 2.2:

**Proof:** For any $\varepsilon > 0$, let us denote by $\Sigma_\varepsilon$ the set of points $x \in \Sigma$ such that there exists an $\varepsilon$-perturbation $B$ over the reduced system $(Orb(x), f)$ verifying: for any integer $k$, all eigenvalues of $M_{f^k(x),B}$ are real, with same modulus.

Let us first remark that $\Delta_\varepsilon = \Sigma \setminus \Sigma_\varepsilon$ is a finite set. Indeed, if $\Delta_\varepsilon$ were an infinite set, we could apply Theorem 2.2 to the linear cocycle $A$ restricted to $\Delta_\varepsilon$, which is strictly without domination; it contradicts the definition of $\Delta_\varepsilon$.

We shall now use the following decomposition

$$\Sigma = \Delta_1 \cup (\Sigma_1 \setminus (\Sigma_1/2)) \cup (\Sigma_1/2 \setminus (\Sigma_1/3)) \cdots \cup (\Sigma_1/n \setminus (\Sigma_1/(n+1))) \cdots \cup (\cap \Sigma_1/n).$$

Let us consider the linear cocycle $B$ defined as follows:

1. $\Delta_1$ being a finite set, the restriction of $B$ to $\Delta_1$ can be any cocycle verifying $M_{x,B} = Id$ for any point $x \in \Delta_1$;

2. by definition, for any integer $k$, there exists an $1/k$-perturbation $B$ of the restriction of $A$ to the set $\Sigma_1/k \setminus (\Sigma_1/(k+1))$ such that for any point $x \in \Sigma_1/k \setminus (\Sigma_1/(k+1))$, $M_{x,B}$ has all eigenvalues real with same modulus;

3. since $A$ satisfies the announced assertion over the set $\cap \Sigma_1/n$, $B$ can be taken equal to $A$ in restriction to the set $\cap \Sigma_1/n$.

The set $\Sigma_1/n \setminus (\Sigma_1/(n+1))$ is included in $\Delta_1/(n+1)$ which is finite, hence the linear cocycle $B$ is a perturbation of $A$.  

**Corollary 2.18.** Given any dimension $d$, any positive constant $K$ and any $\varepsilon > 0$, there exists two integers $\ell$ and $n$ such that, any $K$-bounded linear cocycle $A$ with dimension $d$ over a periodic orbit with period greater than $n$ satisfies one of the following two assertions:

- either $A$ admits an $\ell$-dominated splitting;
- or there exists an $\varepsilon$-perturbation $B$ of $A$ such that $M_{x,B}$ has all eigenvalues real with same modulus.

**Proof:** Assume, arguing by contradiction, that corollary 2.18 is wrong: there is $d, K$ and $\varepsilon > 0$ such that, given any integer $n$, there is a $K$-bounded linear cocycle $A_n$ with dimension $d$, over a periodic orbit $\gamma_n$ with period greater than $n$, verifying the two following properties:
• for every \( k \leq n \), \( A_n \) has no \( k \)-dominated splitting;

• there is no \( \epsilon \)-perturbation \( \tilde{A}_n \) of \( A_n \) such that, for \( x_n \in \gamma_n \), the eigenvalues of \( M_{x_n,\tilde{A}_n} \) are all real with same modulus.

Consider \( \Sigma = \bigcup_{n \in \mathbb{N}} \gamma_n \): this set is a large periods system. Let \( \mathcal{A} \) be the linear cocycle defined over \( \Sigma \), such that its restriction to \( \gamma_n \) is \( A_n \). The cocycle \( \mathcal{A} \) is a \( K \)-bounded linear cocycle which, by construction, is strictly without domination: any invariant set on which \( \mathcal{A} \) admits an \( \ell \)-dominated decomposition is included in \( \bigcup_{n=0}^{\ell} \gamma_n \) and hence is finite. Furthermore, for any perturbation \( \mathcal{B} \) of \( \mathcal{A} \), the set of points \( x \) for which the matrix \( M_{x,\mathcal{B}} \) has all eigenvalues real with same modulus is finite: the cocycle \( \mathcal{B} \) is (by definition of a perturbation) an \( \epsilon \)-perturbation of \( \mathcal{A} \) out of a finite set. This contradicts Theorem 2.2, and this contradiction concludes the proof.

2.6 Statement of the results for diffeomorphisms and flows

A lemma of Franks in [Fr] allows to realize any small perturbation of the derivative of a diffeomorphism along a finite set by a \( C^1 \)-perturbation of the diffeomorphism. This enables us to restate Corollary 2.18 for diffeomorphisms.

**Corollary 2.19.** Let \( f : M \to M \) be a diffeomorphism of a compact manifold, endowed with a Riemannian metric \( \| \cdot \| \). Then for any \( \epsilon > 0 \) there are two integers \( \ell \) and \( n \) such that, for any periodic point \( x \) of period \( p(x) \geq n \):

• either \( f \) admits an \( \ell \)-dominated splitting along the orbit of \( x \);

• or, for any neighborhood \( U \) of the orbit of \( x \), there exists an \( \epsilon \)-perturbation \( g \) of \( f \) for the \( C^1 \)-topology, coinciding with \( f \) outside \( U \) and on the orbit of \( x \), and such that the differential \( Dg^{p(x)}(x) \) has all eigenvalues real and with same modulus. This modulus can furthermore be chosen different from \( 1 \) so that the orbit of \( x \) is a source or a sink of \( g \).

**Remark 2.20.** In fact, the pair \((\ell,n)\) given by the preceding result only depends on \( \epsilon \) and on an upper bound of \( \|Df\| \). The result henceforth holds with the same \((\ell,n)\) for a \( C^1 \)-neighborhood of \( f \).

Let \( f \) be a diffeomorphism on a compact manifold \( M \). Let us recall that a point \( x \in M \) is chain-recurrent if there exist pseudo-orbits starting and ending at \( x \) with arbitrarily small jumps. [BC] proves that any chain-recurrent point can be turned into a periodic point by an arbitrarily small \( C^1 \)-perturbation.

**Corollary 2.21.** Let \( f \) be a diffeomorphism on a compact manifold. Then for any \( \epsilon \), there exist a pair of integers \((\ell,n)\) such that for any chain-recurrent point \( x \), one of the following two assertions holds:

1. either \( x \) belongs to an invariant compact set admitting an \( \ell \)-dominated splitting;

2. or there is an \( \epsilon \)-perturbation \( g \) of \( f \) (in the \( C^1 \)-topology) for which \( x \) is a periodic sink or source.
An equivalent of Franks’ Lemma for $C^1$-vector fields has been used by several authors, however (as far as we know) no reference is available for its proof. For this reason, we give a precise statement (cf Theorem 5.2) and a proof of this lemma in the appendix of this paper. At this stage, let us just give a loose statement: it is possible to consider any small enough perturbation of the differential of a $C^1$-vector field $X$ along an injective arc of trajectory $\gamma$ as the differential of a vector field $Y$, which is $C^1$-close to $X$, preserves the arc of trajectory $\gamma$, and coincides with $X$ outside a small neighborhood of $\gamma$. This is of course only possible under a few conditions on the perturbation (namely, the perturbed differential should preserve the arc of trajectory $\gamma$).

This enables us to restate Corollary 2.18 for $C^1$-vector fields, up to an adaptation of the notion of dominated splitting for a flow given in the appendix of this paper (see Definition 5.1). Let us first introduce a few notations.

Let $X$ be a $C^1$-vector field on a compact Riemannian manifold $M$. Denote by $\text{Sing}(X)$ the set of its singular points and by $\varphi$ the flow induced by $X$ on $M$. Denote by $\varphi_t$ the time-$t$ map of the flow $\varphi$. We denote by $N$ the normal bundle of $X$, defined on $M \setminus \text{Sing}(X)$: the space $N_x \subset T_x M$ is the hyperplane orthogonal to the line $\mathbb{R} \cdot X(x)$. The normal bundle $N_x$ is naturally endowed with the restriction of the euclidean metric of $T_x M$ induced by metric of $M$. The vector field $X$ induces a flow $P_t$ on $N$, called the linear Poincaré flow of $X$: for any time $t$ and any point $x \in M \setminus \text{Sing}(X)$ the linear map $P_t(x) : N_x \rightarrow N_{\varphi_t(x)}$ is obtained by composing the space differential $T_x \varphi_t$ (restricted to $N_x$) with the projection of $T_y M$ on $N_y$, parallel to $X(y)$. In other words, $N_x$ is canonically identified with the quotient space of $T_x M$ by $\mathbb{R} \cdot X(x)$; as the line bundle generated by $X$ is invariant by the differential $T \varphi_t$, $P_t$ is well-defined as the projection of the differential $T \varphi_t$ on the quotient bundle.

In particular (see Corollary 5.1), for any non-recurrent point $x$, any perturbation of $P_T(x)$ can be regarded as the linear Poincaré flow of a vector field $Y$ $C^1$-close to $X$, preserving the orbit of $x$ and coinciding with $X$ outside a small neighborhood of $\varphi_{[0,T]}(x)$.

**Corollary 2.22.** Let $X$ be a $C^1$-vector field of a compact manifold, endowed with a Riemannian metric $\| \|$. Then for any $\varepsilon > 0$ there are two integers $\ell$ and $n$ such that, for any periodic point $x$ of period $p(x) \geq n,$

1. either the linear Poincaré flow $P$ admits an $\ell$-dominated splitting along the orbit of $x$;

2. or, for any neighborhood $U$ of the orbit of $x$, there exists an $\varepsilon$-$C^1$-perturbation $Y$ of $X$, coinciding with $X$ outside $U$ and along the orbit of $x$, and such that the linear Poincaré flow $P_{p(x)}(x)$ has all eigenvalues real and with the same modulus; furthermore this modulus can be chosen different from 1, so that the orbit of $x$ is a source or a sink for $Y$.

As a consequence of Corollary 2.22 one gets that the existence of a dominated splitting is necessary to the robust transitivity. Let us recall the definition:

An invariant compact set $K$ is called transitive if it contains a dense positive orbit; $K$ is called robustly transitive if furthermore there is a neighborhood $U$ of $K$ for which $K$ is the maximal invariant set in the closure $\bar{U}$, and if there is a $C^1$ neighborhood $U$ of $X$ such that, for every $Y \in U$, the maximal invariant set of the flow of $Y$ in $\bar{U}$ is transitive.

**Corollary 2.23.** Let $X$ be a $C^1$-vector field on a compact manifold. Assume that $K$ is an invariant compact set that is robustly transitive. Then the linear Poincaré flow of $X$ on $K \setminus \text{Sing}(X)$ admits a dominated splitting.
Proof: Let $K$ be a robustly transitive set of $X$, and $U$ (resp. $\mathcal{U}$) a neighborhood of $K$ (resp. of $X$), as in the definition. As $K$ is transitive, there is a point $x$ whose positive orbit is dense in $K$. By Pugh’s closing lemma, there is a sequence $Y_n \in \mathcal{U}$ of vector fields converging to $X$ in the $C^1$-topology and having a periodic orbit $\gamma_n \in U$ such that $\gamma_n(0)$ converges to $x$. This implies that $K$ is contained in the lower limit set $\bigcap_N \bigcup_{n=1}^{\infty} \gamma_n$ of the $\gamma_n$.

Notice that no vector field in $\mathcal{U}$ admits a sink or a source contained in $U$. As a consequence of Corollary 2.22, this implies that there is $\ell$ such that, for any large $n$, the Poincaré linear flow of $Y_n$ over $\gamma_n$ admits an $\ell$-dominated splitting $E_n \oplus F_n$. Up to considering a subsequence, one can assume that the dimension $\dim(E_n)$ does not depend on $n$. Then this dominated splitting induces an $\ell$-dominated splitting on the (upper) limit of the $\gamma_n$ for the Poincaré flow of $X$, at each non-singular point, concluding the proof. 

3 Proof of Theorem 2.1

3.1 Perturbations on subbundles and quotient bundles

Throughout this paragraph, we shall denote by $A = (\Sigma, f, E, A)$ a $K$-bounded linear cocycle over a infinite periodic system. Let $F$ be an invariant subbundle of $E$. Let us consider an orthonormal basis of vectors of $F$ and an orthonormal basis of vectors of $F^\perp$: this provides an orthonormal basis of vectors in which we can write $A$ in blocks of the form:

$$
\begin{pmatrix}
A_F & C \\
0 & A/F
\end{pmatrix},
$$

where $C$ is bounded. We thus get the following two lemmas:

**Lemma 3.1.** For any perturbation $B_F$ of the induced cocycle $A_F$ there exists a perturbation $B$ of $A$ with the following properties:

- $F$ is invariant by $B$;
- the induced cocycle obtained by restriction of $B$ to the subbundle $F$ is $B_F$;
- the quotient cocycle $B/F$ coincides with $A/F$. In particular, the eigenvalues of $M_x,A$ associated to eigenvectors out of $F_x$ remain unchanged.

**Lemma 3.2.** For any perturbation $B_{F^\perp}$ of the quotient cocycle $A/F$, there exists a perturbation $B$ of $A$ with the following properties:

- $F$ is invariant by $B$;
- the quotient cocycle of $B$ by $F$ coincides with $B_{F^\perp}$;
- the induced cocycle obtained by restriction of $B$ to $F$ coincides with $A_F$.

**Definition 3.3.** Let $E$ and $E'$ be two bundles of same dimension over a system $(\Sigma, f)$. Any change of basis $P$ from $E$ to $E'$ (that is, for any point $x \in \Sigma$, $P_x \in GL(E_x, E'_x)$) is bounded by $K > 0$ if and only if, for any $x \in \Sigma$, $\|P_x\| \leq K$ and $\|P_x^{-1}\| \leq K$.

We then get the following two lemmas:
Lemma 3.4. Let $E$ and $E'$ be two bundles of same dimension over a system $(\Sigma, f)$, and let $P$ be a bounded change of basis from $E$ to $E'$. Let $B = (\Sigma, f, E', B)$ be the bounded linear cocycle defined by $B_x = P_f(x) \circ A_x \circ P^{-1}_x$ for any point $x \in \Sigma$. Then the following two statements hold:

- if $A$ admits a dominated splitting, then so does $B$;
- for any perturbation $\tilde{A}$ of $A$, the linear cocycle $\tilde{B}$ defined by $P_f(x) \circ \tilde{A}_x \circ P^{-1}_x$ for any point $x \in \Sigma$ is a perturbation of $B$.

Lemma 3.5. Let $F$ and $G$ be two invariant subbundles of $E$ with trivial intersection. Assume that, the angle $F_x, G_x$ is bounded from below by a uniform constant for any point $x \in \Sigma$. Then, there exists a bounded change of basis $P : E \to E$ such that the subbundles $P(F)$ and $P(G)$ are orthogonal. (Notice that the subbundles $P(F)$ and $P(G)$ are necessarily invariant by the linear cocycle $B$ obtained by conjugating $A$ by $P : B_x = P_f(x) \circ A_x \circ P^{-1}_x$ for any point $x \in \Sigma$.)

As the bundles of a dominated splitting have their angles bounded from below, one deduces from the previous lemma:

Corollary 3.6. Let $A$ be a bounded linear cocycle admitting a dominated splitting $E = F \oplus G$. Then, up to a change of basis, we can assume the dominated decomposition to be orthogonal, thus we can write $A$ in blocks of the form:

$$
\begin{pmatrix}
A_F & 0 \\
0 & A_G
\end{pmatrix}.
$$

3.2 Cocycle of dimension 2

Proposition 3.7. Let $A$ be a $K$-bounded linear cocycle of dimension 2 over a large periods system. There exists a perturbation $B$ of $A$ such that, for any point $x \in \Sigma$, $M_{x,B}$ has all eigenvalues real of multiplicity 1, and different modulus.

Furthermore, the Lyapunov exponents of $B$ can be chosen arbitrarily close to those of $A$.

This is a consequence of [BC, lemme 6.6] which is restated below:

Lemma 3.8. For any $\varepsilon > 0$, there exists $N(\varepsilon) \geq 1$ such that, for any integer $n \geq N(\varepsilon)$ and any finite sequence $A_0, \ldots, A_n$ of elements in $SL(2, \mathbb{R})$, there exists a sequence $\alpha_0, \ldots, \alpha_n$ in $]-\varepsilon, \varepsilon[$ such that the following assertion holds:

- for any $i \in \{0, \ldots, n\}$ if we denote by $B_i = R_{\alpha_i} \circ A_i$ the composition of $A_i$ with the rotation $R_{\alpha_i}$ of angle $\alpha_i$, then the matrix $B_n \circ B_{n-1} \circ \cdots \circ B_0$ has real eigenvalues.

Let us deduce the proof of Proposition 3.7:

Proof: Let $A$ be a $K$-bounded linear cocycle of dimension 2 over a large periods system $(\Sigma, f)$. First notice that, if a matrix in $GL(2, \mathbb{R})$ has a real eigenvalue, then there is an arbitrarily small perturbation of this matrix that has two real eigenvalues of multiplicity 1 with different moduli. So we just need to build a perturbation of $A$ such that the matrices $M_{x,B}$ have at least one real eigenvalue (of modulus arbitrarily close to the moduli of the eigenvalues of $M_{x,A}$).

Consider $\Sigma_1 \subset \Sigma$ the set of points $x$ for which the matrix $M_{x,A}$ has a pair of complex (non-real) conjugated eigenvalues. If $\Sigma_1$ is finite, we are done (it suffices to define $B$ on $\Sigma_1$, such that $M_{x,B}, x \in \Sigma_1$ is the homothety transformation whose ratio is the modulus of the complex eigenvalue of $M_{x,A}$.)
Assume now that $\Sigma_1$ is infinite. Fix a sequence $\varepsilon_n$ decreasing to 0 and consider the sets $\Gamma_n = \{ x \in \Sigma_1 \mid p(x) \geq N(\varepsilon_n) \}$. As $\Sigma$ is a large periods system, the complement of each $\Gamma_n$ is finite.

**Remark 3.9.** There is a sequence $\delta_n$ converging to 0 such that, for any $\alpha \in [-\varepsilon_n, \varepsilon_n]$ and any matrix $A \in GL(2, \mathbb{R})$ with $\|A\| < K$ one has $\|A - B\| < \delta_n$ where $B = R_\alpha \circ A$.

For any $x \in \Gamma_n \setminus \Gamma_{n+1}$, Lemma 3.8 gives a sequence $(\alpha_i), i = 0, \ldots, p(x) - 1$, with $|\alpha_i| \leq \varepsilon_n$ such that the matrix $\prod_{i=0}^{p(x)-1} R_{\alpha_i} \circ A(f^i(x))$ has a real eigenvalue. Define $B$ on the orbit of $x$ by

$$B_{f^i(x)} = R_{t(x)\cdot \alpha_i} \circ A(f^i(x))$$

where $t(x)$ is the infimum of the $t \in [0, 1]$ such that the matrix $\prod_{i=0}^{p(x)-1} R_{t\cdot \alpha_i} \circ A(f^i(x))$ has a real eigenvalue. Then $t(x) \in [0, 1]$, and $M_{x,B}$ has a real eigenvalue of multiplicity 2 and whose modulus coincides with the modulus of the eigenvalue of $M_{x,A}$.

Then Remark 3.9 implies that the cocycle $B$ defined that way on $\bigcup_n (\Gamma_n \setminus \Gamma_{n+1})$ is a perturbation of $A$; moreover as $(\Gamma_n)_n$ is a decreasing sequence and $\bigcap_n \Gamma_n = \emptyset$, one gets that $\Gamma_0 = \bigcup_n (\Gamma_n \setminus \Gamma_{n+1})$. Finally $\Sigma_1 \setminus \Gamma_0$ is finite, so that one can complete this perturbation in a perturbation of $A$ on $\Sigma_1$: define $B$ on the finite set $\Sigma_1 \setminus \Gamma_0$ in the same way as when $\Sigma_1$ is finite. We complete this perturbation on $\Sigma$ by defining $B_x = A_x$ for $x \notin \Sigma_1$, thus obtaining the announced perturbation. \hfill $\square$

### 3.3 Proof of Theorem 2.1 and Scholium 2.16

We proceed by induction on the dimension $d$ of the cocycle. The case $d = 1$ is trivial and $d = 2$ is solved by Proposition 3.7.

Assume the result is true for any $d' < d$, and consider a bounded cocycle $A$ of dimension $d$ over a large periods system $(\Sigma, f)$. Notice that, for any $d > 1$, any linear isomorphism of $\mathbb{R}^d$ admits an invariant 2-plane. As a direct consequence, any linear cocycle of dimension $d$ over $(\Sigma, f)$ admits an invariant subbundle $F$ of dimension 2. Applying the induction assumption to the induced cocycle $A_F$ and to the quotient cocycle $A/F$, we obtain the perturbations $B_F$ of $A_F$ and $B_{F^\perp}$ of $A/F$. Lemmas 3.1 and 3.2 ensure then the existence of a perturbation $B$ of $A$ inducing the cocycle $B_F$ on $F$ and whose quotient $B/F$ is $B_{F^\perp}$. Notice that, for any $x$, all eigenvalues of $M_{x,B}$ are real, and that an arbitrarily small additional perturbation can make their moduli pairwise distinct.

### 4 Proof of Theorem 2.2

#### 4.1 Lyapunov diameter

Let $A = (\Sigma, f, E, A)$ be a bounded linear cocycle over a large periods system. Define the **Lyapunov diameter** of $x \in \Sigma$ by:

$$\delta(x, A) = \max \left\{ \frac{|\log(|\lambda_1|) - \log(|\lambda_2|)|}{p(x)}, \lambda_i \in \text{Spec}(M_x) \right\},$$

i.e. the difference between the largest and the smallest Lyapunov exponent of $x$ for the cocycle $A$. We shall use the notation $\delta(x)$ whenever there is no ambiguity on the considered cocycle.
We then denote by $\delta_+(A)$ (the upper Lyapunov diameter of $A$) and $\delta_-(A)$ (the lower Lyapunov diameter) the upper and the lower limit, respectively, of the $\delta(x)$ for $x \in \Sigma$.

**Lemma 4.1.** Let $(\Sigma, f)$ be a large periods system and $A$ a bounded linear cocycle such that $\delta_-(A) = 0$. Then there exists a perturbation $A'$ of $A$ and an infinite invariant subset $\Sigma'$ such that, for any point $x \in \Sigma'$, all eigenvalues of $M_{x,A'}$ are real, with same modulus.

**Proof:** Using Theorem 2.1 and Scholium 2.16, one builds a perturbation $A'$ of $A$ such that $\delta_-(A') = 0$ and such that, for any $x \in \Sigma$, all eigenvalues of $M_{x,A'}$ are real, with multiplicity 1, and different moduli. We can then choose, for any $x \in \Sigma$, an orthonormal basis $b_x$ of each fiber $E_x$, such that each linear map $A'_x$ has an upper triangular matrix in this basis.

As $\delta_-(A') = 0$, there is some infinite invariant subset $\Sigma' \subset \Sigma$ such that $\lim_{x \to \infty} \delta(x, A') = 0$ (that is, for any $\epsilon > 0$, the set $\{ x \in \Sigma' | \delta(x, A') > \epsilon \}$ is finite). For any $x \in \Sigma'$ we consider $C_x: E_x \to E_x$ the linear map whose matrix in the basis $b_x$ is the diagonal matrix $(\alpha_i)$ where $\alpha_i > 0$ verifies:

$$\alpha_i^{p(x)} = \frac{|\det M_{x,A'}|^{\frac{1}{2n}}}{|\lambda_i|}$$

where $\lambda_i$ the $i$th eigenvalue of $M_{x,A'}$.

Since $\delta(x)$ converges to 0 as $p(x)$ goes to $\infty$ for $x \in \Sigma'$, the matrix $C_x$ converges to the Identity matrix.

Define now $B$ by:

- $B_x = A_x$ if $x \notin \Sigma'$;
- $B_x = A_x \circ C_x$ if $x \in \Sigma'$.

The linear cocycle $B$ is a perturbation of $A$ and, for any $x \in \Sigma'$, all eigenvalues of $M_{x,B}$ are real and have the same modulus. \hfill \Box

Given a bounded cocycle $A$ over a large periods system $(\Sigma, f)$, we define the **minimal Lyapunov diameter of $A$**, denoted by $\delta_{\text{min}}(A)$, as the infimum of the $\delta_-(B)$ for all perturbation $B$ of $A$.

**Remark 4.2.** If $\Sigma' \subset \Sigma$ is an infinite invariant subset, then the Lyapunov diameter of the restricted cocycle $A_{|\Sigma'}$ verifies: $\delta_{\text{min}}(A_{|\Sigma'}) \geq \delta_{\text{min}}(A)$.

**Lemma 4.3.** There is a perturbation $B$ of $A$ such that $\delta_-(B) = \delta_{\text{min}}(A)$. Furthermore, $B$ can be chosen so that, for any $x \in \Sigma$, all eigenvalues of $M_{x,B}$ are real, with multiplicity 1 and with different moduli.

**Proof:** For any integer $n > 0$ there is a perturbation $B_n$ of $A$ such that $\delta_-(B_n) < \delta_{\text{min}} + \frac{1}{n}$. Then, by definition of $\delta_-(B_n)$ and of a perturbation of $A$, for any $n > 0$ there is an infinite $f$-invariant subset $\Sigma_n \subset \Sigma$ such that any $x \in \Sigma_n$ verifies:

$$\delta(x) < \delta_{\text{min}} + \frac{2}{n} \quad \text{and} \quad |A_n - B_n| < \frac{1}{n}.$$ 

Choose by iteration an infinite sequence $(x_n)_n \in \Sigma^N$ as follows. Fix $x_1 \in \Sigma_1$. Once $x_1, \ldots, x_n$ chosen, choose $x_{n+1} \in \Sigma_{n+1} \setminus \bigcup_{i=1}^{n} Orb(x_i)$; this is possible because $\Sigma_{n+1}$ is an infinite set whereas the union of the orbits of $x_i, i \in \{1, \ldots, n\}$ is finite.

Define the perturbation $B$ of $A$ as follows:
• if \( x \in \Sigma \) belongs to the orbit of \( x_i \), for some integer \( i \geq 1 \), then \( B_x = B_{i,x} \);
• otherwise, \( B_x = A_x \).

One easily verifies that the so-defined linear cocycle \( B \) is a perturbation of \( A \) verifying \( \delta_-(B) = \delta_{\min}(A) \). A new perturbation of \( B \) given by Theorem 2.1 and Scholium 2.16 allows to turn all eigenvalues into real ones with multiplicity 1, different moduli, without modifying \( \delta_-(B) \). \( \square \)

**Remark 4.4.** In the proof of Lemma 4.3, denote \( \Sigma' = \bigcup_{i=1}^{n} Orb(x_i) \) and let \( B' \) be the restriction of \( B \) to the infinite invariant subset \( \Sigma' \). Then
\[
\delta_+(B') = \delta_-(B') = \delta_{\min}(B') = \delta_{\min}(B).
\]

This remark motivates the following definition

**Definition 4.5.** Let \((\Sigma, f)\) be a large periods system. A bounded linear cocycle \( A \) over \((\Sigma, f)\) is called incompressible if it verifies both following assumptions:

1. \( \delta_+(A) = \delta_-(A) = \delta_{\min}(A) \);
2. for any \( x \in \Sigma \), all eigenvalues of \( M_{x,A} \) are real, with multiplicity 1 and different moduli.

If \( A \) is incompressible, we denote by \( \delta(A) \) the Lyapunov diameter of \( A \) defined by \( \delta(A) = \delta_+(A) = \delta_-(A) = \delta_{\min}(A) \).

**Remark 4.6.** Let \( A = (\Sigma, f, E, A) \) be an incompressible bounded linear cocycle over a large periods system \((\Sigma, f)\), and let \( \Gamma \subset \Sigma \) be an invariant infinite subset. Then the restricted cocycle \( A|\Gamma \) is incompressible.

Theorem 2.2 is now a corollary of the following result:

**Theorem 4.1.** Let \((\Sigma, f)\) be a large periods system and \( A \) a bounded linear cocycle over \((\Sigma, f)\). Assume that \( A \) is incompressible and strictly without domination. Then \( \delta(A) = 0 \).

The proof of Theorem 4.1 is the aim of the next sections. Let us first prove Theorem 2.2 using Theorem 4.1:

**Proof:** Consider a bounded linear cocycle \( A \) over a large periods system \((\Sigma, f)\) and assume that \( A \) is strictly without domination. Then Lemma 4.3 and Remark 4.4 ensure the existence of a perturbation \( B \) of \( A \) and of an infinite \( f \)-invariant subset \( \Sigma' \subset \Sigma \) such that the restriction \( B' = B|_{\Sigma'} \) is incompressible. Corollary 2.15 and Remark 2.8 imply that \( B' \) is strictly without domination.

Then Theorem 4.1 asserts that \( \delta(B') = 0 \), and Lemma 4.1 finally implies the existence of a perturbation \( C' \) of \( B' \) and the existence of an infinite invariant subset \( \Sigma \subset \Sigma' \) such that, for any point \( x \in \Sigma \), all eigenvalues of \( M_{x,C'} \) are real, with same modulus. The cocycle \( C \) on \( \Sigma \) defined by \( C_x = A_x \) when \( x \notin \Sigma \), and \( C_x = C'_x \) when \( x \in \Sigma \), is a perturbation of \( A \) satisfying the conclusion of Theorem 2.2. \( \square \)

**Remark 4.7.** In fact we proved that, if Theorem 4.1 holds for \( d \)-dimensional cocycles, then Theorem 2.2 also holds for \( d \)-dimensional cocycles.

We are left to prove Theorem 4.1. We argue by induction on the dimension \( d \) of the cocycle. We shall first prove that for any incompressible cocycle \( A = (\Sigma, f, E, A) \) strictly without domination, there exists a splitting into two invariant subbundles \( E = F \oplus G \) such that the induced cocycles \( A|_F \) and \( A|_G \) are both strictly without domination.
4.2 Splitting in subbundles strictly without domination

We aim in this section at proving the following:

**Proposition 4.8.** Let \( \mathcal{A} = (\Sigma, f, E, A) \) be a bounded linear cocycle strictly without domination over a large periods system. Assume that, for any \( x \in \Sigma \), all eigenvalues of \( M_x \) are real with multiplicity 1 and with different modulus. Then there exists an invariant partition \( \Sigma = \Sigma_1 \cup \cdots \cup \Sigma_{d-1} \) such that for any \( i \in \{1, \ldots, d-1\} \), either the set \( \Sigma_i \) is empty or the set \( \Sigma_i \) is infinite and in this case, there exists an invariant decomposition \( E = F_i \oplus G_i \) over \( \Sigma_i \) such that

- \( \dim(F_i) = i \);
- the induced cocycles \( \mathcal{A}_F \) and \( \mathcal{A}_G \) over \( \Sigma_i \) are strictly without domination;
- for any \( x \in \Sigma_i \), the moduli of the eigenvalues of \( M_x \) corresponding to \( F_i \) are strictly smaller than the ones corresponding to \( G_i \).

**Lemma 4.9.** Let \( \mathcal{A} \) be a bounded linear cocycle defined over a large periods system \( (\Sigma, f) \). Let \( E = F \oplus G \) be an invariant splitting such that

1. the cocycles induced by restriction \( \mathcal{A}_F \) and \( \mathcal{A}_G \) are strictly without domination;
2. for any \( x \in \Sigma \), the eigenvalues of \( M_x \) corresponding to \( F \) are strictly smaller than the ones corresponding to \( G \);
3. the decomposition \( E = F \oplus G \) is strictly not dominated.

Then the cocycle \( \mathcal{A} \) is strictly without domination.

**Proof:** We proceed by contradiction. Let \( F' \oplus G' \) be a dominated splitting on some infinite invariant subset \( \Sigma' \subset \Sigma \). Then the moduli of the eigenvalues corresponding to \( F' \) are strictly smaller than the ones corresponding to \( G' \). If \( \dim(F') = \dim(F) \) (and hence \( \dim(G') = \dim(G) \)) then the second condition of the lemma implies that \( F' = F \) and \( G' = G \), which is in contradiction with the first condition of the lemma.

Note that

- either \( \dim(F') > \dim(F) \) and \( \dim(G') < \dim(G) \);
- or \( \dim(F') < \dim(F) \) and \( \dim(G') > \dim(G) \).

Both cases are symmetric. We shall consider the first case.

As the moduli of the eigenvalues corresponding to \( F \) are strictly smaller than the ones corresponding to \( G \), we get that \( G' \subset G \) and \( F \subset F' \). We thus deduce that \( F' \) is transverse to \( G \) and that \( G \) admits the following decomposition \( G = (F' \cap G) \oplus G' \) over \( \Sigma' \). This splitting (over the infinite subset \( \Sigma' \)) is dominated by assumption, which is in contradiction with the assumption of \( G \) being strictly without domination.

**Lemma 4.10.** Let \( \mathcal{A} \) be a linear cocycle defined over a large periods system \( (\Sigma, f) \), and \( E = E_1 \oplus \cdots \oplus E_k \) be an invariant splitting such that:

1. for any integer \( i \), the cocycle induced by restriction \( \mathcal{A}_{E_i} \) is strictly without domination;
2. for any integer \( i \), for any point \( x \in \Sigma \), the moduli of the eigenvalues of \( M_x \) corresponding to \( E_i \) are strictly smaller than the ones corresponding to \( E_{i+1} \);

3. the cocycle \( A \) is strictly without domination.

For any \( i \in \{1, \ldots, k-1\} \) define \( F_i = E_i \oplus E_{i+1} \). Then there exists an invariant partition \( \Sigma = \Sigma_1 \cup \cdots \cup \Sigma_{k-1} \) such that, either \( \Sigma_i \) is a finite set, or the induced cocycle \( A_{F_i} \) is strictly without domination over \( \Sigma_i \).

Proof: For any \( x \in \Sigma \), we denote by \( L_i(x) \) the characteristic time (cf Remark 2.4) of the domination of the splitting \( F_i = E_i \oplus E_{i+1} \) over the periodic orbit of \( x \). Let \( i_x \in \{1, \ldots, k-1\} \) be an integer chosen so that the characteristic time \( L_{i_x}(x) \) realizes the maximum of \( \{L_i(x), i \in \{1, \ldots, k-1\} \} \). Note that \( i_x \) can be chosen constant over the periodic orbit of \( x \), hence the function \( x \mapsto i_x \) is invariant by \( f \).

Let \( \Sigma_i = \{x \in \Sigma, i_x = i \} \). Then \( \Sigma_1 \cup \cdots \cup \Sigma_{k-1} \) is an invariant partition of \( \Sigma \). Choose \( i \) such that \( \Sigma_i \) is an infinite set.

Claim 1. The decomposition \( F_i = E_i \oplus E_{i+1} \) is strictly not dominated over \( \Sigma_i \).

Proof: We proceed by contradiction. Assume the decomposition \( F_i = E_i \oplus E_{i+1} \) is dominated over an infinite invariant subset \( \Sigma'_i \subset \Sigma_i \). There exists an integer \( L \) such that \( E_i \prec^L E_{i+1} \) over \( \Sigma'_i \). For any \( x \in \Sigma'_i \), \( L \) is by definition bigger than \( L_i(x) \). As \( i_x = i \), we get that \( L_j(x) \leq L \) for any \( j \in \{1, \ldots, k-1\} \). As a conclusion, \( E_j \prec^L E_{j+1} \) for any \( j \) over \( \Sigma'_i \), hence the decomposition \( E = E_1 \oplus \cdots \oplus E_k \) is dominated over \( \Sigma' \), which contradicts the assumption of \( A \) being strictly without domination.

Note that, thanks to the previous claim, the decomposition \( F_i = E_i \oplus E_{i+1} \) over \( \Sigma_i \) satisfies all the assumptions of the lemma 4.9. We thus get that \( A_{F_i} \) is strictly without domination over \( \Sigma_i \), which concludes the proof of the lemma.

We are now ready to prove Proposition 4.8.

Proof: We are going to show by a decreasing inductive argument that, for any \( k \in \{2, \ldots, d\} \), there exists an integer \( i_k \) and an invariant finite partition \( P_k = \{\Sigma^1_k, \ldots, \Sigma^k_k\} \) of \( \Sigma \) such that for any \( j \in \{1, \ldots, i_k\} \) either the set \( \Sigma^j_k \) is finite, or the set \( \Sigma^k_j \) is infinite and in this case there is an invariant splitting \( E = E_1 \oplus \cdots \oplus E_k \) such that

(P1) for any \( \ell \in \{1, \ldots, k\} \), the induced cocycle \( A_{E_\ell} \) over \( \Sigma^k_j \) is strictly without domination;

(P2) for any \( \ell \in \{1, \ldots, k-1\} \) and for any \( x \in \Sigma^k_j \), the moduli of the eigenvalues of \( M_x \) corresponding to \( E_\ell \) are strictly smaller than the ones corresponding to \( E_{\ell+1} \);

(P3) the restricted cocycle \( A_{\Sigma^k_j} \) is strictly without domination.

For \( k = d \), denote by \( E_1, \ldots, E_d \) the one-dimensional subbundles corresponding, for any \( x \in \Sigma \), to the eigenspaces of \( M_x \) ordered in the increasing way by the moduli of the eigenvalues. The trivial partition \( \Sigma = \Sigma \) and this splitting satisfy all the required conditions.

Assume the proposition verified for \( k+1 \). Let \( \Sigma^{k+1}_j \in P_{k+1} \) be one of the subsets given by the inductive assumption. If this subset is finite, \( \Sigma^{k+1}_j \) remains in the partition \( P_k \). Assume \( \Sigma^{k+1}_j \) is an infinite set. Denote by \( E = E_1 \oplus \cdots \oplus E_{k+1} \) the corresponding invariant splitting. Applying Lemma 4.10 to the restricted cocycle \( A_{\Sigma^{k+1}_j} \), we get a finite invariant partition of the set \( \Sigma^{k+1}_j \) in subsets \( \Sigma' \) verifying the following dichotomy:
• either the subset $\Sigma'$ is finite;

• or the subset $\Sigma'$ is infinite and there is an integer $i$ such that the induced cocycle $A_{E_i \oplus E_{i+1}}$ is strictly without domination. Endow the restricted cocycle to $\Sigma'$ with the decomposition in $k$ subbundles obtained by gathering $E_i$ together with $E_{i+1}$ in the decomposition (in $k+1$ subbundles) associated to $\Sigma_{j}^{k+1}$. In other words, if $F = E_i \oplus E_{i+1}$, the decomposition associated to $\Sigma'$ is $E_1 \oplus \cdots \oplus E_{i-1} \oplus F \oplus E_{i+2} \oplus \cdots \oplus E_{k+1}$. This splitting satisfies the required conditions.

Gathering together the partitions built for each $\Sigma_{j}^{k+1}$ in $\mathcal{P}_{k+1}$, we get the announced partition $\mathcal{P}_{k}$, which ends the induction argument.

We thus get a partition $\mathcal{P}_2$. Let us first denote by $\Sigma'_{0}$ the union of all finite subsets of the partition $\mathcal{P}_2$. Over each infinite set of the partition $\mathcal{P}_2$, there is a splitting of $E$ into two subbundles $E = E_1 \oplus E_2$ verifying the conditions (P1), (P2) and (P3). For any integer $i \in \{1, \ldots, d-1\}$, denote by $\Sigma'_i$ the union all infinite subsets of the partition $\mathcal{P}_2$ such that $\dim(E_1) = i$.

Choose an integer $i_0 \in \{1, \ldots, d-1\}$ such that the set $\Sigma'_{i_0}$ is not empty. Denote by $\Sigma_{i_0}$ the set $\Sigma'_{0} \cup \Sigma'_i$ and extend the subbundle $E_1$ defined over $\Sigma'_i$ to $\Sigma'_0$ by considering the sum of the first $i_0$ eigenspaces with smaller moduli of eigenvalues. We then conclude the proof by taking $\Sigma_i = \Sigma'_{i}$ for any integer $i \neq i_0$. \hfill $\square$

### 4.3 Decreasing the Lyapunov spectrum: proof of Theorem 4.1

We proceed by induction on the dimension $d$ of the cocycle. When $d = 2$, Mañé ([Ma]) ensures the existence of perturbations along arbitrarily long periodic orbits, with complex eigenvalues at the period. Applying Proposition 3.7, there exists a perturbation whose eigenvalues at the period are real, and of moduli arbitrarily close to the modulus of the complex eigenvalues. This concludes the proof of Theorem 4.1 in the 2-dimensional case.

Assume now that Theorem 4.1 holds for any cocycle of dimension strictly less than $d$. We will prove that it also holds for $d$-dimensional cocycles. Notice that this implies that Theorem 2.2 also holds for $d$-dimensional cocycles (see Remark 4.7).

Consider a large periods system $(\Sigma, f)$ and a bounded $d$-dimensional linear cocycle $A = (\Sigma, f, E, A)$ that is incompressible and strictly without domination. We will show, arguing by contradiction, that $\delta(A) = 0$. Assume (by contradiction) that $\delta(A) > 0$. Notice that any cocycle $\mathcal{B}$ obtained by restriction of the cocycle $A$ to an infinite subset of $\Sigma$ verifies the following:

(H1) $\mathcal{B}$ is incompressible;

(H2) $\mathcal{B}$ is strictly without domination;

(H3) $\delta(\mathcal{B}) = \delta(A) > 0$.

Proposition 4.8 implies the existence of an infinite invariant subset $\Gamma \subset \Sigma$ such that the restricted cocycle $\mathcal{B} = A|_{\Gamma}$ admits a (not dominated) splitting $E = F \oplus G$ with the following properties:

(H4) the induced cocycle $\mathcal{B}_F$ and $\mathcal{B}_G$ are strictly without domination;
(H5) for any \( x \in \Gamma \), the moduli of the eigenvalues of \( M_x \) corresponding to \( F \) are strictly smaller than the ones corresponding to \( G \).

**Lemma 4.11.** We have:

\[
\delta_+(B_F) = \delta_+(B_G) = 0.
\]

**Proof:** We proceed by contradiction, assuming (for instance) that \( \delta_+(B_F) = \delta > 0 \). There exists an infinite invariant subset \( \Gamma_0 \subset \Gamma \) such that, for any point \( x \in \Gamma_0 \), \( \delta(x, B_F) > \delta/2 \). The restricted cocycle \( (B_F)_{\Gamma_0} \) is strictly without domination, and its dimension is strictly less than \( d \). Then, the induction assumption allows to apply Theorem 2.2 to the cocycle \( (B_F)_{\Gamma_0} \). Thus there is a perturbation \( \tilde{B}_F \) on \( \Gamma_0 \) such that \( \delta_+ (\tilde{B}_F)_{\Gamma_0} = 0 \). Furthermore, by Remark 2.13, we can assume that this perturbation preserves the determinant of the matrix \( (B_F)_x \) for any point \( x \in \Gamma_0 \).

We can now consider a perturbation \( C \) of the cocycle \( A \) such that

- outside \( \Gamma_0 \), \( C \) coincides with \( A \);
- on \( \Gamma_0 \), \( C_F = (\tilde{B}_F) \) and \( C/F = A/F \);

**Claim 2.** For any point \( x \in \Gamma_0 \),

\[
\delta(x, C) \leq \delta(x, A) - \frac{\delta}{2d}.
\]

**Proof:** Recall that

- the moduli of the eigenvalues of \( M_{x,A} \) corresponding to \( F \) are strictly smaller than the ones corresponding to \( G \);
- the eigenvalues of \( M_{x,C} \) associated to eigenspaces outside \( F \) coincide with the eigenvalues of \( M_{x,A} \) associated to \( G \);
- since the considered perturbation of the matrix \( A_x \) preserves the determinant of \( (A_F)_x \), the modulus of the eigenvalues of \( M_{x,C} \) corresponding to \( F \) is equal to the geometric average of the moduli of the eigenvalues of \( M_{x,A} \).

For any cocycle \( D \), let us denote the smallest and the greatest Lyapunov exponents of a point \( x \) by

\[
\sigma^+(x, D) = \max \left\{ \frac{\log(|\lambda|)}{p(x)} : \lambda \in \text{Spec}(M_{x,D}) \right\}
\]

and

\[
\sigma^-(x, D) = \min \left\{ \frac{\log(|\lambda|)}{p(x)} : \lambda \in \text{Spec}(M_{x,D}) \right\}.
\]

So, the Lyapunov diameter of \( x \) is \( \delta(x, D) = \sigma^+(x, D) - \sigma^-(x, D) \).

As a consequence of the previous remarks, for any \( x \in \Gamma_0 \),

\[
\sigma^+(x, C) = \sigma^+(x, A) \quad \text{and} \quad \sigma^-(x, C) = \frac{\log(|\det(M_{x,A_F})|)}{\dim(F) p(x)}.
\]

Thus, for any point \( x \in \Gamma_0 \), \( \delta(x, A) - \delta(x, C) = \sigma^-(x, C) - \sigma^-(x, A) \). Denote by \( |\lambda_1(x)| < \ldots < |\lambda_{\dim(F)}(x)| \) the moduli of the eigenvalues of \( M_{x,A} \) corresponding to \( F \) and \( \sigma_i(x) = \log(|\lambda_i(x)|)/p(x) \). We then get the following:
We then get by an easy calculus that \( \sigma^- (x, \mathcal{A}) = \sigma_1 (x) \);
\( \sigma^- (x, \mathcal{C}) = \sum_{k=1}^{\dim (F)} \sigma_k (x) \);
and \( \sigma_{\dim (F)} (x) - \sigma_1 (x) > \delta / 2 \) since \( x \in \Gamma_0 \).

We then get by an easy calculus that \( \sigma^- (x, \mathcal{C}) \geq \sigma^- (x, \mathcal{A}) + \frac{\delta}{2 \dim (F)} \), thus \( \delta (x, \mathcal{C}) \leq \delta (x, \mathcal{A}) - \frac{\delta}{2 \dim (F)} \).

Since \( \Gamma_0 \) is an infinite set, we deduce from the preceding claim that \( \delta^- (\mathcal{C}) \leq \delta (\mathcal{A}) - \frac{\delta}{2 \dim (F)} \) which is in contradiction with the assumption of incompressibility of \( \mathcal{A} \). This concludes the proof of the Lemma.

As a direct corollary we get:

**Corollary 4.12.** Let \( F' \subset F \) and \( G' \subset G \) be invariant subbundles defined over an infinite invariant subset \( \Gamma' \subset \Gamma \).

Then the induced cocycle \( (\mathcal{B}_{|\Gamma'})_{F'} \) and \( (\mathcal{B}_{|\Gamma'})_{G'} \) are strictly without domination. Furthermore the quotient cocycles \( \mathcal{B}/F \) and \( \mathcal{B}/G \) verify

\[
\delta_+ (\mathcal{B}/F) = \delta_+ (\mathcal{B}/G) = 0,
\]
hence \( \mathcal{B}/F \) and \( \mathcal{B}/G \) are strictly without domination (and the same holds for any cocycle induced by \( \mathcal{B}/F \) or \( \mathcal{B}/G \) over an infinite invariant subset).

**Proof:** Let us start by two easy criteria given by the Lyapunov diameter:

- If a bounded linear cocycle \( \mathcal{C}_0 \) over a periodic system admits a dominated splitting, then \( \delta^- (\mathcal{C}_0) > 0 \);
- If a bounded linear cocycle \( \mathcal{C} \) over a periodic system is not strictly without domination, then it admits a dominating splitting over an infinite invariant subsystem, hence it verifies \( \delta_+ (\mathcal{C}) > 0 \).

On the one hand, Lemma 4.11 implies that, if \( F' \subset F \) is an invariant subbundle over an infinite invariant subset \( \Gamma' \subset \Gamma \), then \( \delta_+ (\mathcal{B}_{|\Gamma'})_{F'} \leq \delta_+ (\mathcal{B}_F) = 0 \), so that \( (\mathcal{B}_{|\Gamma'})_{F'} \) is strictly without domination.

On the other hand, for any \( x \in \Gamma \), the linear map \( M (x, \mathcal{B}/F) \) is conjugated to \( M (x, \mathcal{B}_G) \). As a consequence they both have same spectrum, thus \( \delta (x, \mathcal{B}/F) = \delta (x, \mathcal{B}_G) \). Similarly, \( \delta (x, \mathcal{B}/G) = \delta (x, \mathcal{B}_F) \). Hence \( \delta_+ (\mathcal{B}/F) = \delta_+ (\mathcal{B}_G) = 0 \) and \( \delta_+ (\mathcal{B}/G) = \delta_+ (\mathcal{B}_F) = 0 \).\( \square \)

**Proposition 4.13.** Let \( \mathcal{B} \) be a linear cocycle over a large periods system \( (\Gamma \subset \Sigma, f) \) satisfying the properties \((H1), (H2), (H3), (H4) and (H5)\). Assume that \( H \subset E \) is a proper \( \mathcal{B} \)-invariant subbundle containing \( F \) as a proper subbundle (i.e. \( 0 \subsetneq F_x \subsetneq H_x \subsetneq E_x \) for any \( x \in \Gamma \)). Then the splitting \( H = F \oplus (G \cap H) \) is a dominated splitting for the cocycle \( \mathcal{B}_H \).

Similarly, if \( L \subset E \) is a proper \( \mathcal{B} \)-invariant subbundle containing \( G \) as a proper subbundle, then the splitting \( L = (F \cap L) \oplus G \) is a dominated splitting of the cocycle \( \mathcal{B}_L \).

21
Proof: The two statements of the proposition being completely symmetrical, we shall just prove the first one. We proceed by contradiction: consider a proper $B$-invariant subbundle $H \subset E$ containing $F$ as a proper subbundle, and assume that the splitting $H = F \oplus (G \cap H)$ is not dominated. We shall contradict the assumption ($H1$).

As the moduli of the eigenvalues of the matrix $M(x, B)$ corresponding to $F$ are strictly smaller than the ones corresponding to $G$ (hence, to $G \cap H$), this splitting is dominated over any finite subsystem. This remark together with the assumption that the splitting $H = F \oplus (G \cap H)$ is not dominated imply, by Lemma 2.9, the existence of an infinite invariant subset $\Gamma' \subset \Gamma$ over which the splitting is strictly not dominated. Corollary 4.12 then states that the cocycles $(B_{|\Gamma'})_F$ and $(B_{|\Gamma'})_{G \cap H}$ are also strictly without domination: we infer by Lemma 4.9 that the cocycle $C = (B_{|\Gamma'})_H$ is strictly without domination.

As $H$ is a proper subbundle of $E$, its dimension is strictly smaller than $d$. The induction assumption hence implies that Theorem 2.2 can be applied to $C$: there is a perturbation $\tilde{C}$ of $C$ over an infinite invariant subset $\tilde{\Gamma} \subset \Gamma'$ such that, for any $x \in \tilde{\Gamma}$, all eigenvalues of $M_{x, \tilde{C}}$ have same modulus. Furthermore, Remark 2.13 allows to assume that this perturbation preserves the determinant for any $x \in \tilde{\Gamma}$.

Using Lemmas 3.1 and 3.2, we build a perturbation $\tilde{B}$ of $B$ verifying the following properties:

1. $\tilde{B}$ coincides with $B$ out of $\tilde{\Gamma}$;
2. $\tilde{B}$ leaves the subbundle $H$ invariant;
3. the quotient cocycle $(\tilde{B})/H$ coincides with $B/H$;
4. the induced cocycle $(\tilde{B}_{|\tilde{\Gamma}})_H$ is $\tilde{C}$.

The following lemma (which contradicts the incompressibility of $B$) will conclude the proof of the proposition:

Lemma 4.14. $\delta_-(\tilde{B}) < \delta(B)$.

Proof: Notice first that $(G \cap H)(x)$, being not reduced to $0$, contains eigenvectors of $M_{x, B}$. As $\delta_+(B_F) = \delta_+(B_G) = 0$, the set

$$\{x \in \Gamma, \ |\delta(x, B_H) - \delta(x, B)| \geq \varepsilon\}$$

is finite for any $\varepsilon > 0$. As a direct consequence, $\delta_-(C) = \delta_+(C) = \delta(B) = \delta > 0$.

Consider the variations of the extremities of the Lyapunov spectrum of $x \in \tilde{\Gamma}$, under the perturbations $B \rightarrow \tilde{B}$. For any $x \in \tilde{\Gamma}$:

- $\sigma^+(x, B) = \sigma^+(x, B)$;
- $\sigma^-(x, B) = \inf \left\{\sigma^-(x, \tilde{C}), \sigma^-(x, B/H)\right\}$.

Notice that $\sigma^+(x, B) - \sigma^+(x, B/H)$ converges to $0$ when $x$ tends to infinity because this difference corresponds to Lyapunov exponents of $B$ associated to $G$ and $\delta_+(B_G) = 0$. In order to prove the lemma, we are left to verify that:

Claim 3.

$$\liminf_{x \rightarrow \infty} \left(\sigma^-(x, \tilde{C}) - \sigma^-(x, C)\right) > 0.$$
The determinant of $M_{x,C}$ and $M_{x,\tilde{C}}$ are equal, and all the eigenvalues of $M_{x,\tilde{C}}$ have the same modulus, so that

$$\sigma^-(x,\tilde{C}) = \frac{1}{\dim(H)} \log |\det(M_{x,C})|.$$  

This determinant is the product of the determinant of $M_{x,B_F}$ by $\dim(H \cap G) = \dim(H) - \dim(F)$ eigenvalues of $M_{x,B}$ corresponding to $G$. Moreover, the difference of any Lyapunov exponent of $M_{x,B_G}$ and any Lyapunov exponent of $M_{x,B_F}$ converges to $\delta$ as $x \to \infty$ (this can be deduced easily from $\delta_+(B_F) = \delta_+(B_G) = 0$ and $\delta_+(B) = \delta_-(B) = \delta(B)$). Now we get:

$$\lim_{x \to \infty} \left( \sigma^-(x,\tilde{C}) - \sigma^-(x,C) \right) = \frac{(\dim(H) - \dim(F)) \delta(B)}{\dim(H)} > 0 \quad (1)$$

This implies the claim and thus concludes the proofs of the lemma and of Proposition 4.13. □

Proposition 4.15. Let $B$ be a linear cocycle over a large periods system $(\Gamma \subset \Sigma, f)$ satisfying the properties $(H1), (H2), (H3), (H4)$ and $(H5)$. Assume that $H \subset E$ is a proper $B$-invariant subbundle containing $F$ as a proper subbundle. The splitting $E = F \oplus G$ induces by projection a natural splitting $E/H = F/H \oplus G/H$ on the quotient cocycle $B/H$. This splitting is then a dominated splitting for the cocycle $B/H$.

Similarly, if $L \subset G$ is a proper $B$-invariant subbundle, then the splitting $E/L = F/L \oplus G/L$ is a dominated splitting for the cocycle $B/L$.

The proof of Proposition 4.15 follows the same argument as the one of Proposition 4.13, so we just give the sketch of the proof.

**Proof:** We proceed by contradiction: consider a proper $B$-invariant subbundle $H \subset F$ and assume that the splitting $E/H = F/H \oplus G/H$ is not dominated. Using Lemma 4.11, one verifies that $\delta_+((B_F)/H) = \delta_+((B_G)/H) = 0$ proving that the two cocycles $(B_F)/H$ and $(B_G)/H$ are strictly without domination. As the splitting $E/H = F/H \oplus G/H$ is not dominated, there is an infinite invariant subset $\Gamma' \subset \Gamma$ over which the splitting is strictly not dominated. Lemma 4.9 implies now that the quotient cocycle $B/H$ is strictly without domination over $\Gamma'$. As the dimension of $E/H$ is strictly less than $d$ the induction hypothesis asserts that we can apply Theorem 2.2 (and Remark 2.13) to $(B/H)|_{\Gamma'}$: there is an infinite invariant subset $\hat{\Gamma} \subset \Gamma'$ and a perturbation $\hat{\tilde{C}}$ of $(B/H)|_{\hat{\Gamma}}$ (preserving the determinant) such that, for any $x \in \hat{\Gamma}$, all eigenvalues of $M_{x,\hat{\tilde{C}}}$ have same modulus. Now, using Lemmas 3.1 and 3.2 we build a perturbation $\hat{B}$ of $B$ verifying the following properties:

1. $\hat{B}$ coincides with $B$ outside $\hat{\Gamma}$;
2. $\hat{B}$ leaves the subbundle $H$ invariant;
3. the induced cocycle $(\hat{B})_H$ coincides with $B_H$;
4. the quotient cocycle $(\hat{B})/H$ coincides with $\hat{\tilde{C}}$ over $\hat{\Gamma}$.

The following lemma (contradicting the incompressibility of $B$) hence concludes the proof of the proposition:

**Lemma 4.16.** $\delta_-(\hat{B}) < \delta(B)$.  

23
The proof of Lemma 4.16 follows exactly the proof of Lemma 4.14 and we let it to the reader.

We conclude the proof of Theorem 4.1 (and hence also of Theorem 2.2) by proving Lemma 4.17 below which contradicts the fact that \( B \) is strictly without domination:

**Lemma 4.17.** The splitting \( E = F \oplus G \) is a dominated splitting for \( B \) over \( \Gamma \).

**Proof:** As \( d \geq 3 \), one of the two subbundles \( F \) or \( G \) has dimension greater than 2. Assume for instance that \( \dim(F) \geq 2 \).

Recall that, for \( x \in \Gamma \), all the eigenvalues of \( M(x, B) \) have multiplicity one, and pairwise distinct moduli. Let \( F_1 \subset F \) be the one-dimensional subbundle corresponding to the eigenvalue of smallest modulus, and \( F_2 \subset F \) be the codimension one subbundle directed by the \( \dim(F) - 1 \) other eigendirections. These subbundles are clearly invariant by \( B \), and we get an invariant splitting \( F = F_1 \oplus F_2 \).

Then, Proposition 4.13 applied to \( L = F_2 \oplus G \) implies that \( F_2 \prec G \). Furthermore, Proposition 4.15 applied to \( H = F_2 \) implies that the splitting \( E/F_2 = F/F_2 \oplus G/F_2 \) is dominated. Notice that \( F/F_2 \) is the projection on \( E/F_2 \) of the subbundle \( F_1 \), and hence coincides with \( F_1/F_2 \) (according to the notations used in Lemma 2.6). Now Lemma 2.6 asserts that \( F_1 \oplus F_2 \prec G \), that is \( F \prec G \).

\[ \square \]

5 Appendix

5.1 Elementary definitions for time-continuous systems

In [Fr][Lemma 1.1], Franks showed how to alter a diffeomorphism in order to achieve a desired derivative at a finite number of points. In particular, his lemma enables to perturb the derivative of a diffeomorphism along the orbit of finitely many periodic points, without interfering with the dynamics involved outside a small neighborhood of these periodic orbits, and while keeping these periodic orbits unchanged. This tool turned out to be the key ingredient (cf [BDP]) when it comes to give a \( C^1 \)-dynamical meaning to results on linear systems. In the present section, we shall prove an analogous result in the case of continuous time systems.

We shall henceforth denote by \( X \) a \( C^1 \)-vector field defined on a compact boundaryless Riemannian manifold \( M \), and by \( \phi \) the flow associated to the vector field \( X \). For any point \( p \in M \), we shall denote by \( D_pX \) the derivative of \( X \) at the point \( p \), by \( \phi_t(x) \) the image of the point \( x \) by the time \( t \) of the flow associated to \( X \), and by \( T_p\phi_t \) the differential of the application \( p \mapsto \phi_t(p) \).

Recall that, for any regular point \( p \in M \) and any time \( t \), the linear Poincaré flow \( P_t(p) \) is defined on the normal bundle \( N_p = X(p)^\perp \) (endowed with the induced Riemannian metric defined on \( T_pM \)) by the following relation:

\[ P_t(p) = \pi_{\phi_t(p)} \circ T_p\phi_t|N_p, \]

where \( \pi_{\phi_t(p)} : T_{\phi_t(p)}M \to N_{\phi_t(p)} \) denotes the projection parallel to \( X(\phi_t(p)) \).

**Definition 5.1.** Let \( F \) and \( G \) be two \( (P_t) \)-invariant subbundles of the normal bundle \( N \). We shall say that \( N = F \oplus G \) is a dominated splitting for the linear Poincaré flow \( P_t \), and we denote
it by $F < G$, if there exists an integer $\ell$ such that, for any point $x \in M \setminus \text{Sing}(X)$ and any pair of non-zero vectors $(u, v) \in F_x \times G_x$, the following inequality holds:

$$\forall t \geq \ell, \quad \frac{|P_t(x)u|}{|u|} < \frac{1}{2} \frac{|P_t(x)v|}{|v|}.$$

For any $\epsilon > 0$ and any linear application $A : N_p \to N_q$, we shall call $\epsilon$-perturbation of $A$ any linear application of the form $A \circ (\text{Id} + E) : N_p \to N_q$ where $E : N_p \to N_p$ is a linear application with norm smaller than $\epsilon$.

Moreover, for any $T > 0$, any injective arc of trajectory $\gamma = \varphi_{[0,T]}(p)$ of $X$, and any tubular neighborhood $U$ of $\gamma$, we shall denote by $\Gamma = \varphi_{[-t_1,t_2]}(p) \supset \gamma$ the continuation of $\gamma$ in $U$ (that is $t_1 = \text{Min}\{t > 0, \varphi_{-t}(p) \in \partial U\}$ and $t_2 = \text{Min}\{t > 0, \varphi_t(p) \in \partial U\}$).

**Theorem 5.1.** Let $\mathcal{U}$ be an $\epsilon$-neighborhood of $X$ in the $C^1$-topology. For any $T > 0$, there exists an $\eta > 0$ such that, for any tubular neighborhood $U$ of an arc of trajectory $\gamma = \varphi_{[0,T]}(p)$ of $X$ and for any $\eta$-perturbation $\mathcal{F}$ of the linear Poincaré map $P_T(p)$, there exists a vector field $Y \in \mathcal{U}$ achieving the perturbation $\mathcal{F}$, (i.e. such that the linear Poincaré map $\tilde{P}_T(p)$ associated to $Y$ coincides with $\mathcal{F}$) and such that $Y$ coincides with $X$ outside $U$ and along $\Gamma$.

### 5.2 Perturbation results for time-continuous systems

Using the relation, explicitly given in the proof, between the linear Poincaré flow of a $C^1$-vector field $X$ and the derivative of the vector field (see [HiSm] for reference), the next statement is a consequence of Theorem 5.1.

Let us first introduce the notion of $\epsilon$-perturbation of the derivative of a vector field $X$. Recall that, since $X : M \to TM$, the derivative $D_pX$ of $X$ is defined on $T_pM$, for any point $p \in M$; more precisely:

$$D_pX : T_pM \to T_{X(p)}(TM).$$

We shall denote by $\Pi : T(TM) \to TM$ the differential of the canonical projection of $(x, u) \in TM \mapsto x \in M$. Notice that $\Pi \circ D_pX$ is the Identity map on $T_pM$. This relation is a restrictive condition for the derivative of any perturbation of $X$. We shall therefore only consider the applications satisfying this property. Let $C : T_pM \to T_{X(p)}(TM)$ be such an application. Then

$$\Pi \circ (D_pX − C) = 0.$$

Notice now that, since $X(p) \in T_pM \subset TM$, the space $T_{X(p)}(T_pM)$ is a subspace of $T_{X(p)}(TM)$.

- This space is the kernel of the application $\Pi$ restricted to $T_{X(p)}(TM)$;
- since $T_pM$ is a vector space, the space $T_{X(p)}(T_pM)$ can be canonically identified with $T_pM$.

Thanks to both these remarks, the previous application $(D_pX − C)$ can be considered as a linear map of $T_pM$ to $T_pM$.

**Definition 5.2.** Fix $\eta > 0$ and denote $\Gamma = \varphi_{[0,\ell]}(p)$ a non-singular arc of trajectory of the vector field $X$. A continuous $\eta$-perturbation $Q$ of $DX_{\Gamma \cdot T\Gamma}$ is a continuous map $p \in \Gamma \mapsto Q_p$ such that the linear map $Q_p : T_pM \to T_pM$ has a norm bounded by $\eta$.
Theorem 5.2. Let $\mathcal{U}$ be an $\varepsilon$-neighborhood of $X$ in the $C^1$-topology. There exists $\delta > 0$ such that, for any $T > 0$, for any tubular neighborhood $U$ of an arc of trajectory $\gamma = \varphi_{[0,T]}(p)$ of $X$ and for any continuous $\delta$-perturbation $Q$ of $DX|_{\Gamma T}$ verifying the following two conditions:

1. $Q_{\varphi_t(p)}(X(\varphi_t(p))) = 0$, $\forall t \in [-t_1, t_2]$;
2. $Q$ vanishes on $(\Gamma \setminus \gamma) = \varphi_{[-t_1, 0]} \cup [T, t_2]$;

there exists a vector field $Y \in \mathcal{U}$ achieving the perturbation $Q$, (i.e. such that $DY$ coincides with $DX + Q$ along $\Gamma$) and such that $Y$ coincides with $X$ outside $U$ and along $\Gamma$.

Remark 5.3. This result enables to perturb the derivative $DX$ along any periodic orbit $\gamma = \varphi_{[0,T]}(p)$ by applying the theorem to the orbit cut into pieces (such that the intersection of more than two pieces is empty).

As we shall see in the proof, $\delta$ does not depend on the time-length $T$ of the arc of trajectory $\gamma$, but only on the variation of velocity (norm and angle) along $\gamma$. Since the arc of trajectory can be easily cut into pieces with arbitrarily small variation of velocity, the previous remark proves that $\delta$ does not depend on $T$.

Let us prove Theorem 5.1 by using Theorem 5.2.

Proof: Let $\delta > 0$ be the constant given by Theorem 5.2. All we need to prove is that, for $\eta$ small enough, any $\eta$-perturbation $F$ of the linear Poincaré map along $\gamma$ induces a $\delta$-perturbation $Q$ of the derivative $DX|_{\Gamma T}$ which verifies both conditions of Theorem 5.2.

The relation between the derivative of the flow of a $C^1$-vector field $X$ and the derivative $DX$ of the vector field is the following: for any point $p \in M$ and any time $t$, and any local coordinates in a neighborhood of $\varphi_t(p)$,

$$D_{\varphi_t(p)}X = \partial_s \left( T_{p\varphi_t+s} \circ (T_{p\varphi_t})^{-1} \right)_{s=0}.$$ (2)

Consider $F : T_pM \to T_{\varphi_T(p)}M$, the linear application defined as follows:

$$\begin{cases} F(X(p)) = T_{p\varphi_T}(X(p)), \\ F|_{N_p} = T_{p\varphi_T}(p)|_{N_p} + (F - P_T(p)). \end{cases}$$

We then construct a family of linear applications $(F_s)_{s \in [0,T]}$:

$$F_s = T_{p\varphi_s} \circ (\rho(s).A + (1 - \rho(s)).Id),$$

where $A = (T_{p\varphi_T})^{-1} \circ F$ and $\rho : [0,T] \to [0,1]$ is a smooth function, that vanishes on $[0,T/3]$, strictly increasing on $[T/3,2T/3]$, and equal to 1 on $[2T/3,T]$. There exists a positive constant $C$ (depending on $T > 0$) such that $A$ is a $(C\eta)$-perturbation of the Identity, hence the $F_s$ is a $(C\eta)$-perturbation of $T_{p\varphi_s}$ for any $s \in [0,T]$.

By construction, we have $F_s(X(p)) = X(\varphi_s(p))$. Moreover, if $\eta$ is small enough, the linear application $F_s$ is invertible for any $s \in [0,T]$. Moreover, the application $s \mapsto F_s$ is differentiable and a straightforward computation gives:

$$\partial_s(F_{t+s})_{s=0} = \partial_s(T_{p\varphi_{t+s}})_{s=0} \circ [\rho(t).A + (1 - \rho(t)).Id] + \rho'(t).T_p\varphi_t \circ (A - Id).$$

This formula allows to establish that, for any $\delta' > 0$, there is $\eta$ sufficiently small for the application $\partial_s(F_{t+s})_{s=0}$ to be an $\delta'$-perturbation of the application $\partial_s(T_{p\varphi_{t+s}})_{s=0}$.

26
Consider now the applications $Q_{\varphi_t}(p) : T_{\varphi_t}(p) M \to T_{\varphi_t}(p) M$ defined by:

$$Q(\varphi_t(p)) = \partial_s(F_t+s)_{s=0} \circ (F_t)^{-1}.$$ 

Notice that there exists $\eta$ sufficiently small so that $Q(\varphi_t(p)) - D_{\varphi_t}(p)X$ is bounded by $\delta$. As we intend to prove that $Q$ is a perturbation of $DX$ along $\gamma$, we shall verify that $Q_{\varphi_t}(p)(X(\varphi_t(p))) = D_{\varphi_t}(p)X(X(\varphi_t(p)))$.

By developing the formula, we get:

$$Q(\varphi_t(p)) = \partial_s (T_p \varphi_{t+s})_{s=0} \circ [\rho(t).A + (1 - \rho(t)).Id] \circ [\rho(t).A + (1 - \rho(t)).Id]^{-1} \circ (T_p \varphi_t)^{-1} + \rho'(t).T_p \varphi_t \circ (A - Id) \circ [\rho(t).A + (1 - \rho(t)).Id]^{-1} \circ (T_p \varphi_t)^{-1}$$

$$= D_{\varphi_t}(p)X + \rho'(t).T_p \varphi_t \circ (A - Id) \circ [\rho(t).A + (1 - \rho(t)).Id]^{-1} \circ (T_p \varphi_t)^{-1}.$$

Moreover, since $(T_p \varphi_t)^{-1}(X(\varphi_t(p))) = X(p)$, the second term of the right-hand of the last equation vanishes:

$$(A - Id) \circ [\rho(t).A + (1 - \rho(t)).Id]^{-1} \circ (T_p \varphi_t)^{-1}(X(\varphi_t(p))) = (A - Id)(X(p)) = 0,$$

thus, for any $t \in [-t_1, t_2]$, we get $Q_{\varphi_t}(p)(X(\varphi_t(p))) = D_{\varphi_t}(p)X(X(\varphi_t(p)))$, hence $Q$ is a perturbation of $DX$ along $\gamma$. Moreover, by the definition of $\rho$, $Q$ coincides with $DX$ along $\varphi[0,T/3 \cup 2T/3,T]$, hence can be continuously extended by $DX$ on $\Gamma \setminus \gamma$. Thus, applying the preceding Theorem, we get a vector field $Y$ which is an $\epsilon$-perturbation of $X$ and which achieves $Q$ as derivative along $\Gamma$. Thanks to the equation 2, $Y$ admits $F$ as derivative of the flow along $\gamma$, and in particular achieves the desired application $F$ as derivative of the flow along $\gamma$. We are thus left to prove that we are able to achieve some particular perturbations of the differential of any real function defined along a $C^2$-curve. Since the 2-dimensional case is a lot easier and moreover useful in the proof of the $n$-dimensional case, we shall first deal with the 2-dimensional case.

### 5.3 Preliminary 2-dimensional analytic result

The key tool in the proof of Theorem 5.2 is the following analytic lemma, which we shall prove in the next paragraph.\(^3\)

**Lemma 5.4.** For any real $\varepsilon > 0$, for any three-uple of positive real numbers $(\eta, T, H)$ such that $\eta < T$, for any continuous function $a : [0, T] \to \mathbb{R}$ such that $a$ is $\varepsilon/4$-bounded (in norm) and equal to zero on $[0, \eta] \cup [T - \eta, T]$, there is a $C^1$-application $f : [0, T] \times [-H, H] \to \mathbb{R}$ verifying the following conditions:

1. $\forall x \in [0, T], \ f(x, 0) = 0$;
2. $\forall x \in [0, T], \ \partial_y f(x, 0) = a(x)$;

\(^3\)For the sake of simplicity of notations, we shall henceforth parametrize the considered arc of trajectory $\Gamma$ by $t \in [0, T]$, instead of $t \in [-t_1, t_2]$, and the arc of trajectory $\gamma$ by $t \in [\eta, T - \eta]$, instead of $t \in [0, T] \subset [-t_1, t_2]$. 27
3. \( \| f_{\partial([0,T] \times [-H,H])} \|_{C^1} = 0; \)
4. \( \| f \|_{C^1} \leq \varepsilon \) on \([0,T] \times [-H,H].\)

Before dealing with the proof of this preliminary result, let us see what this statement implies. Consider a 2-dimensional box \([0,T] \times [-H,H] \subset \mathbb{R}^2\) (endowed with an orthonormal basis \((\partial x, \partial y)\)), and a \(C^1\)-vector field \(X\) defined on this box, such that, for any \(x \in [0,T], X(x,0) = \partial x\). Thus \(DX(x,0) = \left( \frac{\partial x X^x}{\partial x}, \frac{\partial x X^y}{\partial x} \right)\). As we want to alter \((DX(x,0))_{x \in [0,T]}\) while preserving \(X\) along the arc of trajectory \((x,0)_{x \in [0,T]}\), we can only alter \(\partial_y X^x\) and \(\partial_y X^y\) by adding continuous functions \(a_x\) and \(a_y\) satisfying the conditions imposed on the application \(a\) in the previous statement.

Applying twice the announced preliminary result, we then get two functions \(f_x\) and \(f_y\) and the \(C^1\)-perturbed vector field \(\tilde{X}(x,y) = (X^x(x,y) + f_x(x,y)) \partial x + (X^y(x,y) + f_y(x,y)) \partial y\) satisfies both following properties:

- \(\tilde{X}\) coincides with \(X\) outside a box \([0,T] \times [-H,H]\) and along the orbit \([0,T] \times \{0\}\);
- \(D\tilde{X}(x,0) = \left( \frac{\partial x X^x(x,0)}{\partial x} + a_x(x), \frac{\partial x X^y(x,0)}{\partial x} + a_y(x) \right)\).

Thus Theorem 5.2 is proved under special conditions, namely in the 2-dimensional case, when the preserved arc of trajectory \(\Gamma\) is \(\text{straight}\), and the vector field \(X\) constant along \(\Gamma\).

**Proof:** Consider the application \(F : [0,T] \times [-H,H] \to \mathbb{R}\) defined hereafter:

\[ \forall x \in [0,T], \quad F(x,y) = y.a(x). \]

This application, which satisfies the first three conditions of Lemma 5.4, is continuous, but sadly enough not \(C^1\), since \(a\) is not supposed to be more than \(C^0\). We shall improve the regularity of \(F\) by using the convolution product (cf [Schw] p.128). Throughout the rest of this paper, we shall use the following notation \(F_y(x) = F(x,y)\).

Let \(\Phi : \mathbb{R} \to [0,1]\) denotes a smooth bell function, that is a function vanishing outside \([-1,1]\) verifying \(\int_{-1}^{1} \Phi = 1\) and \(\forall k \geq 1, \ \Phi^{(k)}(0) = 0\). For any \(\varepsilon \in ]0,1[\), let \(\Phi_{\varepsilon} : \mathbb{R} \to [\ln(\varepsilon), -\ln(\varepsilon)]\) denote the function defined by\(^4\):

\[ \Phi_{\varepsilon}(x) = -\ln(\varepsilon).\Phi(-\ln(\varepsilon).x) . \]

This function vanishes outside \([1/\ln \varepsilon, -1/\ln \varepsilon]\) and verifies \(\int_{\mathbb{R}} \Phi_{\varepsilon} = 1\).

Consider now the application \(h : [0,T] \times [-H,H] \to \mathbb{R}\) defined hereafter:

- \(\forall x \in [0,T], \ h(x,0) = 0;\)
- \(\forall(x,y) \in [0,T] \times [0,H],\)
  \[ h(x,y) = (\Phi_y \ast F_y)(x) \]
  \[ = \int \Phi_y(x-u).F(u,y)du \]
  \[ = -\ln(y).\int \Phi(-\ln(y).u.a(u)du \]
  \[ = y.\int_{-1}^{1} \Phi(v).a \left( x + \frac{v}{\ln(y)} \right) dv; \]

\(^4\) M.-C. Arnaud signaled us that we could also consider \(\Phi_{\varepsilon} : x \in \mathbb{R} \mapsto 1/\varepsilon.\Phi(x/\varepsilon) \in [-1/\varepsilon, 1/\varepsilon]\). The computation of the partial derivatives of \(h\) is analogous.
satisfies all the announced conditions.

This application clearly satisfies the first two conditions of Lemma 5.4. Let us now determine the partial derivatives of $h$.

On the one hand, for any $x$ and $y > 0$,
\[
\partial_x h(x, y) = \partial_x \left( -y \ln(y) \cdot \int \Phi(-\ln(y) \cdot (x - u)) \cdot a(u) du \right)
\]
\[
= -y \ln(y) \cdot \partial_x \Phi(-\ln(y) \cdot (x - u)) \cdot a(u) du
\]
\[
= y \ln(y)^2 \cdot \int \Phi'(\ln(y) \cdot (x - u)) \cdot a(u) du
\]
\[
= -y \ln(y) \cdot \int \Phi'(v) \cdot a \left( x + \frac{v}{\ln(y)} \right) dv,
\]
thus $\partial_x h(x, y)$ converges to $0 = \partial_x (h(x, 0))$ when $y$ goes to 0.

On the other hand, for any $x$ and any $y > 0$,
\[
\partial_y h(x, y) = \partial_y \left( -y \ln(y) \cdot \int \Phi(-\ln(y) \cdot (x - u)) \cdot a(u) du \right)
\]
\[
= -(\ln(y) + 1) \cdot \int \Phi(-\ln(y) \cdot (x - u)) \cdot a(u) du
\]
\[
+ \ln(y) \cdot \int \Phi'(-\ln(y) \cdot (x - u)) \cdot (x - u) \cdot a(u) du
\]
\[
= \left( 1 + \frac{1}{\ln(y)} \right) \cdot \int \Phi'(v) \cdot a \left( x + \frac{v}{\ln(y)} \right) dv
\]
\[
+ \frac{1}{\ln(y)} \int \Phi'(v) \cdot v \cdot a \left( x + \frac{v}{\ln(y)} \right) dv
\]
hence $\partial_y h(x, y)$ converges to $a(x) = \int_{\mathbb{R}} \Phi(v) \cdot a(x) dv$ when $y$ goes to 0.

We have thus proved that $h$ is a $C^1$-application. Moreover, there is $\tilde{H} \in [0, H]$ such that

- $\|h|_{[\eta/2, T-\eta/2] \times [-\tilde{H}, \tilde{H}]|_{C^1} = 0$;
- $\|h|_{[0, T] \times [-\tilde{H}, \tilde{H}]|_{C^1} \leq \varepsilon/2$.

Let $\psi : [0, H] \to [0, \tilde{H}]$ denotes a smooth step function such that $\psi'(0) = 1$, $\psi'(H) = 0$ with positive derivative bounded by 1. The application $g : [0, T] \times [-H, H] \to \mathbb{R}$ defined by
\[
g(x, y) = h(x, \psi(y))
\]
satisfies the first four conditions of Lemma 5.4.

Consider eventually a function $\rho : [-H, H] \to [0, 1]$ equal to 1 when $y \in [-H/3, H/3]$, and equal to 0 when $y \in [-H, -2H/3] \cup [2H/3, H]$, with a $C^1$-norm bounded by a constant depending on $H$. If $\tilde{H}$ is chosen small enough (recall that $\|h|_{[0, T] \times [-\tilde{H}, \tilde{H}]|_{C^0}$ tends to 0 when $\tilde{H}$ goes to 0), the application $f$ defined by
\[
f(x, y) = \rho(y) \cdot h(x, \psi(y))
\]
satisfies all the announced conditions. □
5.4 Generalization to higher dimension

Here is a statement equivalent to Lemma 5.4 in higher dimension:

**Lemma 5.5.** For any integer \( n \geq 1 \), for any four-uple of real numbers \((\varepsilon, T, H, \eta)\) such that \( \eta < T/2 \), for any continuous function \( a : [0, T] \to \mathbb{R} \) which is \( \varepsilon/4 \)-bounded (in norm) and equal to zero on \([0, \eta] \cup [T - \eta, T]\), for any \( j \in \{2, n\} \), there exists a \( C^1 \)-application \( F(j) : [0, T] \times [-H, H]^n \to \mathbb{R} \) verifying the following assumptions:

1. \( \forall x_1 \in [0, T], \ F(j)(x_1, 0, \ldots, 0) = 0; \)
2. \( \forall x_1 \in [0, T], \ dF(j)(x_1, 0, \ldots, 0) = \partial_{x_1} F(j)(x_1, 0, \ldots, 0) dx_j = a(x_1)dx_j; \)
3. \( \|F|_{\partial([0, T] \times [-H, H]^n)}\|_{C^1} = 0; \)
4. \( \|F\|_{C^1} \leq \varepsilon \) on \([0, T] \times [-H, H]^n\).

**Proof:** In order to simplify the notation, let us consider for example the case \( j = 2 \) and construct the announced function. Using the notations of the proof of Lemma 5.4, there is \( \bar{H} \in [0, H] \) small enough so that the function \( F : [0, T] \times [-H, H] \to \mathbb{R} \) defined by

\[
F(x_1, x_2, \ldots, x_n) = \rho(x_3) \cdot \rho(x_4) \cdots \rho(x_n) \cdot f(x_1, x_2)
\]

satisfies all the announced conditions.

\[ \square \]

5.5 Proof of Theorem 5.2

In order to prove Theorem 5.2, we shall need a more general version of the preceding lemmas. We need to consider an application \( a \) no longer defined on a segment of the type \([0, T] \times \{0\}\), but more generally on a \( C^2 \)-arc of trajectory in \( \mathbb{R}^n \). (In fact, as we shall see in the proof, we only need the arc of trajectory to be \( C^1 \).)

The following lemma states that we can get the preceding result when we consider a \( C^1 \)-arc of trajectory \( \gamma \) as long as the velocity does not change too much along \( \gamma \).

Recall that (using the notation previously introduced), for any neighborhood \( U \) of an arc of trajectory \( \gamma = (\gamma(t))_{t \in [0, T]} \), we denote by \( \Gamma = (\gamma(t))_{t \in [-t_1, t_2]} \) the continuation of \( \gamma \) in \( U \). We denote by \((\partial x_1, \ldots, \partial x_n)\) an orthogonal basis of \( \mathbb{R}^n \).

**Lemma 5.6.** For any integer \( n \geq 2 \), for any four-uple \((\epsilon > 0, \delta \in [0, 1[, \eta < T/2]\), any angle \( \theta \in ]0, \pi/2[, \) and any \( C^1 \)-curve \( \gamma = (x_1(t), \ldots, x_n(t))_{t \in [0, T]} \) in \( \mathbb{R}^n \) verifying the following three assumptions:

- \( \dot{\gamma}(0) = |\dot{\gamma}(0)| \partial x_1 \neq 0; \)
- \( \text{Angle} (\dot{\gamma}(t), \partial x_1) \leq \theta, \ \forall t \in [0, T]; \)
- \( |\ddot{\gamma}(t)|/|\dot{\gamma}(0)| \in [1 - \delta, 1 + \delta], \ \forall t \in [0, T]; \)

for any neighborhood \( U \) of \( \gamma \), for any continuous application \( a : \gamma \to \mathbb{R} \) which is bounded by \( \zeta = \min (\epsilon/8, \epsilon/8((n - 1) \tan \theta + (1 - \delta) \cdot \cos \theta)^{-1}) \) (in norm) and equal to zero on \([\gamma(t)]_{t \in [0, \eta]} \cup [T - \eta, T]\) and for any integer \( j \in \{2, \ldots, n\} \) there exists a \( C^1 \)-function \( f(j) : \mathbb{R}^n \to \mathbb{R} \) verifying the following conditions:
1. $f(j)|_\gamma = 0$;
2. $\partial_x f(j)|_\gamma = a(x)$;
3. $\partial_{x_k} f(j)|_\gamma = 0$, $\forall k \neq j$;
4. $\|f(j)\|_{C^1} \leq \epsilon$;
5. $f(j) = 0$ outside $U$.

**Proof:** For any $h = (0, h_2, \ldots, h_n) \in Vector(\partial x_2, \ldots, \partial x_n) \subset \mathbb{R}^n$, we denote by $\gamma_h$ the image of $\gamma$ by the translation application $T_h : x \mapsto x + h$ (thus $\gamma_h(t) = T_h(\gamma(t))$ for any $t$). Let us denote by $H$ a positive real number such that

$$\left\{\begin{array}{l}
V = \bigcup_{-H \leq h \leq H} \gamma_h \subset U, \\
V \cap \Gamma = \gamma.
\end{array}\right.$$ 

Let $\epsilon' = \min(\epsilon/2, \epsilon/(n-1) \tan \theta + ((1 - \delta) \cos \theta)^{-1})$. By Lemma 5.5, for any $j \in [2, n]$, there exists a $C^1$-application $g(j) : [-t_1, t_2] \times [-H, H]^{n-1} \rightarrow \mathbb{R}$ satisfying the following properties:

1. $\forall t \in [-t_1, t_2], g(j)(t, 0) = 0$;
2. $\forall t \in [0, T]$, $\partial_x g(j)(t, 0, \ldots, 0) = a(t)$;
3. $\partial_{x_k} g(j)|_\gamma = 0$, $\forall k \neq j$;
4. $\|g(j)\|_{[0, T] \times [-H, H]^{n-1}}_{C^1} = 0$;
5. $\|g(j)\|_{C^1} \leq \epsilon'$.

Consider now the continuous application $f(j) : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

- $f(j)(\gamma_h(t)) = g(t, \partial x_1 + h)$, for any $t \in [-t_1, t_2]$ and any $h \in Vector(\partial x_2, \ldots, \partial x_n)$,
- $f(j) = 0$ outside $V$.

This application clearly satisfies the first and the last condition of Lemma 5.6. Since $\partial_{x_i} f(j)(\gamma_h(t)) = \partial_{x_i} g(j)(t, h)$ for any $i \in [2, n]$, the function $f(j)$ verifies the second condition as well.

Moreover, on the one hand $df(j)(\gamma_h(t)) = \partial x_1 g(j)(t, h)$, and on the other hand $df(j)(\gamma_h(t)) = \partial x_1 f(j)(\gamma_h(t))dx_1 + \sum_{k=2}^n \partial_{x_k} f(j)(\gamma_h(t))dx_k$ implies

$$df(j)(\gamma_h(t)) = \|\partial x_1 f(j)(\gamma_h(t))\| \cdot \langle \dot{\gamma}(t), \partial x_1 \rangle + \sum_{k=2}^n \|\partial_{x_k} f(j)(\gamma_h(t))\| \cdot \langle \dot{\gamma}(t), \partial x_k \rangle.$$ 

Thus $f$ is a $C^1$-application, and we can write:

$$\|\partial x_1 f(j)(\gamma_h(t))\| \leq \|\partial x_1 g(j)\| \cdot \frac{1}{\langle \dot{\gamma}(t), \partial x_1 \rangle} + \sum_{k=2}^n \|\partial_{x_k} g(j)\| \cdot \frac{\langle \dot{\gamma}(t), \partial x_k \rangle}{\langle \dot{\gamma}(t), \partial x_1 \rangle} \leq \epsilon' \left( \frac{1}{(1 - \delta) \cos \theta} + (n - 1) \tan \theta \right),$$

which concludes the proof.

Thanks to the following proposition, Theorem 5.2 is a consequence of the Lemma 5.6.
Proposition 5.7. Let $X$ be a $C^1$-vector field on a boundaryless compact Riemannian manifold $M$. For any $\delta > 0$ and any angle $\theta \in [0, \pi/2]$, there is $T > 0$ such that for any regular point $p$, both the following assertions hold:

- $\forall t \in [0, T], \text{Angle}(X(\varphi_t(p), X(p))) \leq \theta$;
- $\forall t \in [0, T], |X(\varphi_t(p))| > \delta |X(p)|$.

Proof: Thanks to the Long Tubular Neighborhood theorem (cf [PalMel]), the compacity of $M$ implies, for any real $T > 0$, the existence of an atlas of finitely many coordinates charts $(U_i)$, such that any arc of trajectory $\gamma = \varphi_{[0,T]}(\{p\})$ admits a tubular neighborhood $U_i$. The proposition is then the consequence of the convergence of the derivative $T_p\varphi_t$ to the Identity, uniformly in $p \in M$, when $t$ goes to zero, together with the equation $X(\varphi_t(p)) = T_p\varphi_t(X(p))$.

References


Christian Bonatti (bonatti @ u-bourgogne.fr)
Nikolaz Gourmelon (Nicolas.Gourmelon @ u-bourgogne.fr)
Thérèse Vivier (therese.vivier @ u-bourgogne.fr)
I.M.B., UMR 5584 du CNRS, B.P. 47 870, 21078 Dijon Cedex, France