Interference Alignment via Message-Passing

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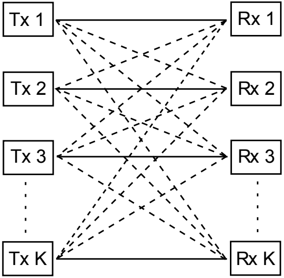
Outline

- Motivation of the Interference Alignment Problem
- Message-passing and optimization
- Numerical Results
- Conclusion

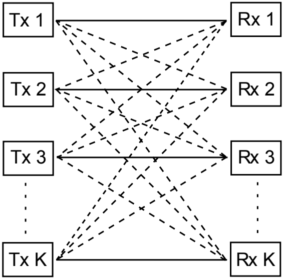
Communication Problem Motivation

- K transmitters (Tx), K receivers (Rx)
- Each equipped with an antenna array; each antenna sends/receives a complex scalar
- Rx i is only interested in the message from the corresponding Tx i
- Other transmitters create (unwanted) interference



Communication Problem Motivation

- At each (discrete) time instant, Tx j sends a (N-dimensional) signal $\underline{x}_j$, Rx i receives $\underline{y}_i$ (M-dimensional)
- The gains of the wireless propagation channel between Tx j and Rx i is $\mathbf{H}_{ij}$ ($M \times N$ matrix)



$$\underline{y}_i = \sum_{j=1}^{K} \mathbf{H}_{ij} \underline{x}_j \quad \forall i = 1 \dots K$$

- Rx i wants to infer $\underline{x}_i$ from $\underline{y}_i$ (all $\mathbf{H}_{ij}$ are assumed known at all nodes)

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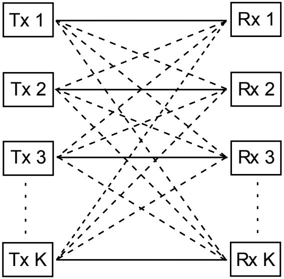

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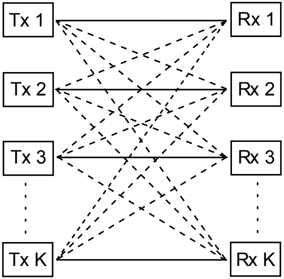

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Inteference Alignment

Simple, linear transmission scheme that mitigates interference:

- Tx signals are restricted to a d-dimensional subspace of the N-dimensional space of the antenna array, spanned by a truncated unitary matrix $\mathbf{V}_j$. $\underline{s}_j$ is the data to transmit:

$$\underline{x}_j = \mathbf{V}_j \underline{s}_j \qquad (\mathbf{V}_j \in \mathcal{G}_{N,d})$$

- At Rx i, signal is projected onto a d-dimensional subspace spanned by a truncated unitary matrix $\mathbf{U}_i$ ($\in \mathcal{G}_{M,d}$)

$$\underline{s}'_i = \mathbf{U}_i^\dagger \underline{y}_i = \sum_{j=1}^{K} \mathbf{U}_i^\dagger \mathbf{H}_{ij} \mathbf{V}_j \underline{s}_j$$

- Choose the $\mathbf{U}_i$, $\mathbf{V}_j$ such that $\mathbf{U}_i^\dagger \mathbf{H}_{ij} \mathbf{V}_j$ vanishes for $i \neq j$ (interference) but not for $i = j$

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HUAWEI

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Mathematical Formulation

- Given the $M \times N$ matrices $\{\mathbf{H}_{ij}\}_{i,j \in \{1,\ldots K\}}$ and $d < M, N$
- find $\{\mathbf{U}_i\}_{i \in \{1,\ldots K\}}$ in $\mathcal{G}_{M,d}$ and $\{\mathbf{V}_i\}_{i \in \{1,\ldots K\}}$ in in $\mathcal{G}_{N,d}$
- such that $\mathbf{U}_i^{\dagger}\mathbf{H}_{ij}\mathbf{V}_j = \mathbf{0} \quad \forall i \neq j$.
- Depending on the relative values of M, N, K and d, the problem can be trivial, difficult, or provably impossible to solve.
- Example of non-trivial case: $d = 2, M = N = 4, K = 3$.

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Matrix Decomposition Formulation

$$\underbrace{\begin{bmatrix} \mathbf{U}_1^\dagger & & \\ & \ddots & \\ & & \mathbf{U}_K^\dagger \end{bmatrix}}_{Kd \times KM} \underbrace{\begin{bmatrix} \mathbf{H}_{11} & \dots & \mathbf{H}_{1K} \\ \vdots & \ddots & \vdots \\ \mathbf{H}_{K1} & \dots & \mathbf{H}_{KK} \end{bmatrix}}_{KM \times KN} \underbrace{\begin{bmatrix} \mathbf{V}_1 & & \\ & \ddots & \\ & & \mathbf{V}_K \end{bmatrix}}_{KN \times Kd} = \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}$$

Available Solutions

State of the art:

- No closed-form solution known for general dimensions
- Iterative solutions exist[1] but have some undesirable properties

We propose to **use a message-passing (MP) algorithm**:

- distributed solution
- use *local* data ($\mathbf{H}_{ij}$ is known at Rx i and Tx j)

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Generalized Distributive Law

- Efficiently compute functions involving *many variables* that can be decomposed (factorized) into terms involving *subsets of the variables*
- Variable–Factor dependencies captured by a bipartite graph
- General MP formulation for an arbitrary commutative semi-ring[2]

TABLE I
SOME COMMUTATIVE SEMIRINGS. HERE A DENOTES AN ARBITRARY COMMUTATIVE RING, S IS AN ARBITRARY FINITE SET, AND Λ DENOTES AN ARBITRARY DISTRIBUTIVE LATTICE

	K	"$(+,0)$"	"$(\cdot,1)$"	short name
1.	A	$(+,0)$	$(\cdot,1)$	
2.	$A[x]$	$(+,0)$	$(\cdot,1)$	
3.	$A[x,y,\ldots]$	$(+,0)$	$(\cdot,1)$	
4.	$[0,\infty)$	$(+,0)$	$(\cdot,1)$	sum-product
5.	$(0,\infty]$	$(\min,\infty)$	$(\cdot,1)$	min-product
6.	$[0,\infty)$	$(\max,0)$	$(\cdot,1)$	max-product
7.	$(-\infty,\infty]$	$(\min,\infty)$	$(+,0)$	min-sum
8.	$[-\infty,\infty)$	$(\max,-\infty)$	$(+,0)$	max-sum
9.	$\{0,1\}$	$(\texttt{OR},0)$	$(\texttt{AND},1)$	Boolean
10.	2^S	$(\cup,\emptyset)$	$(\cap,S)$	
11.	Λ	$(\vee,0)$	$(\wedge,1)$	
12.	Λ	$(\wedge,1)$	$(\vee,0)$.	

[2] Aji and McEliece, *The Generalized Distributive Law*, IEEE Trans. Inf. Theory, vol. 46, no. 2, Mar. 2000.

HUAWEI

Generalized Distributive Law

- Algorithm operates by "propagating" messages on the graph
- If the graph has no cycle, computation is exact and terminates in finite number of steps
- Otherwise, iterate and hope that it converges to something reasonable
- BCJR decoder, Viterbi, FFT algorithms as special cases

Min-Sum Algorithm[3]

- GDL applied to $(\mathbb{R}, \min, +)$
- Example:

$$C(x_1, x_2, x_3, x_4) = C_a(x_1, x_2, x_3) + C_b(x_2, x_4) + C_c(x_3, x_4)$$

- Min-Sum can solve efficiently

$$\min_{x_1, x_2, x_3, x_4} C(x_1, x_2, x_3, x_4)$$

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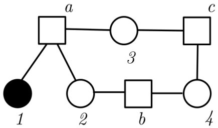

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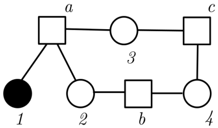

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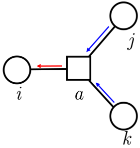
Min-Sum Implementation

- factor-to-variable update



$$m_{a\to i}(x_i) = \min_{x_j, x_k} \left[C_a(x_i, x_j, x_k) + m_{j\to a}(x_j) + m_{k\to a}(x_k) \right]$$

- variable-to-factor update

$$m_{i\to a}(x_i) = \sum_{b\in\mathbb{N}(i)\setminus a} m_{b\to i}(x_i)$$

(illustrations from [Yedidia 2011])

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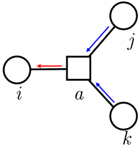

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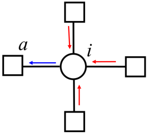

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HUAWEI

Back to Our Interference Problem...

- Reformulate the IA equations

$$\mathbf{U}_i^\dagger \mathbf{H}_{ij} \mathbf{V}_j = \mathbf{0} \quad \forall i \neq j \qquad \Leftrightarrow \qquad \sum_{i \neq j} \left|\left| \mathbf{U}_i^\dagger \mathbf{H}_{ij} \mathbf{V}_j \right|\right|_F^2 = 0$$

$$\Updownarrow$$

$$\sum_{i=1}^{K} \sum_{j \neq i} \mathrm{trace} \left(\mathbf{U}_i^\dagger \mathbf{H}_{ij} \mathbf{V}_j \mathbf{V}_j^\dagger \mathbf{H}_{ij} \mathbf{U}_i \right) = 0$$

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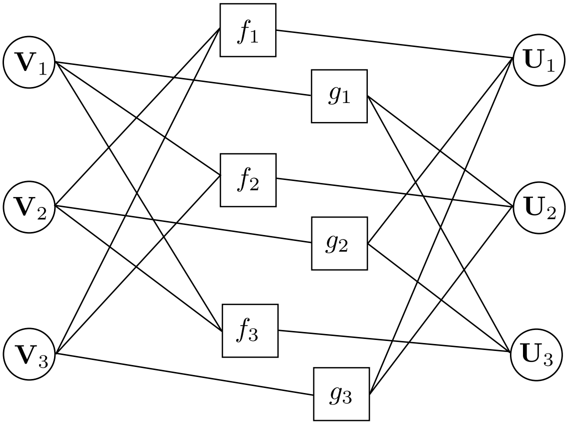
- **Solve via Min-Sum**

$$\min_{\substack{\mathbf{U}_{i=1\ldots K} \in \mathcal{G}_{M,d}, \\ \mathbf{V}_{j=1\ldots K} \in \mathcal{G}_{N,d}}} \quad \sum_{i=1}^{K} f_i \left(\mathbf{U}_i, \{\mathbf{V}_j\}_{j \neq i} \right) + \sum_{j=1}^{K} g_j \left(\mathbf{V}_j, \{\mathbf{U_i}\}_{i \neq j} \right)$$

Graphical representation of IA on the 3-user IC



Challenge #1: Continuity

- Consider one message $m_{f_i \to \mathbf{U}_i}$:

$$m_{f_i \to \mathbf{U}_i}(\mathbf{U}_i) = \min_{\mathbf{V}_{j \neq i}} \left[f_i(\mathbf{U}_i, \mathbf{V}_{j \neq i}) + \sum_{j \neq i} m_{\mathbf{V}_j \to f_i}(\mathbf{V}_j) \right]$$

- This message is a *function* with argument in $\mathcal{G}_{M,d}$
- Compute and pass $m_{f_i \to \mathbf{U}_i}(\mathbf{U}_i)$ for each possible value of $\mathbf{U}_i$?
- Parameterize the function!
- We propose (for $\mathbf{X} \in \mathcal{G}_{M,d}$)

$$m_{a \to b}(\mathbf{X}) = \mathrm{trace}\left(\mathbf{X}^\dagger \mathbf{Q}_{a \to b} \mathbf{X} \right)$$

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Challenge #2: The Importance of being Closed-Form

- Simple variable-to-functions message computation:

$$m_{\mathbf{U}_i \to f_i}(\mathbf{U}_i) = \sum_{j \neq i} m_{g_j \to \mathbf{U}_i}(\mathbf{U}_i) \quad \Leftrightarrow \quad \mathbf{Q}_{\mathbf{U}_i \to f_i} = \sum_{j \neq i} \mathbf{Q}_{g_j \to \mathbf{U}_i} \ldots$$

- Function-to-variable messages are more tricky:

$$m_{f_i \to \mathbf{U}_i}(\mathbf{U}_i) = \sum_{j \neq i} \min_{\mathbf{V}_j} \operatorname{trace}\left(\mathbf{V}_j^\dagger \left[\mathbf{H}_{ij}^\dagger \mathbf{U}_i \mathbf{U}_i^\dagger \mathbf{H}_{ij} + \mathbf{Q}_{\mathbf{V}_j \to f_i}\right] \mathbf{V}_j\right)$$

  - Each minimization is an eigenspace problem
  - Resort to approximation to express $m_{f_i \to \mathbf{U}_i}$ as $\operatorname{trace}\left(\mathbf{X}^\dagger \mathbf{Q}_{f_i \to \mathbf{U}_i} \mathbf{X}\right)$
  - Approximation becomes exact in the vicinity of the solution

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$$m_{\mathbf{U}_i \to f_i}(\mathbf{U}_i) = \sum_{j \neq i} m_{g_j \to \mathbf{U}_i}(\mathbf{U}_i) \quad \Leftrightarrow \quad \mathbf{Q}_{\mathbf{U}_i \to f_i} = \sum_{j \neq i} \mathbf{Q}_{g_j \to \mathbf{U}_i} \dots$$

- Function-to-variable messages are more tricky:

$$m_{f_i \to \mathbf{U}_i}(\mathbf{U}_i) \quad = \quad \sum_{j \neq i} \min_{\mathbf{V}_j} \operatorname{trace}\left(\mathbf{V}_j^\dagger \left[\mathbf{H}_{ij}^\dagger \mathbf{U}_i \mathbf{U}_i^\dagger \mathbf{H}_{ij} + \mathbf{Q}_{\mathbf{V}_j \to f_i}\right] \mathbf{V}_j\right)$$

  - Each minimization is an eigenspace problem
  - Resort to approximation to express $m_{f_i \to \mathbf{U}_i}$ as $\operatorname{trace}\left(\mathbf{X}^\dagger \mathbf{Q}_{f_i \to \mathbf{U}_i} \mathbf{X}\right)$
  - Approximation becomes exact in the vicinity of the solution

HUAWEI

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HUAWEI

Some Observations on the Distributed Aspect

- Factors $f_i(\mathbf{U}_i, \mathbf{V}_{j\neq i})$ and $g_j(\mathbf{V}_j, \mathbf{U}_{i\neq j})$ rely on local data only (respectively available at Rx i and Tx j)

- Mapping of the graph nodes to the physical devices?

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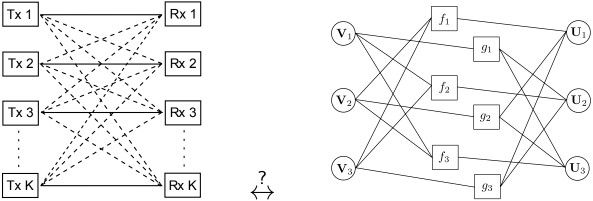
- Mapping of the graph nodes to the physical devices?



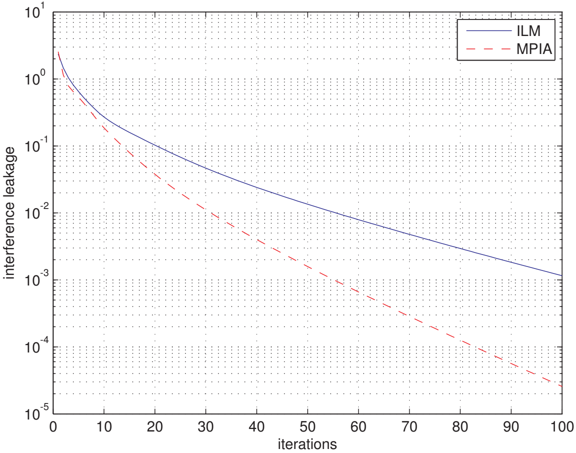
Numerical Results

- Metric of interest: Residual interference power (leakage)

$$\sum_{i=1}^{K} f_i \left(\mathbf{U}_i, \{\mathbf{V}_j\}_{j \neq i} \right) + \sum_{j=1}^{K} g_j \left(\mathbf{V}_j, \{\mathbf{U}_\mathbf{i}\}_{i \neq j} \right)$$

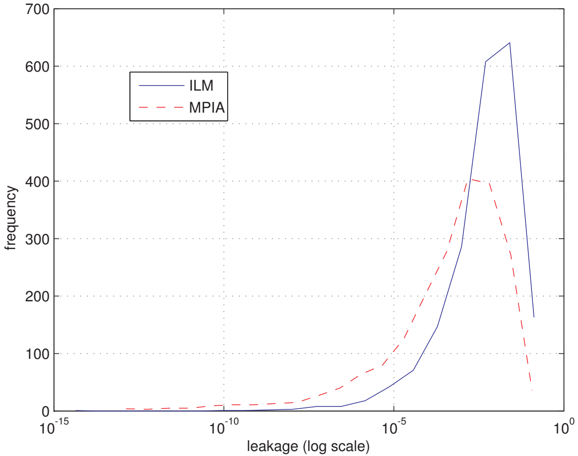
- Converges reliably to 0 despite lack of optimality proof for MP when the graph has cycles
- Convergence speed compares favorably to existing centralized algorithms (ILM...)

Numerical Results



Leakage vs. iterations for one realization of $\mathbf{H}_{ij}$, $K = 3$, $M = N = 4$, $d = 2$.

Numerical Results



Empirical distribution of leakage after 100 iterations over random realizations of $\mathbf{H}_{ij}$, $K = 3$, $M = N = 4$, $d = 2$.

Conclusion / Open Questions

- Systematic method to *distributedly* solve the interference alignment problem
- ...and possibly other simultaneous orthogonalization problems formulated on the Grassmann manifold
- Open issues
  - Optimality proof is missing
  - Effect of quantizing the "messages" $\mathbf{Q}_{a \to b}$?
  - Optimize mapping of the graph nodes to the devices

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Backup Slides

About the approximation in $m_{f_i \to \mathbf{U}_i}$

$$
\begin{aligned}
m_{f_i \to \mathbf{U}_i}(\mathbf{U}_i) &= \sum_{j \neq i} \min_{\mathbf{V}_j} \operatorname{trace}\left(\mathbf{V}_j^\dagger \left[\mathbf{H}_{ij}^\dagger \mathbf{U}_i \mathbf{U}_i^\dagger \mathbf{H}_{ij} + \mathbf{Q}_{\mathbf{V}_j \to f_i}\right] \mathbf{V}_j\right) \\
&\approx \sum_{j \neq i} \operatorname{trace}\left(\mathbf{V}_j^{0\,\dagger} \left[\mathbf{H}_{ij}^\dagger \mathbf{U}_i \mathbf{U}_i^\dagger \mathbf{H}_{ij} + \mathbf{Q}_{\mathbf{V}_j \to f_i}\right] \mathbf{V}_j^0\right) \\
& \text{where} \quad \mathbf{V}_j^0 = \arg\min_{\mathbf{V}_j} \operatorname{trace}\left(\mathbf{V}_j^\dagger \mathbf{Q}_{\mathbf{V}_j \to f_i} \mathbf{V}_j\right)
\end{aligned}
$$

- Approximation becomes exact in the vicinity of the solution

Numerical Results: Benchmark

Benchmark: Iterative leakage minimization (ILM) algorithm from [Gomadam,Cadambe,Jafar 2011].

- Iterative
- Assumes channel reciprocity
- Over-the-air training phases
- No formal proof of convergence

Algorithm 1 Iterative Interference Alignment

1: Start with arbitrary precoding matrices $\mathbf{V}^{[j]} : M^{[j]} \times d^{[j]}, \mathbf{V}^{[j]}\mathbf{V}^{[j]\dagger} = \mathbf{I}_{d^{[j]}}$.

2: Begin iteration

3: Compute interference covariance matrix at the receivers:

$$\mathbf{Q}^{[k]} = \sum_{j=1, j\neq k}^{K} \frac{P^{[j]}}{d^{[j]}} \mathbf{H}^{[kj]}\mathbf{V}^{[j]}\mathbf{V}^{[j]\dagger}\mathbf{H}^{[kj]\dagger}$$

4: Compute the interference suppression matrix at each receiver:

$$\mathbf{U}_{\star d}^{[k]} = \boldsymbol{\nu}_d[\mathbf{Q}^{[k]}], \quad d = 1, \ldots, d^{[k]}$$

5: Reverse the communication direction and set $\overleftarrow{\mathbf{V}}^{[k]} = \mathbf{U}^{[k]}$.

6: Compute interference covariance matrix at the new receivers:

$$\overleftarrow{\mathbf{Q}}^{[j]} = \sum_{k=1, k\neq j}^{K} \frac{\overleftarrow{P}^{[k]}}{d^{[k]}} \overleftarrow{\mathbf{H}}^{[jk]}\overleftarrow{\mathbf{V}}^{[k]}\overleftarrow{\mathbf{V}}^{[k]\dagger}\overleftarrow{\mathbf{H}}^{[jk]\dagger}$$

7: Compute the interference suppression matrix at each receiver:

$$\overleftarrow{\mathbf{U}}_{\star d}^{[j]} = \boldsymbol{\nu}_d[\overleftarrow{\mathbf{Q}}^{[j]}], \quad d = 1, \ldots, d^{[k]}$$

8: Reverse the communication direction and set $\mathbf{V}^{[k]} = \overleftarrow{\mathbf{U}}^{[k]}$.

9: Continue till convergence.