université de BORDEAUX	ANNÉE UNIVERSITAIRE 2018 / 2019 SESSION 1 D'AUTOMNE **PARCOURS / ÉTAPE : 4TMA903U** **Code UE : 4TTN901S, 4TTN901S** **Épreuve :** Algebraic number theory **Date :** 5/11/2018 **Heure :** 8h30 **Durée :** 1h30 Documents : non autorisés Épreuve de Mr Brinon	**Collège Sciences et technologies**

Documents are not allowed.
The quality of writing will be an important assessment factor.

Exercise 1

(1) Find the limit of the sequence of integers $(3, 33, 333, 3333, \ldots)$ (written in base 10) in $\mathbf{Q}_5$.
(2) Let p be a prime integer and $x \in \mathbf{Z}$ such that $\mathsf{gcd}(p, x) = 1$. Show that the sequence $(x^{p^n})_{n \in \mathbf{Z}_{\geqslant 0}}$ converges in $\mathbf{Q}_p$. Show that the limit is a root of unity, that depends only on the image of x in $\mathbf{F}_p^\times$.
(3) Let p be a prime integer. Prove that $X^p - X - 1$ is irreducible in $\mathbf{Q}_p[X]$.

Exercise 2

Let $(K, |.|)$ be a non archimedean valued field such that $\mathsf{char}(K) = \mathsf{char}(\kappa_K)$. Show that if $x, y \in K$ are distinct roots of unity, then $|x - y| = 1$.

Exercise 3

Show that the polynomial $(X^2-2)(X^2-17)(X^2-34)$ has a root in $\mathbf{R}$ and in $\mathbf{Z}_p$ for every prime p, but no root in $\mathbf{Q}$.

Exercise 4

Let K be a field. A subring $A \subset K$ is a *valuation ring* of K when $(\forall x \in K)\, x \notin A \Rightarrow x^{-1} \in A$ (this implies in particular that $K = \mathsf{Frac}(A)$).
(1) Show if A is a valuation ring of K and I, J are ideals in A, then either $I \subset J$ or $J \subset I$. Deduce that A is local (we denote henceforth its maximal ideal by $\mathfrak{m}_A$).
(2) Let F be a field, $A = F[\![X, Y]\!]$ the ring of formal series and $K = F(\!(X, Y)\!) = \mathsf{Frac}(A)$ the field of formal Laurent series. Is the local ring A a valuation ring of K?
(3) Show that a valuation ring of K is integrally closed.
(4) Let $A \subset K$ be a subring and $\mathfrak{p} \subset A$ a maximal ideal. The aim of this question is to show that there exists a valuation ring R of K such that $A \subset R$ and $A \cap \mathfrak{m}_R = \mathfrak{p}$.

- (a) Show that the set $\mathscr{E}$ of subrings $B \subset K$ such that $A \subset B$ and $1 \notin \mathfrak{p}B$ contains an element R which is maximal for the inclusion [hint: Zorn].
- (b) Show that R is local, and that its maximal ideal $\mathfrak{m}_R$ satisfies $A \cap \mathfrak{m}_R = \mathfrak{p}$ [hint: consider the localization of R at maximal ideal $\mathfrak{m} \subset R$ such that $\mathfrak{p}R \subset \mathfrak{m}$].
- (c) Let $x \in K^\times$ be such that $x, x^{-1} \notin R$. Using the fact that $R[x], R[x^{-1}] \notin \mathscr{E}$, show that there exist relations $1 = a_1 x + \cdots + a_n x^n$ and $1 = b_1 x^{-1} + \cdots + b_m x^{-m}$ with $a_1, \ldots, a_n, b_1, \ldots, b_m \in \mathfrak{m}_R$. Assuming $n, m \in \mathbf{Z}_{>0}$ minimal, derive a contradiction an deduce that R is a valuation ring.

(5) Let $A \subset K$ be a subring, $B \subset K$ the integral closure of A in K, and B' the intersection of all the valuation rings of K that contain A.

- (a) Show that $B \subset B'$.
- (b) Let $x \in K$ such that x is not integral over A. Show that $x^{-1}A[x^{-1}]$ is a strict ideal in $A[x^{-1}]$. Conclude that there exists a valuation ring R such that $x \notin R$ [hint: use question (4)].
- (c) Conclude that $B' = B$.

(6) Let A be a PID, $K = \mathsf{Frac}(A)$. Show that the valuation rings of K that contain A and are distinct from K are the localizations A_{pA} where p is a prime element in A.
(7) Let $A \subset K$ be a valuation ring such that there exists a prime ideal $\mathfrak{p} \subset A$ such that $\{0\} \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}_A$. Show that the ring $R = A[\![X]\!]$ is not integrally closed [hint: take $a \in \mathfrak{m}_A \backslash \mathfrak{p}$ and $b \in \mathfrak{p} \backslash \{0\}$, and show that the polynomial $T^2 + aT + X$ has a root f such that $bf \in XR$ but $f \notin R$].