université de BORDEAUX	ANNÉE UNIVERSITAIRE 2019 / 2020 SESSION 1 D'AUTOMNE **PARCOURS / ÉTAPE : 4TMA903U** **Code UE : 4TTN901S, 4TTN901S** **Épreuve :** Algebraic number theory **Date :** 24/01/2020 **Heure :** 13h **Durée :** 4h Documents : autorisés (notes de cours) Épreuve de Mr Brinon	**Collège Sciences et technologies**

Apart from lecture notes, documents are not allowed.
The quality of writing will be an very important assessment factor.

Exercise 1

Let p be a prime number.

(1) Assume p is odd. What is the p-adic development of $\frac{1}{2}$ (*i.e.* write $\frac{1}{2} = \sum\limits_{i=0}^{\infty} a_i p^i$ with $a_i \in \{0, 1, \ldots, p-1\}$ for all $i \in \mathbf{Z}_{\geqslant 0}$).
(2) Is $\mathbf{Q}_p^{\mathrm{ur}}$ (the maximal unramified extension of $\mathbf{Q}_p$ in $\overline{\mathbf{Q}}_p$) complete for the p-adic absolute value?
(3) For which primes p is -1 a square in $\mathbf{Q}_p$?
(4) For $i \in \mathbf{Z}_{>0}$, put $U_i = 1 + p^i \mathbf{Z}_p$. Let $n \in \mathbf{Z}_{>0}$. Show that if $y \in U_{2v_p(n)+1}$, there exists $x \in U_1$ such that $x^n = y$. Deduce that $\mathbf{Q}_p^\times / \mathbf{Q}_p^{\times n}$ is a finite group, and give its order when p does not divide n.

Exercise 2

(1) Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Show that $\mathfrak{m}/\mathfrak{m}^2$ is a κ-vector space of finite dimension, and that $d = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$ is the minimal number of generators of the ideal $\mathfrak{m}$.
(2) Let A be a noetherian integral domain which is not a field. Show that A is a Dedekind ring if and only if for every maximal ideal $\mathfrak{p}$ of A, there are no ideals $I \subset R$ such that $\mathfrak{p}^2 \subsetneq I \subsetneq \mathfrak{p}$.

Exercise 3

Let $(K, |.|)$ be a complete discretely valued field and $\overline{K}$ an algebraic closure of K. We assume that the residue field κ_K of K contains the finite field $\mathbf{F}_q$ (where $q = p^f$ with $f \in \mathbf{Z}_{>0}$). Fix a uniformizer π of K and let $P(X) = X^q + \pi X \in K[X]$. Choose a sequence $(\pi_n)_{n \in \mathbf{Z}_{\geqslant 0}}$ in $\overline{K}$ such that $\pi_0 = 0$, $\pi_1 \neq 0$ and $P(\pi_n) = \pi_{n-1}$ for all $n \in \mathbf{Z}_{>0}$. For $n \in \mathbf{Z}_{\geqslant 0}$, we put $K_n = K(\pi_n)$.
(1) Explain why the group $\mu_{q-1}(K)$ of $(q-1)$-th roots of unity is cyclic of order $q-1$.
(2) Show that K_1/K is totally ramified and that π_1 is a uniformizer of K_1.
(3) Show that K_1/K is Galois and describe its Galois group.
(4) Show that for all $n \in \mathbf{Z}_{>0}$, the extension K_{n+1}/K_n is separable, totally ramified of degree q, and that π_{n+1} is a uniformizer of K_{n+1} [hint: use induction].
(5) Show that $\mathcal{O}_{K_n} = \mathcal{O}_K[\pi_n]$ for all $n \in \mathbf{Z}_{\geqslant 0}$.
(6) Compute the different $\mathfrak{D}_{K_{n+1}/K_n}$ [do the case $n = 0$ separately], and deduce $\mathfrak{D}_{K_n/K}$ and the discriminant $\mathfrak{d}_{K_n/K}$ for all $n \in \mathbf{Z}_{\geqslant 0}$.

Exercise 4

Let $(K, |.|)$ be a complete discretely valued field of characteristic 0, with perfect residue field κ_K of characteristic p. We denote by v the normalized valuation on K and by $e_K = v(p)$ its absolute ramification index. Let $n \in \mathbf{Z}_{>0}$ be such that $\mathbf{F}_{p^n} \subset \kappa_K$ and $\alpha \in K$ such that $v(\alpha) > -\frac{p^n e_K}{p^n - 1}$. Put $P(X) = X^{p^n} - X - \alpha \in K[X]$, let $\lambda \in \overline{K}$ be a root of P and $L = K(\lambda)$. We still denote by v its extension to L.

(1) Recall why there is a unique multiplicative map $[.] \colon \mathbf{F}_{p^n} \to \mathcal{O}_K$ such that $\pi \circ [.] = \mathsf{Id}_{\mathbf{F}_{p^n}}$, where $\pi \colon \mathcal{O}_K \to \kappa_K$ is the projection.

Put $Q(X) = P(X + \lambda) \in L[X]$.

(2) Assume $v(\alpha) < 0$. Show that $v(\lambda) = \frac{v(\alpha)}{p^n}$. Deduce that $Q(X) \in \mathcal{O}_L[X]$ and compute the image $\overline{Q}(X)$ of $Q(X)$ in $\kappa_L[X]$.

(3) For $x \in \mathbf{F}_{p^n}$, compute the images of $Q([x])$ and $Q'([x])$ in κ_L. Deduce that P is split in L.

What precedes shows that L/K is Galois: put $G = \mathsf{Gal}(L/K)$.

(4) Show that if $\sigma \in G\backslash\{\mathsf{Id}_L\}$, we have $|\sigma(\lambda) - \lambda| = 1$.

(5) Assume now that $p \nmid v(\alpha)$ and $v(\alpha) < 0$.

- (a) Show that L/K is totally ramified, and give a uniformizer π_L in terms of a uniformizer π_K of K and λ [hint: use the fact that $\mathsf{gcd}(p^n, v(\alpha)) = 1$].
- (b) Show that the ramification filtration with lower numbering is given by

$$G_i = \begin{cases} G & \text{if } i \leqslant -v(\alpha) \\ \{\mathsf{Id}_L\} & \text{if } i > -v(\alpha) \end{cases}.$$

- (c) Compute the different $\mathfrak{D}_{L/K}$ and the discriminant $\mathfrak{d}_{L/K}$.

(6) Show that if $\alpha_1 \in K$ satisfies $|\alpha - \alpha_1| < 1$ and λ_1 is a root of $P_1(X) = X^{p^n} - X - \alpha_1$, then $K(\lambda) = K(\lambda_1)$.

(7) Assume now that $\alpha_1, \alpha_2 \in K$ are such that $v(\alpha_1), v(\alpha_2) > -e_K$ and $|\alpha - \alpha_1 - \alpha_2| < 1$. Show that $L = K(\lambda)$ lies in the compositum of $K(\lambda_1)K(\lambda_2)$.