université de BORDEAUX

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SESSION 2 DE PRINTEMPS
PARCOURS / ÉTAPE : 4TMA903U
Code UE : 4TTN901S, 4TTN901S
Épreuve : Algebraic number theory
Date : 20/06/2019 **Heure :** 9h **Durée :** 3h
Documents : non autorisés
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Documents are not allowed.
The quality of writing will be an important assessment factor.
Everywhere in the text, p denotes a prime number.

Exercise 1

(1) Let $(K, |.|)$ be a non archimedean valued field. Is the map $|.| : K \to \mathbf{R}_{\geqslant 0}$ continuous when $\mathbf{R}_{\geqslant 0}$ is endowed with its "usual" topology? What if $\mathbf{R}_{\geqslant 0}$ is endowed with the discrete topology?
(2) What is the cardinality of $\mathbf{Z}_p$?
(3) Show that the unique unramified extension of degree n of $\mathbf{Q}_p$ (in a fixed algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$) is the decomposition field of $X^{p^n} - X$.
(4) Show that if $p \neq 2$, then 1 is the only p-th root of unity in $\mathbf{Q}_p$.
(5) Let $x = \sum\limits_{k=0}^{\infty} 2^{k!} \in \mathbf{Q}_2$. Show that x is transcendental over $\mathbf{Q}$.

Exercise 2

Let K be a complete discretely valued field of characteristic 0, whose residue field κ_K has characteristic $p > 0$. We denote by $v_K \colon K \to \mathbf{Z} \cup \{\infty\}$ its normalized valuation.
(1) Let L/K be a totally ramified finite extension and $E(X) = X^e + a_{e-1}X^{e-1} + \cdots + a_0 \in \mathcal{O}_K[X]$ the minimal polynomial over K of a uniformizer π_L of L. Put $c(L) = v_L(\mathfrak{D}_{L/K}) - e + 1$ (where $v_L \colon L \to \mathbf{Z} \cup \{\infty\}$ is the normalized valuation and $\mathfrak{D}_{L/K}$ the different of L/K). Show that $c(L) \in \mathbf{Z}_{\geqslant 0}$ and that $c(L) = 0$ if and only if L/K is tamely ramified [hint: use the equality $\mathfrak{D}_{L/K} = E'(\pi_L)\mathcal{O}_L$].
(2) Show that if L/K is not tamely ramified, then $c(L) = \min\{ev_K(e), ev_K(a_i) - e + i\}_{1 \leqslant i < e}$.

Let $\overline{K}$ be a fixed separable closure of K and π a uniformizer of K. We denote by $U_K = 1 + \pi\mathcal{O}_K$ the group of principal units of K. Henceforth, we assume that κ_K is *finite*: let q be its order.
(3) Show that an element $u \in \mathcal{O}_K^\times$ can be written uniquely $u = [\alpha]\widetilde{u}$ where $\alpha \in \kappa_K^\times$, $[\alpha] \in \mathcal{O}_K^\times$ is the unique $(q-1)$-th root of unity lifting α and $\widetilde{u} \in U_K$.

We denote by Σ_e the set of subextensions L/K of $\overline{K}$ that are totally ramified of degree $e \in \mathbf{Z}_{>0}$.
(4) Assume that $p \nmid e$. Recall that, being tamely ramified over K, elements in Σ_e are of the form $K_\theta := K(\theta)$ where $\theta \in \overline{K}$ is a root of the polynomial $X^e - u\pi$ for some $u \in \mathcal{O}_K^\times$.
- (a) Let $\widetilde{u} \in U_K$. Show that there exists $\lambda \in U_K$ such that $\lambda^e = \widetilde{u}$. Deduce that we may restrict to elements u of the form $[\alpha]$ with $\alpha \in \kappa_K^\times$.
- (b) Let $\alpha, \alpha' \in \kappa_K^\times$ and $\theta, \theta' \in \overline{K}$ such that $\theta^e = [\alpha]\pi$ and $\theta'^e = [\alpha']\pi$. Show that $K_\theta = K_{\theta'}$ if and only if there exists $\beta \in \kappa_K^\times$ such that $\alpha' = \beta^e\alpha$ and an e-th root of unity $\zeta \in K$ such that $\theta' = [\beta]\zeta\theta$. Deduce that it is equivalent to the existence of $\gamma \in \kappa_K^\times$ such that $\theta' = [\gamma]\theta$.
- (c) Show that $\#\Sigma_e = e$.

(5) In this question, we assume that $p \mid e$: by question (1), we have $L \in \Sigma_e \Rightarrow c(L) \in \{1, \ldots, ev_K(e)\}$.
- (a) For each $j \in \{1, \ldots, e-1\}$, construct an element $L \in \Sigma_e$ such that $c(L) = j$.
- (b) Deduce that $\#\Sigma_e \geqslant e$.
- (c) Assume $\#\Sigma_e = e$. Using (2), show that $v_K(e) = 1$, then that $e = p$ is a uniformizer of K [hint: consider the extension generated by the roots of $X^e - \pi$, then that generated by a root of $X^e - u\pi$ for an appropriate root of unity $u \in \mathcal{O}_K^\times$].
- (d) Deduce $\#\Sigma_e > e$.

Exercise 3

Assume $p > 2$ and let $K/\mathbf{Q}_p$ be a totally ramified Galois extension of degree p. Denote by π a uniformizer of K and v_K its normalized valuation. Let $E(X) = X^p + a_{p-1}X^{p-1} + \cdots + a_0 \in \mathbf{Z}_p$ be the minimal polynomial of π over $\mathbf{Q}_p$. Recall that $v_K(\mathfrak{D}_{K/\mathbf{Q}_p}) = \min\{2p-1, v_K(a_i)+i-1\}_{1\leqslant i<p}$ (where $\mathfrak{D}_{K/\mathbf{Q}_p}$ denotes the different ideal of $K/\mathbf{Q}_p$).

(1) Show that $p-1 \mid v_K(\mathfrak{D}_{K/\mathbf{Q}_p})$ [hint: use the ramification filtration].

(2) Deduce that $v_K(\mathfrak{D}_{K/\mathbf{Q}_p}) = 2p-2$.

(3) Compute $\mathsf{Gal}(K/\mathbf{Q}_p)_x$ for $x \in [-1, +\infty[$.

(4) Deduce $\mathsf{Gal}(K/\mathbf{Q}_p)^y$ for $y \in [-1, +\infty[$.

(5) Assume $L/\mathbf{Q}_p$ is a totally ramified Galois extension such that $\mathsf{Gal}(L/\mathbf{Q}_p) \simeq (\mathbf{Z}/p\mathbf{Z})^2$.
  - (a) Show that $L = K_1K_2$ where $K_i/\mathbf{Q}_p$ is totally ramified Galois of degree p for $i \in \{1, 2\}$.
  - (b) Show that $\mathsf{Gal}(L/\mathbf{Q}_p)^y \hookrightarrow \mathsf{Gal}(K_1/\mathbf{Q}_p)^y \times \mathsf{Gal}(K_2/\mathbf{Q}_p)^y$ for all $y \in [-1, +\infty[$.
  - (c) Compute $\mathsf{Gal}(L/\mathbf{Q}_p)^y$ for all $y \in [-1, +\infty[$.
  - (d) Deduce $\mathsf{Gal}(L/\mathbf{Q}_p)_1/\mathsf{Gal}(L/\mathbf{Q}_p)_2$.
  - (e) Derive a contradiction and conclude that no such L exists.

Exercise 4

Let L/K be a totally ramified Galois extension of local fields of characteristic 0. Assume that its Galois group $G \simeq \{\pm1, \pm i, \pm j, \pm k\}$ is the quaternion group[1] (so that $C := \mathsf{Z}(G) \simeq \{\pm 1\}$), and that $G_4 = \{\mathsf{Id}_L\}$. Show that $G = G_0 = G_1$, and $G_2 = G_3 = C$. What is the different of L/K? Show that

$$G^y = \begin{cases} G & \text{if } y \leqslant 1 \\ C & \text{if } 1 < y \leqslant \frac{3}{2} \\ \{\mathsf{Id}_L\} & \text{if } \frac{3}{2} < y \end{cases}$$

[1] Recall that $i^2 = j^2 = k^2 = ijk = -1$.