FILTERED (φ, N)-MODULES AND SEMI-STABLE REPRESENTATIONS

by

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Abstract. — The aim of these notes is to give a short introduction on the theory of semi-stable representations and filtered (φ, N)-modules, assuming some knowledge of crystalline and de Rham theories. It corresponds to a course given at Rennes in May 2014.

Contents

1. Introduction

We fix the notations that will be used throughout the text. Let K be a complete discrete valuation field of characteristic 0, whose residue field k is perfect of characteristic $p > 0$. We choose a uniformizer π, an algebraic closure $\overline{K}$ of K and put $G_K = \mathsf{Gal}(\overline{K}/K)$. The valuation $v\colon K \to \mathbb{R}\cup\{+\infty\}$ (normalized by $v(p) = 1$) extends to a (non-discrete) valuation $v\colon \overline{K} \to \mathbb{R}\cup\{+\infty\}$: we denote by C the completion of $\overline{K}$ for this valuation. The action of G_K extends to C by continuity. We will write $\mathcal{O}_L$ (resp. $\mathfrak{m}_L$) for the ring of integers (resp. the maximal ideal) of a subfield $L \subset C$. Put $W = \mathsf{W}(k)$ and σ the Witt vectors Frobenius. It extends to $K_0 = \mathsf{Frac}(W)$. Following Bourbaki, $0 \in \mathbf{N}$.

In what follows, we assume that the reader is familiar with the period rings formalism (*cf.* [36, §1&2] or [18, §1.3&1.4]), and the period rings $\mathsf{B}_{\mathrm{cris}}$ and B_{dR} (*cf.* [35] or [18, §3]).

Let X be a proper and smooth K-variety. There exists a comparison isomorphism

$$\mathsf{B}_{\mathrm{dR}} \otimes_{\mathbf{Z}_p} \mathsf{H}^\bullet(X_{\overline{K}}, \mathbf{Z}_p) \xrightarrow{\sim} \mathsf{B}_{\mathrm{dR}} \otimes_K \mathsf{H}^\bullet_{\mathrm{dR}}(X/K)$$

(compatible with the Hodge filtrations on the de Rham cohomology[(1)] and B_{dR}) analogous to the period isomorphism between Betti and de Rham cohomology in the complex analytic case (*cf.*

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(1) That comes from the Hodge to de Rham spectral sequence $E_1^{ij} = \mathsf{H}^j(X, \Omega^i_{X_K/K}) \Rightarrow \mathsf{H}^{i+j}_{\mathrm{dR}}(X/K)$ (that degenerates at E_1, *cf.* [29, §5] and [30]).

[18, §4.1]). When X has good reduction, *i.e.* admits a proper and smooth model $\mathcal{X}$ over $\mathcal{O}_K$, it is refined by an isomorphism

$$\mathsf{B}_{\mathrm{cris}} \otimes_{\mathbf{Z}_p} \mathsf{H}^\bullet(X_{\overline{K}}, \mathbf{Z}_p) \xrightarrow{\sim} \mathsf{B}_{\mathrm{cris}} \otimes_K \mathsf{H}^\bullet_{\mathrm{cris}}(\mathcal{X}_k/W)$$

where $\mathcal{X}_k$ denotes the special fiber of $\mathcal{X}$, compatible with the Frobenius operators on the crystalline cohomology and $\mathsf{B}_{\mathrm{cris}}$, and with the Hodge filtrations after extending the scalars to K (via the Berthelot-Ogus isomorphism $K \otimes_W \mathsf{H}^\bullet_{\mathrm{cris}}(\mathcal{X}_k/W) \simeq \mathsf{H}^\bullet_{\mathrm{dR}}(X/K)$, *cf.* [13, Theorem 2.4]).

Assume now that X has semi-stable reduction, *i.e.* admits a proper and flat model $\mathcal{X}$ with semi-stable reduction (which means that $\mathcal{X}$ is regular, generically smooth, and the special fiber $Y = \mathcal{X}_k$ is a reduced divisor with normal crossings(2)). Logarithmic geometry (briefly reviewed in appendix 6) is the right context to study this case: in the log-geometric context, $\mathcal{X} \to \mathsf{Spec}(\mathcal{O}_K)$ is smooth. Here again, the de Rham comparison isomorphism can be refined (*cf.* §4). Of course, crystalline cohomology does not provide the right object on the de Rham side. It is replaced by its logarithmic analogue: one attaches to $\mathcal{X}$ its log-crystalline cohomology groups $\mathsf{H}^m_{\mathrm{log\text{-}cris}}(Y/W)$ (*cf.* §6.4) that are finitely generated W-modules, endowed with a Frobenius map $\varphi\colon \mathsf{H}^m_{\mathrm{log\text{-}cris}}(Y/W) \to \mathsf{H}^m_{\mathrm{log\text{-}cris}}(Y/W)$ (which is σ-semi-linear and an isomorphism after inverting p), and with a new structure: a monodromy operator(3) $N\colon \mathsf{H}^m_{\mathrm{log\text{-}cris}}(Y/W) \to \mathsf{H}^m_{\mathrm{log\text{-}cris}}(Y/W)$ such that $N\varphi = p\varphi N$. It is related to de Rham cohomology by the Hyodo-Kato isomorphism(4)

$$\rho_\pi\colon \mathsf{H}^i_{\mathrm{log\text{-}cris}}(Y/W) \otimes_W K \xrightarrow{\sim} \mathsf{H}^i_{\mathrm{dR}}(X/K)$$

(that depends on the choice of π). As $\mathsf{H}_{\mathrm{dR}}(X/K)$ comes equipped with its Hodge filtration, the space

$$D := \mathsf{H}^i_{\mathrm{log\text{-}cris}}(Y/W) \otimes_W K_0$$

is an object of the following category, which thus plays a central role in what follows.

Definition 1.1. — A *filtered* (φ, N)*-module* is a quadruple $(D, \varphi, N, \mathsf{Fil}^\bullet D_K)$ where D is a finite dimensional K_0-vector space, $\varphi\colon D \to D$ a σ-semi-linear automorphism, *i.e.* such that $(\forall \lambda \in K_0)$ $(\forall d \in D)$ $\varphi(\lambda d) = \sigma(\lambda)\varphi(d)$ (the Frobenius map), $N\colon D \to D$ a K_0-linear endomorphism (the monodromy operator) such that $N\varphi = p\varphi N$ and $\mathsf{Fil}^\bullet D_K$ a decreasing filtration of $D_K := K \otimes_{K_0} D$ by sub-K-vector spaces, which is separated (*i.e.* $\cap_{i\in\mathbf{Z}} \mathsf{Fil}^i D_K = \{0\}$) and exhaustive (*i.e.* $\cup_{i\in\mathbf{Z}} \mathsf{Fil}^i D_K = D_K$). A morphism of filtered (φ, N)-modules is a K_0-linear map compatible with Frobenius maps, monodromy operators and filtrations after extending the scalars to K. The category of filtered (φ, N)-modules is denoted by $\mathbf{MF}_K(\varphi, N)$. This is an additive (but not abelian) category.

The comparison isomorphism in this context requires to enlarge the period ring $\mathsf{B}_{\mathrm{cris}}$ into a $\mathsf{B}_{\mathrm{cris}}$-algebra B_{st} (*cf.* §3.1). Of course, it is perfectly possible to use other period rings, that are handier in some situations when one has to look under the hood (*cf.* [22, p.512] and [23, 0.3.4]).

As in the de Rham or crystalline case, the period ring B_{st} defines a full subcategory of the category of p-adic representations: that of *semi-stable representations*. The associated functor has values in $\mathbf{MF}_K(\varphi, N)$. A nice feature of this construction is the fundamental fact that this functor provides an *equivalence* between the category of semi-stable representations and an explicit subcategory $\mathbf{MF}^{\mathsf{ad}}_K(\varphi, N)$ of $\mathbf{MF}_K(\varphi, N)$ (theorem 3.28). This classification result has now various proofs. Section 3.6 provides an overview of one of these proofs, due to Kisin.

The relevance of the "semi-stable case" also comes from its relationship with the "general case". It is not known if a semi-stable reduction theorem(5) holds, but it turns out to be true at

(2) That is $\mathcal{X}$ is étale locally isomorphic to $\mathsf{Spec}\left(\mathcal{O}_K[T_1, \ldots, T_n]/(T_1 \cdots T_r - \pi)\right)$.
(3) Analogous to the monodromy of a family of complex analytic varieties parametrized by the unit disc, *cf.* §2
(4) Which is a generalization of Berthelot-Ogus isomorphism.
(5) *I.e.* X acquires semi-stable reduction after base change to a suitable finite extension of K.

the cohomological level. More generally, any de Rham representation is potentially semi-stable, *i.e.* becomes semi-stable when restricted to G_L for an suitable finite extension L/K (this deep result, the *p-adic monodromy theorem*, has several proofs as well, *cf.* theorem 5.3). As a result, one can slightly generalize the classification of semi-stable representations mentioned before, to get a complete classification of de Rham representations (which is not possible using the functor D_{dR} alone). In particular, this allows to recover the étale cohomology of a proper and smooth K-variety in terms of a hidden structure of its de Rham cohomology, *cf.* §5.3.

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2. Analogies with the ℓ-adic/complex analytic case

We assume in this section that K is a finite extension of $\mathbf{Q}_p$. Recall the exact sequences

$$\{1\} \to I_K \to G_K \to \mathsf{Gal}(\bar{k}/k) \to \{1\}$$

$$\{1\} \to P_K \to I_K \xrightarrow{(t_\ell)_\ell} \prod_{\ell \neq p} \mathbf{Z}_\ell(1) \to \{0\}$$

where $I_K = \mathsf{Gal}(\overline{K}/K^{\mathrm{nr}})$ is the inertia subgroup (one has $K^{\mathrm{nr}} = \cup_{p \nmid n} K(\mu_n)$) and $P_K = \mathsf{Gal}\left(\overline{K}/\cup_{p\nmid n} K^{\mathrm{nr}}(p^{1/n})\right)$ the wild inertia subgroup (*i.e.* the pro-p-Sylow subgroup of I_K). Here $t_\ell \colon I_K \to \mathbf{Z}_\ell(1) = \varprojlim_n \mu_{\ell^n}(\overline{K})$ is the cocycle defined by $\frac{g(\pi_{\ell,n})}{\pi_{\ell,n}} = t_\ell(g) \mod \ell^n$ for all $g \in I_K$ (where $(\pi_{\ell,n})_{n\geq 0}$ is a sequence in $\overline{K}$ such that $\pi_{\ell,0} = \pi$ and $\pi_{\ell,n+1}^\ell = \pi_{\ell,n}$ for all $n \in \mathbf{N}$).

***Theorem 2.1* (Grothendieck's monodromy theorem).** — *Let $\ell \neq p$ be a prime integer, and V an ℓ-adic representation of G_K. Then V is* quasi-unipotent, *i.e. there exists a unique nilpotent endomorphism $N \colon V(1) \to V$ and an open subgroup $I \subseteq I_K$ such that*

$$(\forall g \in I)(\forall v \in V)\ g(v) = \exp(t_\ell(g)N)(v).$$

The étale cohomology groups $\mathsf{H}^\bullet(X_{\overline{K}}, \mathbf{Q}_\ell)$ of a proper and smooth K-variety X provide such ℓ-adic representations of G_K. Grothendieck's theorem implies that $\mathsf{H}^\bullet(X_{\overline{K}}, \mathbf{Q}_\ell)$ is quasi-unipotent. When X has good reduction, then $\mathsf{H}^\bullet(X_{\overline{K}}, \mathbf{Q}_\ell)$ is unramified (*i.e.* the action of I_K is trivial). As mentioned above, the p-adic étale cohomology of X is potentially semi-stable (crystalline in the good reduction case).

Similarly there is an analogy with the complex analytic case. Let Δ be the open unit disc, and $\Delta^* = \Delta \setminus \{0\}$. Recall that from the point of view of the étale topology, Δ (resp. Δ^*) is the analogue of the spectrum of a strictly henselian DVR (resp. its generic fiber). Let $f \colon X \to \Delta$ be a proper morphism of complex analytic spaces, which is smooth above Δ^*. Let $t \in \Delta^*$: as f induces a fiber bundle of Δ^*, the positive generator of $\pi_1(\Delta^*, t)$ provides an automorphism

$$T \colon \mathsf{H}^\bullet(X_t, \mathbf{Z}) \to \mathsf{H}^\bullet(X_t, \mathbf{Z})$$

(the *monodromy operator*) which is of course trivial when f is smooth. In general it is quasi-unipotent: there exists $a \in \mathbf{Z}_{>0}$ such that $T^a - 1$ is nilpotent (local monodromy theorem). When the special fiber X_0 is a reduced normal crossing divisor, then it is even unipotent, *i.e.* $N = T - 1$ is nilpotent (*cf.* [44, §2] for more details).

Hence there are the following analogies:

	complex analytic	ℓ-adic	p-adic
good reduction	trivial	unramified	crystalline
semi-stable	unipotent	unipotent	semi-stable
general case	quasi-unipotent	quasi-unipotent	potentially semi-stable
	local monodromy	ℓ-adic monodromy theorem	p-adic monodromy theorem

Remark 2.2. — A notable difference is that in the p-adic case, unlike the complex analytic case, the monodromy operator lives on the differential side (log-crystalline cohomology), and not the topological side (étale cohomology). This had been predicted by Jannsen.

The preceding analogies are reinforced by partial converses of comparison theorems in the case of abelian varieties. More precisely, if A is an abelian variety over K and $\mathsf{V}_\ell(A)$ its ℓ-adic Tate module, we have the following results:

Theorem 2.3 **(Néron-Ogg-Shafarevich criterion, *cf.* Serre-Tate** [57, Theorem 1]**, Grothendieck** [41, Exposé IX, Proposition 3.5]**)**
A has good (resp. semi-stable) reduction if and only if for some (all) $\ell \neq p$, the ℓ-adic representation $\mathsf{V}_\ell(A)$ is unramified (resp. unipotent of level 2).

The p-adic analogue is:

Theorem 2.4 **(Coleman-Iovita, *cf.*** [21, Theorem 4.7]**, Breuil** [16, Corollaire 1.6]**)**
A has good (resp. semi-stable) reduction if and only if the p-adic representation $\mathsf{V}_p(A)$ is crystalline (resp. semi-stable).

Remark 2.5. — There are ℓ-adic and p-adic anabelian good reduction criteria for proper, smooth and geometrically irreducible K-curves with semi-stable reduction (*cf.* [5]).

3. The ring B_{st} and semi-stable representations

We use the standard notations: $\mathcal{R} = \varprojlim_{x\mapsto x^p} \mathcal{O}_C$, and $\varepsilon = (\varepsilon^{(n)})_{n\geq 0}$ (resp. $\widetilde{p} = (p^{(n)})_{n\geq 0}$ and $\widetilde{\pi} = (\pi^{(n)})_{n\geq 0}$) is an element of $\mathcal{R}$ such that $\varepsilon^{(1)}$ is a primitive p-th root of the unity (resp. $p^{(0)} = p$ and $\pi^{(0)} = \pi$). These elements are denoted $\underline{\varepsilon}$, $p^\flat$ and $\pi^\flat$ respectively in [18]. We denote by χ the p-adic cyclotomic character. Most of what follows is contained in [35] & [36] (*cf.* also the exhaustive book [40]).

3.1. Construction of B_{st}. — An easy example of curve with semi-stable reduction is given by the Tate curve(6). There exist elements $b_2(q), b_3(q) \in q\,\mathbf{Z}_p[\![q]\!]$ (hence rigid analytic functions converging on the open unit disc), such that for any complete extension F of $\mathbf{Q}_p$, for any $q \in F$ such that $0 < |q| < 1$, the cubic curve $E_q \subset \mathbf{P}^2_F$ whose equation is $y^2 + xy = x^3 - b_2(q)x - b_3(q)$ is non-singular, and $E_q(F) \simeq F^\times/q^{\mathbf{Z}}$ (*cf.* [60, Theorem 1]). Let's consider the case $q \in K$. If $n \in \mathbf{N}_{>0}$, and $x \in E_q(C) \simeq C^\times/q^{\mathbf{Z}}$ is killed by p^n, let $\widetilde{x} \in C^\times$ be a lift of x : there exists $f(\widetilde{x}) \in \mathbf{Z}$ such that $\widetilde{x}^{p^n} = q^{f(\widetilde{x})}$. The image of $f(\widetilde{x})$ in $\mathbf{Z}/p^n\mathbf{Z}$ only depends on x. One gets a map $E_q[p^n] \to \mathbf{Z}/p^n\mathbf{Z}$, which is a surjective morphism of groups, whose kernel is $\mu_{p^n}(\overline{K})$. Thus, there is an exact sequence $0 \to \mu_{p^n}(\overline{K}) \to E_q[p^n] \to \mathbf{Z}/p^n\mathbf{Z} \to 0$. Passing to the limit, it provides an exact sequence $0 \to \mathbf{Z}_p(1) \to \mathsf{T}_p(E_q) \to \mathbf{Z}_p \to 0$, hence an exact sequence of p-adic representations

$$(*)\qquad 0 \to \mathbf{Q}_p(1) \to \mathsf{V}_p(E_q) \to \mathbf{Q}_p \to 0.$$

This extension corresponds to a class $a \in \mathsf{H}^1(G_K, \mathbf{Q}_p(1))$. Explicitly, let $\widetilde{q} = (q^{(n)})_{n\in\mathbf{N}} \in \mathcal{R}$ be an element such that $q^{(0)} = q$. The action of G_K on $\widetilde{q}$ is given by $g(\widetilde{q}) = \varepsilon^{c_{\widetilde{q}}(g)}\widetilde{q}$ for $g \in G_K$, where $c_{\widetilde{q}}\colon G_K \to \mathbf{Z}_p(1)$ is a continuous cocycle, whose class is a. By Kummer's theory, one has $K^\times/K^{\times p^n} \xrightarrow{\sim} \mathsf{H}^1(G_K, \mu_{p^n}(\overline{K}))$, hence $\delta\colon \varprojlim_n K^\times/K^{\times p^n} \xrightarrow{\sim} \mathsf{H}^1(G_K, \mathbf{Z}_p(1))$ (because $\{\mathsf{H}^0(G_K, \mu_{p^n}(\overline{K}))\}_{n\in\mathbf{N}}$ has the Mittag-Leffler property, *cf.* [59, Proposition 2.2]). The class of $c_{\widetilde{q}}$ in $\mathsf{H}^1(G_K, \mathbf{Z}_p(1))$ is nothing but $\delta(q)$. The valuation induces a morphism of groups $K^\times/K^{\times p^n} \to \mathbf{Z}/p^n\mathbf{Z}$, hence a morphism $\mathsf{H}^1(G_K, \mathbf{Z}_p(1)) \to \mathbf{Z}_p$. As $v(q) > 0$, the class of $c_{\widetilde{q}}$ has infinite order

(6) That provides the uniformization of the elliptic curves with non-integral j-invariant over $\mathbf{Q}_p$.

in $\mathsf{H}^1(G_K, \mathbf{Z}_p(1))$, in particular, its image $a \in \mathsf{H}^1(G_K, \mathbf{Q}_p(1))$ is not trivial, and the sequence $(*)$ does not split.

In the basis (e, f) (where $e =$ "$\log(\varepsilon)$" and $f =$ "$\log(\widetilde{q})$") of $\mathsf{V}_p(E_q)$, the matrix of the action of G_K is

$$\begin{pmatrix} \chi & c_{\widetilde{q}} \\ 0 & 1 \end{pmatrix}.$$

A basis of $\mathsf{D}_{\mathrm{dR}}(\mathsf{V}_p(E_q))$ is thus $(t^{-1} \otimes e, 1 \otimes f - t^{-1} b \otimes e)$ where $b \in \mathsf{B}_{\mathrm{dR}}$ is such that $c_{\widetilde{q}}(g) = (g-1)(b)$ for all $g \in G_K$, *i.e.* behaves like "$\log(\widetilde{q})$": the construction of B_{st} requires that of a logarithm map

$$\log[.] \colon \mathsf{Frac}(\mathcal{R})^\times \to \mathsf{B}_{\mathrm{dR}}^+ .$$

Recall that $\mathcal{R}$ is a valuation ring: denote its maximal ideal by $\mathfrak{m}_{\mathcal{R}}$. First, if $x \in 1 + \mathfrak{m}_C$ (resp. $x \in 1 + \mathfrak{m}_{\mathcal{R}}$), one has $x^{p^r} \in 1 + p\mathcal{O}_C$ (resp. $x^{(0)} \in 1 + \mathfrak{m}_C$, so $\big(x^{(0)}\big)^{p^r} \in 1 + p\mathcal{O}_C$ *i.e.* $\big[x^{p^r}\big] - 1 \in \mathsf{Ker}(\theta) + p\,\mathsf{W}(\mathcal{R})$) for large r. As $1 + p\mathcal{O}_C$ (resp. $\mathsf{Ker}(\theta) + p\,\mathsf{W}(\mathcal{R})$) has divided powers in $\mathcal{O}_C$ (resp. $\mathsf{A}_{\mathrm{cris}}$), the series

$$\log x = \frac{\log\big(x^{p^r}\big)}{p^r} = \frac{1}{p^r} \sum_{n=1}^{+\infty} (-1)^{n-1}(n-1)!\big(x^{p^r} - 1\big)^{[n]}$$

$$\Big(\text{resp.} \quad \log[x] = \frac{\log\big[x^{p^r}\big]}{p^r} = \frac{1}{p^r} \sum_{n=1}^{+\infty} (-1)^{n-1}(n-1)!\big(\big[x^{p^r}\big] - 1\big)^{[n]}\Big)$$

converges in C (resp. $\mathsf{B}_{\mathrm{cris}}^+ = \mathsf{A}_{\mathrm{cris}}[p^{-1}]$), defining a group homomorphism

$$\log \colon 1 + \mathfrak{m}_C \to C \quad (\text{resp.} \log[.] \colon 1 + \mathfrak{m}_{\mathcal{R}} \to \mathsf{B}_{\mathrm{cris}}^+).$$

It is injective because $1 + \mathfrak{m}_C$ (resp. $1 + \mathfrak{m}_{\mathcal{R}}$) does not contain p^r-th roots of unity. As $\mathcal{O}_C^\times \simeq \bar{k}^\times \times (1 + \mathfrak{m}_C)$ (resp. $\mathcal{R}^\times = \bar{k}^\times \times (1 + \mathfrak{m}_{\mathcal{R}})$), one extends it to $\mathcal{O}_C^\times$ (resp. $\mathcal{R}^\times$) by putting $\log[\alpha]x = \log x$ for $\alpha \in \bar{k}^\times$ and $x \in 1 + \mathfrak{m}_C$ (resp. $\log[\alpha x] = \log[x]$ for $\alpha \in \bar{k}^\times$ and $x \in 1 + \mathfrak{m}_{\mathcal{R}}$). Now, if $x \in C^\times$ (resp. $x \in \mathsf{Frac}(\mathcal{R})^\times$) has valuation $\frac{a}{b}$ (with $a \in \mathbf{Z}$ and $b \in \mathbf{Z}_{>0}$), then $y = \frac{x^b}{p^a} \in \mathcal{O}_C^\times$ (resp. $y = \frac{x^b}{\widetilde{p}^a} \in \mathcal{R}^\times$), and we put

$$\log x = \frac{\log y}{b} \quad \Big(\text{resp.} \quad \log[x] = \frac{\log[y] + a\log[\widetilde{p}]}{b}\Big)$$

where *we choose*

$$\log(p) = 0 \quad \text{and} \quad \log[\widetilde{p}] = \log\left(\frac{[\widetilde{p}]}{p}\right) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} \left(\frac{[\widetilde{p}]}{p} - 1\right)^n = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{np^n} \xi^n \in \mathsf{Fil}^1 \mathsf{B}_{\mathrm{dR}}^+ .$$

Of course, the maps log and $\log[.]$ are G_K-equivariant group homomorphisms. Also, if $x \in \mathsf{Frac}(\mathcal{R})^\times$, one has $\theta(\log[x]) = \log(x^{(0)})$. Indeed, if y, a and b are as above, one has $\theta(\log[x]) = \frac{1}{b}\theta(\log[y] + a\log[\widetilde{p}]) = \frac{1}{b}\log\theta([y]) = \frac{\log y^{(0)}}{b} = \log x^{(0)}$.

Remark 3.1. — If moreover $x^{(0)} \in \overline{K}^\times$, one has $\frac{[x]}{x^{(0)}} - 1 \in \mathsf{Fil}^1 \mathsf{B}_{\mathrm{dR}}$, and

$$\log[x] - \log(x^{(0)}) = \sum_{n=1}^{+\infty} \tfrac{(-1)^{n-1}}{n} \big(\tfrac{[x]}{x^{(0)}} - 1\big)^n \in \mathsf{Fil}^1 \mathsf{B}_{\mathrm{dR}} .$$

Coming back to $\mathsf{V}_p(E_q)$, we can now write $\mathsf{D}_{\mathrm{dR}}(\mathsf{V}_p(E_q)) = Kx \oplus Ky$ with $x = t^{-1} \otimes e$ and $y = 1 \otimes f - t^{-1}\log[\widetilde{q}] \otimes e$. As $\theta(\log[\widetilde{q}] - \log q) = 0$, one has

$$\mathsf{Fil}^i \mathsf{D}_{\mathrm{dR}}(\mathsf{V}_p(E_q)) = \begin{cases} \mathsf{D}_{\mathrm{dR}}(\mathsf{V}_p(E_q)) & \text{if } i < 0; \\ K(y + \log(q)x) & \text{if } i = 0; \\ 0 & \text{if } i > 0. \end{cases}$$

Definition 3.2. — We put $\mathsf{B}_{\mathrm{st}} = \mathsf{B}_{\mathrm{cris}}\left[\log[\widetilde{p}]\right] \subset \mathsf{B}_{\mathrm{dR}}$.

Put $U := \{\log[x],\ x \in 1+\mathfrak{m}_{\mathcal{R}}\}$. As the image of $\log[.]$ is $U + \mathbf{Q}\log[\widetilde{p}] \subset \mathsf{B}_{\mathrm{cris}}^{+} + \mathbf{Q}\log[\widetilde{p}]$, the periods of Tate curves lie in B_{st}, so a $\mathsf{B}_{\mathrm{cris}}$-algebra which is a period ring for the semi-stable case should contain B_{st}. It turns out that B_{st} is such a ring, *cf.* §4.

Remark 3.3. — (a) As $p = \pi^{e_K} u$ with $u \in \mathcal{O}_K^\times$, one has $\log[\widetilde{\pi}] \in \frac{\log[\widetilde{p}]}{e_K} + U$ (*cf.* above), one has $\mathsf{B}_{\mathrm{st}} = \mathsf{B}_{\mathrm{cris}}\left[\log[\widetilde{\pi}]\right]$ as well.
(b) The construction of the map $\log[.]$, *i.e.* that of $\log[\widetilde{p}]$ depends on the choice of $\log(p)$ (here we chose $\log(p) = 0$), that is of an extension of the p-adic logarithm to $C^\times$. Thus the ring B_{st}, *seen as a subring of* B_{dR}, depends on this choice.
(c) More generally, any extension $0 \to \mathbf{Q}_p(1) \to V \to \mathbf{Q}_p \to 0$ is semi-stable (*i.e.* B_{st}-admissible, *cf.* definition 3.13). It is crystalline precisely when its class in $\mathsf{H}^1(G_K, \mathbf{Q}_p(1)) \simeq \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \varprojlim_n K^\times / K^{\times p^n}$ belongs to the image of $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \varprojlim_n \mathcal{O}_K^\times / \mathcal{O}_K^{\times p^n}$ (because $U \subset \mathsf{B}_{\mathrm{cris}}$).
(d) One can show that $U = \mathsf{B}_{\mathrm{cris}}^{\varphi=1} \cap \mathsf{Fil}^0 \mathsf{B}_{\mathrm{dR}}$ (*cf.* [25, Proposition 1.3]).

3.2. Properties of B_{st}. — In what follows, we put $u_\pi = \log[\widetilde{\pi}]$ (so that $\mathsf{B}_{\mathrm{st}} = \mathsf{B}_{\mathrm{cris}}[u_\pi]$), and $c = c_{\widetilde{\pi}} \colon G_K \to \mathbf{Z}_p$ for short. For $g \in G_K$, one has $g(\widetilde{\pi}) = \varepsilon^{c(g)}\widetilde{\pi}$, so $g([\widetilde{\pi}]) = [\varepsilon]^{c(g)}[\widetilde{\pi}]$: taking logarithms, we have

$$g(u_\pi) = u_\pi + c(g)t.$$

The natural map $K \otimes_{K_0} \mathsf{B}_{\mathrm{cris}} \to \mathsf{B}_{\mathrm{dR}}$ is injective (*cf.* [17, Proposition 2.47]). It extends into a map $K \otimes_{K_0} \mathsf{B}_{\mathrm{st}} \to \mathsf{B}_{\mathrm{dR}}$.

Proposition 3.4 (cf. [35, Théorème 4.2.4]**).** — *The natural map*

$$\begin{aligned} \iota_\pi \colon (K \otimes_{K_0} \mathsf{B}_{\mathrm{cris}})[X] &\to \mathsf{B}_{\mathrm{dR}} \\ X &\mapsto u_\pi \end{aligned}$$

is injective. In particular, $\mathsf{B}_{\mathrm{st}} \simeq \mathsf{B}_{\mathrm{cris}}[X]$*, and* $K \otimes_{K_0} \mathsf{B}_{\mathrm{st}} \to \mathsf{B}_{\mathrm{dR}}$ *is injective.*

Corollary 3.5 (cf. [36, Proposition 5.1.2]**).** — $\mathsf{B}_{\mathrm{st}}^{G_K} = (\mathsf{Frac}(\mathsf{B}_{\mathrm{st}}))^{G_K} = K_0$.

Remark 3.6. — One has $\bar{k} \hookrightarrow \mathcal{R}$, so $\mathsf{W}(\bar{k}) \hookrightarrow \mathsf{W}(\mathcal{R}) \subset \mathsf{A}_{\mathrm{cris}}$, hence $\widehat{K_0^{\mathrm{nr}}} = \mathsf{W}(\bar{k})[p^{-1}] \hookrightarrow \mathsf{B}_{\mathrm{cris}}^{+}$. The preceding proposition implies that the map $\overline{K} \otimes_{K_0^{\mathrm{nr}}} \mathsf{B}_{\mathrm{st}} \to \mathsf{B}_{\mathrm{dR}}$ induced by ι_π is injective.

We endow $K \otimes_{K_0} \mathsf{B}_{\mathrm{st}}$ with the filtration induced by that of B_{dR}. Also, we can extend the Frobenius map on $\mathsf{B}_{\mathrm{cris}}$ to a map

$$\varphi\colon \mathsf{B}_{\mathrm{st}} \to \mathsf{B}_{\mathrm{st}}$$

by putting $\varphi(u_\pi) = pu_\pi$ (since $u_\pi = \log[\widetilde{\pi}]$ and $\varphi([\widetilde{\pi}]) = [\widetilde{\pi}]^p$). This is licit because $\mathsf{B}_{\mathrm{st}} = \mathsf{B}_{\mathrm{cris}}[u_\pi] \simeq \mathsf{B}_{\mathrm{cris}}[X]$. Also, we endow B_{st} with the *monodromy operator*

$$N\colon \mathsf{B}_{\mathrm{st}} \to \mathsf{B}_{\mathrm{st}}$$

which is the unique $\mathsf{B}_{\mathrm{cris}}$-derivation of $\mathsf{B}_{\mathrm{st}} = \mathsf{B}_{\mathrm{cris}}[u_\pi]$ such that $N(u_\pi) = -1$. Of course, φ and N commute with the action of G_K. Also, one has $\mathsf{B}_{\mathrm{st}}^{N=0} = \mathsf{B}_{\mathrm{cris}}$, so the fundamental exact sequence (*cf.* [35, Théorème 5.3.7]) can be rewritten as follows:

Proposition 3.7. — *The sequence*

$$0 \to \mathbf{Q}_p \to \mathsf{B}_{\mathrm{st}}^{\substack{\varphi=1 \\ N=0}} \to \mathsf{B}_{\mathrm{dR}} / \mathsf{B}_{\mathrm{dR}}^{+} \to 0$$

is exact.

For $i \in \mathbf{Z}_{\geq 0}$, we have $N(\varphi(u_\pi^i)) = N((pu_\pi)^i) = ip^i u_\pi^{i-1} = p(i(pu_\pi)^{i-1}) = p\varphi(iu_\pi^{i-1}) = p\varphi(N(u_\pi^i))$. By $\mathsf{B}_{\mathrm{cris}}$-(semi-)linearity, we get:

Proposition 3.8 (cf. [35, 3.2.2]**).** — $N\varphi = p\varphi N$ *in* B_{st}.

Remark 3.9. — As a $\mathsf{B}_{\mathrm{cris}}$-algebra, with G_K-action and Frobenius map, the ring B_{st} does not depend on the choice of $\log(p)$, whereas the monodromy operator and the embedding $K\otimes_{K_0}\mathsf{B}_{\mathrm{cris}}\to\mathsf{B}_{\mathrm{dR}}$ do, *cf.* remark 3.3.

3.3. The tensor structure of $\mathbf{MF}_K(\varphi,N)$. — The category $\mathbf{MF}_K(\varphi,N)$ has a tensor product and internal Hom.

• Let $(D',\varphi',N',\mathsf{Fil}^\bullet D'_K)$ and $(D'',\varphi'',N'',\mathsf{Fil}^\bullet D''_K)$ be filtered (φ,N)-modules. The tensor product $D'\otimes D''$ is the K_0-vector space $D'\otimes_{K_0}D''$ endowed with the Frobenius map $\varphi'\otimes\varphi''$, monodromy operator $N'\otimes\mathsf{Id}_{D''}+\mathsf{Id}_{D'}\otimes N''$ and filtration $\mathsf{Fil}^r(D'\otimes_{K_0}D'')_K=\sum_{i\in\mathbf{Z}}\mathsf{Im}\left(\mathsf{Fil}^i D'_K\otimes_K\mathsf{Fil}^{r-i}D''_K\to(D'\otimes_{K_0}D'')_K\right)$ for $r\in\mathbf{Z}$.

• There is a unit object $\mathbf{1}:=(K_0,\sigma,N=0,\mathsf{Fil}^\bullet K)$ where $\mathsf{Fil}^i K=\begin{cases}K & \text{if } i\leq 0;\\ 0 & \text{if } i>0.\end{cases}$

• The internal Hom, denoted by $\underline{\mathsf{Hom}}(D',D'')$, is the K_0-vector space $\mathsf{Hom}_{K_0}(D',D'')$ equipped with the Frobenius map (resp. monodromy operator) defined by $\varphi(f)=\varphi''\circ f\circ\varphi'^{-1}$ (resp. $N(f)=N''\circ f-f\circ N'$) for $f\in\mathsf{Hom}_{K_0}(D',D'')$ and filtration $\mathsf{Fil}^r\,\underline{\mathsf{Hom}}(D',D'')_K=\left\{f\colon D'_K\to D''_K,\ (\forall i\in\mathbf{Z})\ f(\mathsf{Fil}^i D'_K)\subset\mathsf{Fil}^{r+i}D''_K\right\}$ for $r\in\mathbf{Z}$. In particular, every $D\in\mathbf{MF}_K(\varphi,N)$ has a dual $D^\vee=\underline{\mathsf{Hom}}(D,\mathbf{1})$.

Remark 3.10. — The category $\mathbf{MF}_K(\varphi,N)$ is not abelian, yet it has a notion of exact sequence: $0\to D'\to D\to D''\to 0$ is exact when it is in the category of K_0-vector spaces endowed with a Frobenius map and a monodromy operator, and when the filtrations on D'_K and D''_K are those induced by that D_K.

3.4. Semi-stable representations. — As usual, we denote by $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ the category of p-adic representations. There are full sub-categories

$$\mathrm{Rep}_{\mathrm{cris}}(G_K)\subset\mathrm{Rep}_{\mathrm{dR}}(G_K)\subset\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$$

(*cf.* [36]). If $V\in\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, we put:

$$\mathsf{D}_{\mathrm{st}}(V)=(\mathsf{B}_{\mathrm{st}}\otimes_{\mathbf{Q}_p}V)^{G_K}.$$

As $\mathsf{B}_{\mathrm{st}}^{G_K}=K_0$, this is a K_0-vector space, endowed with a σ-semi-linear Frobenius map $\varphi\colon\mathsf{D}_{\mathrm{st}}(V)\to\mathsf{D}_{\mathrm{st}}(V)$ and a K_0-linear monodromy operator $N\colon\mathsf{D}_{\mathrm{st}}(V)\to\mathsf{D}_{\mathrm{st}}(V)$ (induced by $\varphi\otimes\mathsf{Id}_V$ and $N\otimes\mathsf{Id}_V$ respectively, since they are G_K-equivariant). The relation $N\varphi=p\varphi N$ on $\mathsf{D}_{\mathrm{st}}(V)$ follows from proposition 3.8. Also, the injective map $K\otimes_{K_0}\mathsf{B}_{\mathrm{st}}\to\mathsf{B}_{\mathrm{dR}}$ provides an injective map

$$K\otimes_{K_0}\mathsf{D}_{\mathrm{st}}(V)\to\mathsf{D}_{\mathrm{dR}}(V).$$

We endow $\mathsf{D}_{\mathrm{st}}(V)_K:=K\otimes_{K_0}\mathsf{D}_{\mathrm{st}}(V)$ with the filtration $\mathsf{Fil}^\bullet\mathsf{D}_{\mathrm{st}}(V)$ induced by that of $\mathsf{D}_{\mathrm{dR}}(V)$.

Remark 3.11. — Of course, the monodromy operator and the embedding $\mathsf{D}_{\mathrm{st}}(V)_K\to\mathsf{D}_{\mathrm{dR}}(V)$ depend on the choice of π and log (*cf.* remark 3.9). One can explicitly describe how they vary when one changes π and log (*cf.* [36, §5.2]).

By B_{st}-linearity, the inclusion $\mathsf{D}_{\mathrm{st}}(V)\subset\mathsf{B}_{\mathrm{st}}\otimes_{\mathbf{Q}_p}V$ induces a B_{st}-linear and G_K-equivariant map

$$\alpha_{\mathrm{st}}(V)\colon\ \mathsf{B}_{\mathrm{st}}\otimes_{K_0}\mathsf{D}_{\mathrm{st}}(V)\to\mathsf{B}_{\mathrm{st}}\otimes_{\mathbf{Q}_p}V.$$

Proposition 3.12. — *The map $\alpha_{\mathrm{st}}(V)$ is injective and $\mathsf{D}_{\mathrm{st}}(V)\in\mathbf{MF}_K(\varphi,N)$.*

Proof. — As $\mathsf{Frac}(\mathsf{B}_{\mathrm{st}})$ is a field with invariants K_0 (*cf.* corollary 3.5), the natural map

$$a_{\mathrm{st}}(V)\colon\ \mathsf{Frac}(\mathsf{B}_{\mathrm{st}})\otimes_{K_0}(\mathsf{Frac}(\mathsf{B}_{\mathrm{st}})\otimes_{\mathbf{Q}_p}V)^{G_K}\to\mathsf{Frac}(\mathsf{B}_{\mathrm{st}})\otimes_{\mathbf{Q}_p}V$$

is injective by the standard argument (*cf.* [36, Proposition 1.6.1]). The injectivity of $\alpha_{\mathrm{st}}(V)$ follows from the following diagram

$$\begin{tikzcd}
\mathsf{B}_{\mathrm{st}} \otimes_{K_0} \mathsf{D}_{\mathrm{st}}(V) \arrow[r, "\alpha_{\mathrm{st}}(V)"] \arrow[d] & \mathsf{B}_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V \arrow[d] \\
\mathsf{Frac}(\mathsf{B}_{\mathrm{st}}) \otimes_{K_0} \mathsf{D}_{\mathrm{st}}(V) \arrow[r, "1\otimes\alpha_{\mathrm{st}}(V)"] \arrow[d] & \mathsf{Frac}(\mathsf{B}_{\mathrm{st}}) \otimes_{\mathbf{Q}_p} V \arrow[d] \\
\mathsf{Frac}(\mathsf{B}_{\mathrm{st}}) \otimes_{K_0} (\mathsf{Frac}(\mathsf{B}_{\mathrm{st}}) \otimes_{\mathbf{Q}_p} V)^{G_K} \arrow[r, "\alpha_{\mathrm{st}}(V)"] & \mathsf{Frac}(\mathsf{B}_{\mathrm{st}}) \otimes_{\mathbf{Q}_p} V
\end{tikzcd}$$

and the injectivity of $\mathsf{B}_{\mathrm{st}} \otimes_{K_0} \mathsf{D}_{\mathrm{st}}(V) \to \mathsf{Frac}(\mathsf{B}_{\mathrm{st}}) \otimes_{K_0} (\mathsf{Frac}(\mathsf{B}_{\mathrm{st}}) \otimes_{\mathbf{Q}_p} V)^{G_K}$ (deduced from the inclusions $\mathsf{B}_{\mathrm{st}} \subset \mathsf{Frac}(\mathsf{B}_{\mathrm{st}})$ and $\mathsf{D}_{\mathrm{st}}(V) \subset (\mathsf{Frac}(\mathsf{B}_{\mathrm{st}}) \otimes_{\mathbf{Q}_p} V)^{G_K}$). This implies that $\dim_{K_0}(\mathsf{D}_{\mathrm{st}}(V)) < +\infty$, so that $\mathsf{D}_{\mathrm{st}}(V) \in \mathbf{MF}_K(\varphi, N)$. □

Furthermore, the map $\alpha_{\mathrm{st}}(V)$ is compatible to Frobenius maps ($\varphi \otimes \varphi$ on the LHS, and $\varphi \otimes \mathsf{Id}_V$ on the RHS), monodromy operators ($N \otimes \mathsf{Id}_{\mathsf{D}_{\mathrm{st}}(V)} + \mathsf{Id}_{\mathsf{B}_{\mathrm{st}}} \otimes N$ on the LHS, and $N \otimes \mathsf{Id}_V$ on the RHS) and filtrations after extending the scalars to K (where $\mathsf{Fil}^r(K \otimes_{K_0} \mathsf{B}_{\mathrm{st}} \otimes_{K_0} \mathsf{D}_{\mathrm{st}}(V)) = \sum_{i\in\mathbf{Z}} \mathsf{Im}\big(\mathsf{Fil}^i(K \otimes_{K_0} \mathsf{B}_{\mathrm{st}}) \otimes_K \mathsf{Fil}^{r-i}(\mathsf{D}_{\mathrm{st}}(V)_K) \to K \otimes_{K_0} \mathsf{B}_{\mathrm{st}} \otimes_{K_0} \mathsf{D}_{\mathrm{st}}(V)\big)$ and $\mathsf{Fil}^r(K \otimes_{K_0} \mathsf{B}_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V) = \mathsf{Fil}^r(K \otimes_{K_0} \mathsf{B}_{\mathrm{st}}) \otimes_{\mathbf{Q}_p} V)$.

Definition 3.13. — $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ is called *semi-stable* if $\alpha_{\mathrm{st}}(V)$ is an isomorphism. The full subcategory of $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ of semi-stable representations is denoted by $\mathrm{Rep}_{\mathrm{st}}(G_K)$.

The good properties of $\mathrm{Rep}_{\mathrm{st}}(G_K)$ follow from:

***Proposition 3.14* (*cf.* [18, §1.4.1] & [36, Proposition 5.1.2]).** — *B_{st} is G_K-regular.*

Knowing proposition 3.12, this follows from:

Lemma 3.15. — *Let $b \in \mathsf{B}_{\mathrm{st}} \setminus \{0\}$ such that the line $\mathbf{Q}_p b \subset \mathsf{B}_{\mathrm{st}}$ is stable by G_K. Then $b \in t^{\mathbf{Z}} \widehat{K_0^{\mathrm{nr}}}^{\times} \subset \mathsf{B}_{\mathrm{cris}}^{\times}$.*

Proof. — We only use the stability under the inertia $I_K = \mathsf{Gal}(\overline{K}/K^{\mathrm{nr}})$: we may assume $G_K = I_K$, *i.e.* $k = \bar{k}$ hence $K_0 = \widehat{K_0^{\mathrm{nr}}}$. Multiplying b by the appropriate power of t, we may also assume $b \in \mathsf{B}_{\mathrm{dR}}^{+} \setminus t\,\mathsf{B}_{\mathrm{dR}}^{+}$: the map θ induces a G_K-equivariant morphism $\mathbf{Q}_p b \xrightarrow{\sim} K_0\theta(b) \subset C$. This implies that the representation $\mathbf{Q}_p b$ is C-admissible. By [55, Corollary 1] (*cf.* also [18, Theorem 2.2.1]), the action of G_K on the line $\mathbf{Q}_p b$ is finite: there exists a finite extension L/K_0 such that $b \in \mathsf{B}_{\mathrm{st}}^{G_L} = L_0 = K_0$ (recall $K_0 = \widehat{K_0^{\mathrm{nr}}}$ by assumption). □

Proposition 3.14 and lemma 3.15 imply (*cf.* [36, Proposition 1.5.2]):

Corollary 3.16. — (i) *$V \in \mathrm{Rep}_{\mathrm{st}}(G_K) \Leftrightarrow \dim_{K_0}(\mathsf{D}_{\mathrm{st}}(V)) = \dim_{\mathbf{Q}_p}(V)$;*

(ii) *if $V_1, V_2 \in \mathrm{Rep}_{\mathrm{st}}(G_K)$, then $V_1 \otimes V_2 \in \mathrm{Rep}_{\mathrm{st}}(G_K)$ and the natural map $\mathsf{D}_{\mathrm{st}}(V_1) \otimes_{K_0} \mathsf{D}_{\mathrm{st}}(V_2) \to \mathsf{D}_{\mathrm{st}}(V_1 \otimes V_2)$ is an isomorphism compatible with Frobenius maps, monodromy operators and filtrations after extending the scalars to K; similarly, if $V \in \mathrm{Rep}_{\mathrm{st}}(G_K)$, then $V^{\vee}, \wedge^{\bullet} V \in \mathrm{Rep}_{\mathrm{st}}(G_K)$ and $\mathsf{D}_{\mathrm{st}}(V^{\vee}) \to \mathsf{D}_{\mathrm{st}}(V)^{\vee}$, $\wedge^{\bullet} \mathsf{D}_{\mathrm{st}}(V) \to \mathsf{D}_{\mathrm{st}}(\wedge^{\bullet} V)$ are isomorphisms compatible with extra structures;*

(iii) *if $V \in \mathrm{Rep}_{\mathrm{st}}(G_K)$ and $0 \to V' \to V \to V'' \to 0$ is an exact sequence in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ then $V', V'' \in \mathrm{Rep}_{\mathrm{st}}(G_K)$ and the sequence $0 \to \mathsf{D}_{\mathrm{st}}(V') \to \mathsf{D}_{\mathrm{st}}(V) \to \mathsf{D}_{\mathrm{st}}(V'') \to 0$ is exact in $\mathbf{MF}_K(\varphi, N)$;*

(iv) *Let $\eta \in \mathsf{Hom}_{\mathrm{cont}}(G_K, \mathbf{Q}_p^{\times})$. Then η is semi-stable (i.e. $\mathbf{Q}_p(\eta) \in \mathrm{Rep}_{\mathrm{st}}(G_K)$) if and only if $\eta = \eta_{\mathrm{nr}} \chi^i$ with $\eta_{\mathrm{nr}} \in \mathsf{Hom}_{\mathrm{cont}}(G_K, \mathbf{Q}_p^{\times})$ unramified and $i \in \mathbf{Z}$.*

Remark 3.17. — (a) Properties (i)-(iii) in the preceding corollary can be summarized by saying that $\mathrm{Rep}_{\mathrm{st}}(G_K)$ is a sub-tannakian category of $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, with fiber functor D_{st}.

(b) Of course, if V is semi-stable, it is de Rham and $\mathsf{D}_{\mathrm{st}}(V)_K \xrightarrow{\sim} \mathsf{D}_{\mathrm{dR}}(V)$ (by a dimension argument).

(c) If $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ then $\mathsf{D}_{\mathrm{cris}}(V) = \mathsf{D}_{\mathrm{st}}(V)^{N=0}$. In particular, if V is crystalline, then V is semi-stable, $\mathsf{D}_{\mathrm{cris}}(V) \xrightarrow{\sim} \mathsf{D}_{\mathrm{st}}(V)$ and $N = 0$.

(d) $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ is semi-stable if and only if $V_{|I_K}$ is semi-stable (in $\mathrm{Rep}_{\mathbf{Q}_p}(G_{\widehat{K^{\mathrm{nr}}}})$). This follows from $\mathsf{H}^1(\mathsf{Gal}(\bar{k}/k), \mathsf{GL}_n(\mathsf{W}(\bar{k}))) = \{1\}$ (Hilbert 90), which implies that $\widehat{K_0^{\mathrm{nr}}} \otimes_{K_0} \mathsf{D}_{\mathrm{st}}(V) \xrightarrow{\sim} \mathsf{D}_{\mathrm{st}}(V_{|I_K})$.

Definition 3.18 (cf. [36, §5.3]). — If $(D, \varphi, N, \mathsf{Fil}^\bullet D_K) \in \mathbf{MF}_K(\varphi, N)$, one puts

$$\mathsf{V}_{\mathrm{st}}(D) = (\mathsf{B}_{\mathrm{st}} \otimes_{K_0} D)^{\substack{\varphi=1\\ N=0}} \cap \mathsf{Fil}^0(\mathsf{B}_{\mathrm{dR}} \otimes_K D_K)$$

where the intersection is taken in $\mathsf{B}_{\mathrm{dR}} \otimes_K D_K = \mathsf{B}_{\mathrm{dR}} \otimes_{K_0} D$ and $\mathsf{B}_{\mathrm{dR}} \otimes_K D_K$ is endowed with the usual tensor product filtration. This provides a functor on $\mathbf{MF}_K(\varphi, N)$ with values in the category of topological $\mathbf{Q}_p$-vector spaces endowed with a continuous linear action of G_K (since G_K commutes to the Frobenius maps, the monodromy operators and filtrations).

Proposition 3.19 (cf. [36, Théorème 5.3.5]). — *If* $V \in \mathrm{Rep}_{\mathrm{st}}(G_K)$, *then* $V \xrightarrow{\sim} \mathsf{V}_{\mathrm{st}}(\mathsf{D}_{\mathrm{st}}(V))$. *The functor*

$$\mathsf{D}_{\mathrm{st}} \colon \mathrm{Rep}_{\mathrm{st}}(G_K) \to \mathbf{MF}_K(\varphi, N)$$

induced by D_{st} *is fully faithful.*

Proof. — Let $V \in \mathrm{Rep}_{\mathrm{st}}(G_K)$: one has $\mathsf{B}_{\mathrm{st}} \otimes_{K_0} \mathsf{D}_{\mathrm{st}}(V) \xrightarrow{\sim} \mathsf{B}_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V$. Extending the scalars to B_{dR} yields an isomorphism $\mathsf{B}_{\mathrm{dR}} \otimes_K \mathsf{D}_{\mathrm{st}}(V)_K \xrightarrow{\sim} \mathsf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$, so that

$$V = \big(\mathsf{B}_{\mathrm{st}}^{\substack{\varphi=1\\ N=0}} \cap \mathsf{Fil}^0 \mathsf{B}_{\mathrm{dR}}\big) \otimes_{\mathbf{Q}_p} V \xrightarrow{\sim} (\mathsf{B}_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V)^{\substack{\varphi=1\\ N=0}} \cap \mathsf{Fil}^0(\mathsf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V) = \mathsf{V}_{\mathrm{st}}(\mathsf{D}_{\mathrm{st}}(V))$$

(the first equality follows from the fundamental exact sequence, *cf.* proposition 3.7).
This implies that if $V_1, V_2 \in \mathrm{Rep}_{\mathrm{st}}(G_K)$, the composite

$$\mathsf{Hom}(V_1, V_2) \to \mathsf{Hom}(\mathsf{D}_{\mathrm{st}}(V_1), \mathsf{D}_{\mathrm{st}}(V_2)) \to \mathsf{Hom}(\mathsf{V}_{\mathrm{st}}(\mathsf{D}_{\mathrm{st}}(V_1)), \mathsf{V}_{\mathrm{st}}(\mathsf{D}_{\mathrm{st}}(V_2))) \simeq \mathsf{Hom}(V_1, V_2)$$

is the identity. Also, one has $f = \mathsf{D}_{\mathrm{st}}(\mathsf{V}_{\mathrm{st}}(f))$ for all $f \in \mathsf{Hom}(\mathsf{D}_{\mathrm{st}}(V_1), \mathsf{D}_{\mathrm{st}}(V_2))$ (because $\mathsf{D}_{\mathrm{st}}(V_i) \xrightarrow{\sim} \mathsf{D}_{\mathrm{st}}(\mathsf{V}_{\mathrm{st}}(\mathsf{D}_{\mathrm{st}}(V_i)))$): the second map is injective, and D_{st} is fully faithful. □

Remark 3.20. — Of course, the category $\mathrm{Rep}_{\mathrm{st}}(G_K)$ is not stable by extension. For instance, assuming k finite, a non trivial extension

$$0 \to \mathbf{Q}_p(i) \to V \to \mathbf{Q}_p \to 0$$

is semi-stable if $i \geq 1$ (even crystalline when $i \geq 2$), but it is not de Rham (hence not semi-stable) when $i < 0$. For $i = 0$, there are extensions that are not de Rham (*cf.* [15, Example 3.9]).

3.5. Admissible filtered (φ, N)-modules. — Recall (Dieudonné-Manin theory, *cf.* [31, Chapter 4]) that when $k = \bar{k}$, the category of φ-modules over K_0 (*i.e.* the category of F-isocrystals over k) is semi-simple, with simple objects $\{D_{[\alpha]}\}_{\alpha \in \mathbf{Q}}$ where, if $\alpha = \frac{r}{h}$ with $r \in \mathbf{Z}$, $h \in \mathbf{Z}_{>0}$ and $\mathsf{gcd}(r, h) = 1$,

$$D_{[\alpha]} = K_0[T]/(T^h - p^r)$$

is endowed with the σ-semi-linear Frobenius given by the multiplication by T. This means that if D is a φ-module over K_0, there exists a unique sequence $\alpha_1 < \cdots < \alpha_r$ of rationals (the *slopes* of D) and a unique decomposition $D = \oplus_{i=1}^r D(\alpha_i)$ where $D(\alpha_i)$ is isomorphic to a finite sum of copies of $D_{[\alpha_i]}$. The *multiplicity* of the slope α_i is the integer $\dim_{K_0}(D(\alpha_i)) \in \mathbf{Z}$. The *slope sequence* of D is the non-decreasing sequence $\lambda_1 \leq \cdots \leq \lambda_n$ of slopes of D, each one repeated according to its multiplicity (hence $n = \dim_{K_0}(D)$). The *Newton polygon* $P_N(D)$ of D is the piecewise linear curve of the plane starting at the origin whose vertices have coordinates $(j, \lambda_1 + \cdots + \lambda_j)$

for $0 \leq j \leq n$. The slopes of its segments are precisely $\alpha_1 < \cdots < \alpha_r$, and its break-points have integral coordinates (because $\alpha_i \dim_{K_0}(D_{[\alpha_i]}) \in \mathbf{Z}$). In the general case (*i.e.* when k is not algebraically closed), the slopes and Newton polygon of D are those of $\widehat{K_0^{\mathrm{nr}}} \otimes_{K_0} D$. The Newton polygon of D is nothing but the Newton polygon of the characteristic polynomial of the matrix of φ in any K_0-base of D.

Similarly, one attaches the *Hodge polygon* to the filtration $\mathsf{Fil}^\bullet D_K$. Its slopes are the integers $i \in \mathbf{Z}$ such that $\mathsf{gr}^i D_K \neq 0$ (the Hodge-Tate weights), with multiplicity $\dim_K(\mathsf{gr}^i(D_K))$: let $i_1 \leq \cdots \leq i_n$ be the sequence of slopes, each one repeated according to its multiplicity. The *Hodge polygon* $P_H(D)$ of D is the piecewise linear curve of the plane starting at the origin whose vertices have coordinates $(j, i_1 + \cdots + i_j)$ for $0 \leq j \leq n$. The slopes of its segments are precisely the Hodge-Tate weights, and its break-points have integral coordinates.

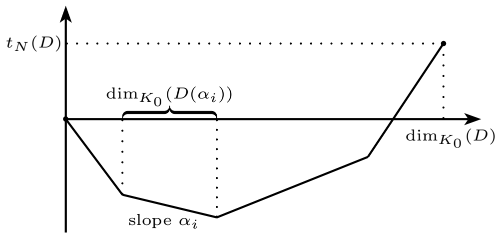


FIGURE 1. The Newton polygon

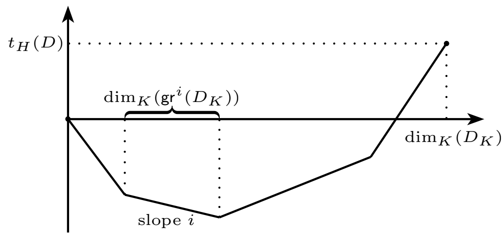


FIGURE 2. The Hodge polygon

***Definition 3.21* (*cf.* [36, §4.4.1]).** — Let $D \in \mathbf{MF}_K(\varphi, N)$. If $\dim_{K_0}(D) = 1$, one can write $D = K_0 e$ and $\varphi(e) = \lambda e$ with $\lambda \in K_0$: we put $t_N(D) = v(\lambda) \in \mathbf{Z}$. Also, there exists $t_H(D) \in \mathbf{Z}$ such that $\mathsf{Fil}^i D_K = \begin{cases} D_K & \text{if } i \leq t_H(D) \\ 0 & \text{if } i > t_H(D) \end{cases}$. For general D, we put $t_H(D) = t_H(\det(D))$ and $t_N(D) = t_N(\det(D))$. One easily that $t_N(D) = \sum_{\alpha \in \mathbf{Q}} \alpha \dim_{K_0}(D(\alpha))$ is the valuation of the determinant of the matrix of the Frobenius map (in any base), and is the ordinate of the endpoint of $P_N(D)$. Also $t_H(D) = \sum_{i \in \mathbf{Z}} i \dim_K(\mathsf{gr}^i D_K)$ is the ordinate of the endpoint of $P_H(D)$.

Proposition 3.22. — *The functions t_N and t_H are additive on $\mathbf{MF}_K(\varphi, N)$, i.e. for any exact sequence*

$$0 \to D' \to D \to D'' \to 0,$$

one has $t_N(D) = t_N(D') + t_N(D'')$ and $t_H(D) = t_H(D') + t_H(D'')$. Also, they are invariant by base change to $\widehat{K_0^{\mathrm{nr}}}$.

***Definition 3.23* (*cf.* [36, Définition 4.4.3]).** — A filtered (φ, N)-module D is called *admissible*[(7)] if for every sub-object $D' \subset D$ (in the category $\mathbf{MF}_K(\varphi, N)$), one has

$$t_H(D') \leq t_N(D')$$

with equality when $D' = D$. The full sub-category of $\mathbf{MF}_K(\varphi, N)$ made of admissible objects is denoted by $\mathbf{MF}_K^{\mathsf{ad}}(\varphi, N)$.

Remark 3.24. — (1) One can show (*cf.* [32, Proposition 4.3.3]) that D is admissible if and only for every sub-object D' of D, the polygon $P_N(D')$ lies above $P_H(D')$, with same end-points when $D' = D$.

[(7)] Historically, these objects were called *weakly admissible*, admissible objects being those in the essential image of D_{st}. As they are *a posteriori* the same (*cf.* theorem 3.28), we call them admissible from the start.

(2) An object $D\in\mathbf{MF}_K(\varphi,N)$ is admissible if and only if $\widehat{K_0^{\mathrm{nr}}}\otimes_{K_0}D\in\mathbf{MF}^{\mathsf{ad}}_{K\otimes_{K_0}\widehat{K_0^{\mathrm{nr}}}}(\varphi,N)$ (*cf.* [52, Proposition 1.7]).

Proposition 3.25 (cf. [25, Proposition 4.4]). — *Assume $D\in\mathbf{MF}_K(\varphi,N)$ has dimension 1. Then*

$$\dim_{\mathbf{Q}_p}(\mathsf{V}_{\mathrm{st}}(D))=\begin{cases}0 & \text{if } t_H(D)<t_N(D);\\ 1 & \text{if } t_H(D)=t_N(D);\\ +\infty & \text{if } t_H(D)>t_N(D).\end{cases}$$

In particular, D lies in the essential image of $\mathsf{D}_{\mathrm{st}}\colon \mathrm{Rep}_{\mathrm{st}}(G_K)\to\mathbf{MF}_K(\varphi,N)$ if and only if D is admissible.

Proof. — One has $N=0$ since it is nilpotent. Write $D=K_0e$ and $\varphi(e)=p^{t_N(D)}\lambda e$ with $\lambda\in W^\times$. By successive approximations, one can construct $\beta\in\mathsf{W}(\bar k)^\times$ such that $\sigma(\beta)=\lambda\beta$, *i.e.*

$$\varphi(\beta^{-1}\otimes e)=\sigma(\beta)^{-1}p^{t_N(D)}\lambda\otimes e=p^{t_N(D)}\beta^{-1}\otimes e.$$

One has $\mathsf{V}_{\mathrm{st}}(D)=(\mathsf{B}_{\mathrm{cris}}\otimes e)^{\varphi=1}\cap\mathsf{Fil}^0(\mathsf{B}_{\mathrm{dR}}\otimes e)$, and $\mathsf{Fil}^i(\mathsf{B}_{\mathrm{dR}}\otimes e)=t^{i-t_H(D)}\,\mathsf{B}^+_{\mathrm{dR}}\otimes e$ hence

$$\mathsf{V}_{\mathrm{st}}(D)=\big\{t^{-t_H(D)}b\beta^{-1}\otimes e\,\big|\,b\in\mathsf{B}_{\mathrm{cris}},\ \varphi(b)=p^{t_H(D)-t_N(D)}b,\ b\in\mathsf{B}^+_{\mathrm{dR}}\big\}.$$

Put $i=t_H(D)-t_N(D)$: one has $\mathsf{V}_{\mathrm{st}}(D)=t^{-t_H(D)}\beta^{-1}\big(\mathsf{B}_{\mathrm{cris}}^{\varphi=p^i}\cap\mathsf{Fil}^0\,\mathsf{B}_{\mathrm{dR}}\big)\otimes e$, which is $\{0\}$ when $i<0$, has dimension 1 if $i=0$, and has infinite dimension when $i>0$ (*cf.* [25, Proposition 1.3]).

In the case $i=0$, the character $\eta\colon G_K\to\mathbf{Q}_p^\times$ corresponding to $\mathsf{V}_{\mathrm{st}}(D)=\mathbf{Q}_p\,t^{-t_H(D)}\beta^{-1}\otimes e$ is $\eta=\chi^{-t_H(D)}\eta_0$ with η_0 unramified (since $\beta\in\mathsf{W}(\bar k)^\times$), hence $\mathsf{V}_{\mathrm{st}}(D)$ is semi-stable (and even crystalline). Moreover, one has $\mathsf{B}_{\mathrm{st}}\otimes_{\mathbf{Q}_p}\mathsf{V}_{\mathrm{st}}(D)=\mathsf{B}_{\mathrm{st}}(t^{-t_H(D)}\beta^{-1}\otimes e)\xrightarrow{\sim}\mathsf{B}_{\mathrm{st}}\otimes_{K_0}D$, thus $\mathsf{D}_{\mathrm{st}}(\mathsf{V}_{\mathrm{st}}(D))\xrightarrow{\sim}D$ lies in the essential image of $\mathsf{D}_{\mathrm{st}}\colon\mathrm{Rep}_{\mathrm{st}}(G_K)\to\mathbf{MF}_K(\varphi,N)$. Conversely if $D\simeq\mathsf{D}_{\mathrm{st}}(V)$ with $V\in\mathrm{Rep}_{\mathrm{st}}(G_K)$ of dimension 1, then $\dim_{K_0}(D)=1$ and $V\simeq\mathsf{V}_{\mathrm{st}}(\mathsf{D}_{\mathrm{st}}(V))\simeq\mathsf{V}_{\mathrm{st}}(D)$ has dimension 1, hence $t_H(D)=t_N(D)$ by what precedes. □

Proposition 3.26. — *If $V\in\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ is semi-stable, then $\mathsf{D}_{\mathrm{st}}(V)$ is admissible, hence the functor D_{st} induces a fully faithful functor $\mathrm{Rep}_{\mathrm{st}}(G_K)\to\mathbf{MF}^{\mathsf{ad}}_K(\varphi,N)$.*

Proof. — Let D' be a sub-object of $D:=\mathsf{D}_{\mathrm{st}}(V)$. If $r=\dim_{K_0}(D')$, one has $\det(D')\subset\wedge^rD\simeq\mathsf{D}_{\mathrm{st}}(\wedge^rV)$ (*cf.* corollary 3.16 (ii)), with equality if $D'=D$. Applying the functor V_{st} gives the inclusion (equality when $D'=D$)

$$\mathsf{V}_{\mathrm{st}}(\det(D'))\subset\mathsf{V}_{\mathrm{st}}(\mathsf{D}_{\mathrm{st}}(\wedge^rV))\simeq\wedge^rV$$

so that $\dim_{\mathbf{Q}_p}\big(\mathsf{V}_{\mathrm{st}}(\det(D'))\big)<+\infty$ (and $\dim_{\mathbf{Q}_p}\big(\mathsf{V}_{\mathrm{st}}(\det(D'))\big)=1$ when $D'=D$). As $\dim_{K_0}(\det(D'))=1$, proposition 3.25 implies that $t_H(\det(D'))\leq t_N(\det(D'))$, *i.e.* $t_H(D')\leq t_N(D')$, with equality when $D'=D$. □

A rather elementary consequence of proposition 3.25 is the following useful fact.

Lemma 3.27 (cf. [25, Proposition 4.5]). — *Let $D\in\mathbf{MF}_K(\varphi,N)$ and $V:=\mathsf{V}_{\mathrm{st}}(D)$. Assume that for all sub-object D' of D (in $\mathbf{MF}_K(\varphi,N)$), one has $t_H(D')\leq t_N(D')$. Then $\dim_{\mathbf{Q}_p}(V)\leq\dim_{K_0}(D)$, the p-adic representation V is semi-stable and $\mathsf{D}_{\mathrm{st}}(V)$ is a sub-object of D. In particular, D lies in the essential image of $\mathsf{D}_{\mathrm{st}}\colon\mathrm{Rep}_{\mathrm{st}}(G_K)\to\mathbf{MF}_K(\varphi,N)$ if and only if $\dim_{\mathbf{Q}_p}(V)=\dim_{K_0}(D)$.*

In fact, Colmez and Fontaine proved (*cf.* [25, Théorème A]) that $\mathbf{MF}^{\mathsf{ad}}_K(\varphi,N)$ is precisely the essential image of D_{st} on $\mathrm{Rep}_{\mathrm{st}}(G_K)$ (*cf.* §5), hence:

Theorem 3.28. — *The functor*

$$\mathsf{D}_{\mathrm{st}}\colon \mathrm{Rep}_{\mathrm{st}}(G_K)\to \mathbf{MF}_K^{\mathsf{ad}}(\varphi,N)$$

is an equivalence of categories, with quasi-inverse V_{st}.

3.6. A proof of theorem 3.28. — The first proofs of theorem 3.28 are of analytic nature (*cf.* [25], [24], [10] and [51]). Here we review that of Kisin's article [51], which is based on ideas of Berger (*cf.* [10]). Filtered (φ,N)-modules are seen as fibers of vector bundles over the open unit analytic disc Δ over K_0 (whose $\overline{K}$-points are G_{K_0}-conjugacy classes of $\{x\in\overline{K},\ |x|<1\}$, where $|.|$ is the absolute value associated with the valuation v). Denote by u the coordinate of Δ, and by $\mathcal{O}$ the ring of K_0-analytic functions on Δ:

$$\mathcal{O}=\Big\{f=\sum_{n\geq 0}a_nu^n\in K_0[\![u]\!]\ \Big|\ (\forall r\in[0,1[)\ \lim_{n\to+\infty}|a_n|r^n=0\Big\}.$$

Let $\mathfrak{S}=W[\![u]\!]$ be the sub-ring of functions whose coefficients have absolute values bounded by 1. Finally, let

$$\mathscr{R}=\Big\{f=\sum_{n\in\mathbf{Z}}a_nu^n\in K_0[\![u,u^{-1}]\!]\ \Big|\ (\exists r_f\in[0,1[)\ (\forall r\in[r_f,1[)\ \lim_{n\to\pm\infty}|a_n|r^n=0\Big\}$$

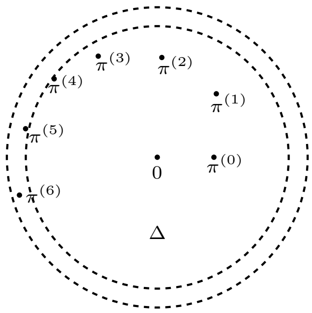
be the Robba ring: the ring of functions defined on some annulus (depending on the function) of outer radius 1. Of course one has the inclusions $\mathfrak{S}\subset\mathcal{O}\subset\mathscr{R}$. All these rings are endowed with the σ-semi-linear Frobenius endomorphism φ defined by

$$\varphi(u)=u^p.$$

Recall we fixed an element $\widetilde{\pi}\in\mathcal{R}$, *i.e.* a sequence $(\pi^{(n)})_{n\in\mathbf{N}}$ such that $\pi^{(0)}=\pi$ and $(\pi^{(n+1)})^p=\pi^{(n)}$ for all $n\in\mathbf{N}$. Let $E(u)\in W[u]$ be the minimal polynomial of π over K_0 (this is an Eisenstein polynomial). One has $\frac{E(u)}{E(0)}\in 1+\mathfrak{S}[p^{-1}]$, so the infinite product

$$\lambda=\prod_{n=0}^{\infty}\varphi^n\big(\tfrac{E(u)}{E(0)}\big)$$

converges in $\mathcal{O}$, providing a function whose divisor is the set of (conjugacy classes of) the $\pi^{(n)}$. One has $\lambda=\frac{E(u)}{E(0)}\varphi(\lambda)$, so $\varphi(\lambda^{-1})=\lambda^{-1}\frac{E(u)}{E(0)}\in\mathcal{O}[\lambda^{-1}]$, and φ extends to $\mathcal{O}[\lambda^{-1}]$.



Let $N_\nabla\colon\mathcal{O}\to\mathcal{O}$ be the derivation given by $N_\nabla=\lambda u\frac{\mathrm{d}}{\mathrm{d}u}$. It extends to $\mathcal{O}[\lambda^{-1}]$, and

$$N_\nabla\varphi=p\tfrac{E(u)}{E(0)}\varphi N_\nabla.$$

Definition 3.29. — (1) $D\in\mathbf{MF}_K(\varphi,N)$ is said *effective* if $\mathsf{Fil}^0 D_K=D_K$. We denote by $\mathbf{MF}_K^{\mathsf{eff}}(\varphi,N)$ (resp. $\mathbf{MF}_K^{\mathsf{eff,ad}}(\varphi,N)$) the sub-category of effective (admissible) filtered (φ,N)-modules.

(2) A *φ-module* over $\mathcal{O}$ is a finite and free $\mathcal{O}$-module $\mathcal{M}$ endowed with an injective φ-semi-linear Frobenius endomorphism $\varphi\colon\mathcal{M}\to\mathcal{M}$. It is of *finite E-height* if the cokernel of the linearization $\varphi^*\mathcal{M}\to\mathcal{M}$ is killed by some power of E.

(3) A (φ, N_∇)-*module* over $\mathcal{O}$ is a φ-module $(\mathcal{M}, \varphi)$ endowed with derivation $N_\nabla \colon \mathcal{M} \to \mathcal{M}$ (*i.e.* such that $(\forall f \in \mathcal{O})$ $(\forall m \in \mathcal{M})$ $N_\nabla(fm) = N_\nabla(f)m + fN_\nabla(m)$) and such that $N_\nabla \varphi = p\frac{E(u)}{E(0)}\varphi N_\nabla$. We denote by $\mathbf{Mod}_{\mathcal{O}}(\varphi, N_\nabla)$ the category of (φ, N_∇)-modules over $\mathcal{O}$ that are of finite E-height.

(4) A (φ, N)-*module* over $\mathcal{O}$ is a φ-module $(\mathcal{M}, \varphi)$ over $\mathcal{O}$ endowed with a K_0-linear endomorphism $N \colon \mathcal{M}/u\mathcal{M} \to \mathcal{M}/u\mathcal{M}$ such that $N\varphi = p\varphi N$ modulo u. We denote by $\mathbf{Mod}_{\mathcal{O}}(\varphi, N)$ the category of (φ, N)-modules over $\mathcal{O}$ that are of finite E-height.

(5) A (φ, N)-*module* over $\mathfrak{S}$ is a finite and free $\mathfrak{S}$-module $\mathfrak{M}$ endowed with an injective and φ-semi-linear endomorphism $\varphi \colon \mathfrak{M} \to \mathfrak{M}$ and a K_0-linear endomorphism $N \colon K_0 \otimes_W (\mathfrak{M}/u\mathfrak{M}) \to K_0 \otimes_W (\mathfrak{M}/u\mathfrak{M})$ such that $N\varphi = p\varphi N$ modulo u. We denote by $\mathbf{Mod}_{\mathfrak{S}}(\varphi, N)$ the category of (φ, N)-modules over $\mathfrak{S}$ that are of finite E-height (*i.e.* whose cokernel of the linearization $\varphi^*\mathfrak{M} \to \mathfrak{M}$ is killed by some power of E), and by $\mathbf{Mod}_{\mathfrak{S}}(\varphi, N)_{\mathbf{Q}}$ its isogeny category (*i.e.* $\mathbf{Mod}_{\mathfrak{S}}(\varphi, N)$ with Hom groups tensored by $\mathbf{Q}$).

3.6.1. Kedlaya's slope filtration. — To translate the admissibility of a filtered (φ, N)-module in terms of the associated bundle over Δ (*cf.* below), one needs Kedlaya's results on the slope filtration of φ-modules on the Robba ring.

Definition 3.30. — A φ-*module* over $\mathscr{R}$ is a finite and free $\mathscr{R}$-module M endowed with a φ-semi-linear endomorphism $\varphi \colon M \to M$ whose linearization is an isomorphism. We denote by $\mathbf{Mod}_{\mathscr{R}}(\varphi)$ the corresponding category.

In [49], Kedlaya constructs a $\mathscr{R}$-algebra $\mathscr{R}^{\mathrm{alg}}$ such that for all $M \in \mathbf{Mod}_{\mathscr{R}}(\varphi)$, there exists a finite extension $L/\widehat{K_0^{\mathrm{nr}}}$ such that

$$M \otimes_{\mathscr{R}} \mathscr{R}^{\mathrm{alg}} \otimes_{\widehat{K_0^{\mathrm{nr}}}} L$$

admits a basis $(m_1, \dots, m_n)$ such that there exist $\alpha_1, \dots, \alpha_n \in L$ with $\varphi(m_i) = \alpha_i m_i$ for all $i \in \{1, \dots, n\}$. The valuations of $\alpha_1, \dots, \alpha_n$ only depend on M and are called the *slopes* of M (when they are all equal to $s \in \mathbf{Q}$, M is said *pure of slope* s). Moreover, there is a Dieudonné-Manin type filtration in this context:

Theorem 3.31 **(*cf.*** [49, Theorem 6.10]**).** — *Let $M \in \mathbf{Mod}_{\mathscr{R}}(\varphi)$. There exists a sequence of rationals $s_1 < s_2 < \cdots < s_r$ (the* slopes *of M) and a canonical filtration (the* slope filtration*)*

$$\{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$$

by sub-φ-modules such that M_i/M_{i-1} is pure of slope s_i for all $i \in \{1, \dots, r\}$.

Remark 3.32. — This theorem is similar to Harder-Narasimhan filtration on vector bundles over smooth projective curves. These are special cases of a very general formalism of slope filtrations and semi-stability, developed by André (*cf.* [2]).

Definition 3.33. — We denote by $\mathbf{Mod}^0_{\mathcal{O}}(\varphi, N_\nabla)$ (resp. $\mathbf{Mod}^0_{\mathcal{O}}(\varphi, N)$) the subcategory of $\mathbf{Mod}_{\mathcal{O}}(\varphi, N_\nabla)$ (resp. $\mathbf{Mod}_{\mathcal{O}}(\varphi, N)$) of $\mathcal{M}$ such that the φ-module $\mathscr{R}$ defined by $\mathscr{R} \otimes_{\mathcal{O}} \mathcal{M}$ is pure of slope 0.

In [51], Kisin constructs functors between the preceding categories:

$$\begin{tikzcd}
\mathbf{MF}_K^{\mathsf{eff}}(\varphi,N) \arrow[r, shift left, "\underline{\mathcal{M}}"] & \mathbf{Mod}_{\mathcal{O}}(\varphi,N_\nabla) \arrow[l, shift left, "\underline{D}"] \\
\mathbf{MF}_K^{\mathsf{eff},\mathsf{ad}}(\varphi,N) \arrow[u, hook] \arrow[r, shift left] & \mathbf{Mod}_{\mathcal{O}}^0(\varphi,N_\nabla) \arrow[u, hook] \arrow[l, shift left] \arrow[d, hook, "\Psi"] \\
\mathbf{Mod}_{\mathfrak{S}}(\varphi,N)_{\mathbf{Q}} \arrow[r, "\approx", "\Theta"'] & \mathbf{Mod}_{\mathcal{O}}^0(\varphi,N).
\end{tikzcd}$$

3.6.2. Indications on how the functors are defined. — To construct $\underline{\mathcal{M}}$, one considers the polynomial ring $\mathcal{O}[\lambda^{-1},\ell_u]$, where ℓ_u is a variable that corresponds to "$\log(u)$". The maps φ and N_∇ extend to $\mathcal{O}[\lambda^{-1},\ell_u]$ putting

$$\varphi(\ell_u)=p\ell_u \qquad \text{and} \qquad N_\nabla(\ell_u)=-\lambda.$$

We denote by N the derivation with respect to ℓ_u: one has $NN_\nabla=N_\nabla N$ and $N\varphi=p\varphi N$ on $\mathcal{O}[\lambda^{-1},\ell_u]$.

For $n\in\mathbf{N}$, let $\widehat{\mathfrak{S}}_n$ be the completion of the localization of $K_0(\pi_n)\otimes_W\mathfrak{S}$ at the ideal generated by $u-\pi^{(n)}$: it is a complete discrete valuation ring, whose fraction field is $\widehat{\mathfrak{S}}_n\left[(u-\pi^{(n)})^{-1}\right]$. We endow it with the $(u-\pi^{(n)})$-adic filtration. This ring corresponds to germs of functions at $\pi^{(n)}$. There are natural inclusions

$$\mathfrak{S}[p^{-1}]\subset\mathcal{O}\subset\mathcal{O}[\ell_u]\subset\widehat{\mathfrak{S}}_n$$

where the last inclusion maps ℓ_u to

$$\log\left(1+\frac{u-\pi^{(n)}}{\pi^{(n)}}\right)=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left(\frac{u-\pi^{(n)}}{\pi^{(n)}}\right)^m\in\widehat{\mathfrak{S}}_n.$$

If $D\in\mathbf{MF}_K^{\mathsf{eff}}(\varphi,N)$, the $\mathcal{O}$-module $\underline{\mathcal{M}}(D)$ is defined as a sub-$\mathcal{O}$-module of

$$\left(\mathcal{O}[\ell_u,\lambda^{-1}]\otimes_{K_0}D\right)^{N=0}$$

defined by conditions (related to the filtration) on the fibers at the points of Δ corresponding to the $\pi^{(n)}$ (for $n\in\mathbf{N}$) as follows.

The Frobenius map on $\mathcal{O}[\ell_u]$ factorizes as $\varphi=\varphi_W\circ\varphi_{\mathcal{O}/W}$ where $\varphi_W\colon\mathcal{O}[\ell_u]\to\mathcal{O}[\ell_u]$ is $\mathbf{Z}_p[\![u]\!]$-linear and acts by φ on coefficients, whereas $\varphi_{\mathcal{O}/W}$ is W-linear and maps u to u^p. We consider the composite

$$\begin{tikzcd}
\mathcal{O}[\ell_u]\otimes_{K_0}D \arrow[r, "\varphi_W^{-n}\otimes\varphi^{-n}"] \arrow[rr, bend right, "\iota_n"'] & \mathcal{O}[\ell_u]\otimes_{K_0}D \arrow[r] & \widehat{\mathfrak{S}}_n\otimes_{K_0}D=\widehat{\mathfrak{S}}_n\otimes_K D_K\ .
\end{tikzcd}$$

As the image of $\varphi_W^{-n}(\lambda)=\prod\limits_{m\in\mathbf{N}}\varphi_W^{m-n}\Big(\frac{E(u^{p^m})}{E(0)}\Big)$ in $\widehat{\mathfrak{S}}_n$ belongs to $\frac{E(u^{p^n})}{E(0)}\widehat{\mathfrak{S}}_n^\times=(u-\pi^{(n)})\widehat{\mathfrak{S}}_n^\times$, it extends into a localization map:

$$\iota_n\colon\mathcal{O}[\lambda^{-1},\ell_u]\otimes_{K_0}D\to\widehat{\mathfrak{S}}_n\big[(u-\pi^{(n)})^{-1}\big]\otimes_K D_K.$$

The $\mathcal{O}$-module $\mathcal{O}[\lambda^{-1},\ell_u]\otimes_{K_0}D$ is endowed with the operator N defined by $N\otimes\mathsf{Id}_D+\mathsf{Id}_{\mathcal{O}[\lambda^{-1},\ell_u]}\otimes N$. Furthermore, the $\widehat{\mathfrak{S}}_n$-module $\widehat{\mathfrak{S}}_n\left[\left(u-\pi^{(n)}\right)^{-1}\right]\otimes_K D_K$ is equipped with the tensor product filtration, given by:

$$\mathsf{Fil}^i\left(\widehat{\mathfrak{S}}_n\left[(u-\pi^{(n)})^{-1}\right]\otimes_K D_K\right)=\sum_{j\in\mathbf{Z}}\mathsf{Im}\left((u-\pi^{(n)})^{-j}\widehat{\mathfrak{S}}_n\otimes_K\mathsf{Fil}^{i+j}(D_K)\right)$$

for all $i\in\mathbf{Z}$. Then

$$\underline{\mathcal{M}}(D)=\left\{x\in\left(\mathcal{O}[\ell_u,\lambda^{-1}]\otimes_{K_0}D\right)^{N=0}\ \middle|\ (\forall n\in\mathbf{N})\ \iota_n(x)\in\mathsf{Fil}^0\left(\widehat{\mathfrak{S}}_n\left[(u-\pi_n)^{-1}\right]\otimes_K D_K\right)\right\}.$$

Basically, $\underline{D}(\mathcal{M})$ is just $\mathcal{M}/u\mathcal{M}$ endowed with the induced Frobenius and the monodromy operator N given by the reduction of N_∇ modulo u. The filtration on $\underline{D}(\mathcal{M})_K$ is more difficult to define (it is related to the fibers at $\{\pi^{(n)}\}_{n\in\mathbf{N}}$, *cf.* [51, §1.2.7]).

The functor Ψ consists in replacing N_∇ by its reduction N modulo u. As for Θ, it is merely the scalar extension from $\mathfrak{S}$ to $\mathcal{O}$.

The functors $\underline{\mathcal{M}}$ and $\underline{D}$ are quasi-inverse equivalences of categories (*cf.* [51, Theorem 1.2.15]), and induce quasi-inverse equivalences of categories between $\mathbf{MF}_K^{\mathsf{eff,ad}}(\varphi,N)$ and $\mathbf{Mod}_{\mathcal{O}}^0(\varphi,N_\nabla)$ (*cf.* [51, Theorem 1.3.8]). This provides a fully faithful functor

$$\mathbf{MF}_K^{\mathsf{eff,ad}}(\varphi,N)\to\mathbf{Mod}_{\mathfrak{S}}(\varphi,N)_{\mathbf{Q}},$$

hence an "integral" p-adic Hodge theory (p is not invertible in $\mathfrak{S}$), without restriction on the absolute ramification index of K or the length of the filtration. The interest of such a theory is that it is convenient to deal with deformation problems, finite flat group-schemes over $\mathcal{O}_K$, *etc.*

Remark 3.34. — The p-adic completion $\mathcal{O}_{\mathcal{E}}$ of $\mathfrak{S}[u^{-1}]$ is a discrete valuation ring, whose residue field is $k(\!(u)\!)$. The $\mathfrak{S}$-algebra structure of $\mathsf{W}(\mathcal{R})$ given by $u\mapsto[\widetilde{\pi}]$ extends to a $\mathcal{O}_{\mathcal{E}}$-algebra structure of $\mathsf{W}(\mathsf{Frac}(\mathcal{R}))$. Put $\mathfrak{S}^{\mathrm{nr}}=\mathcal{O}_{\mathcal{E}}^{\mathrm{nr}}\cap\mathsf{W}(\mathcal{R})\subset\mathsf{W}(\mathsf{Frac}(\mathcal{R}))$. It is an extension of $\mathfrak{S}$ endowed with an action of $G_{K(\pi^{(n)}|n\in\mathbf{N})}\simeq\mathsf{Gal}\left(k(\!(u)\!)^{\mathsf{sep}}/k(\!(u)\!)\right)$.

As every (φ,N)-module becomes effective after an appropriate Tate twist, Theorem 3.28 then follows from:

Proposition 3.35 (cf. [51, Proposition 2.1.5]). — *Let $D\in\mathbf{MF}_K^{\mathsf{eff,ad}}(\varphi,N)$ and $\mathfrak{M}\in\mathbf{Mod}_{\mathfrak{S}}(\varphi,N)_{\mathbf{Q}}$ such that $\Theta(\mathfrak{M})\simeq D$. Then there is a canonical ($G_{K(\pi^{(n)}|n\in\mathbf{N})}$-equivariant) isomorphism*

$$\mathsf{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M},\mathfrak{S}^{\mathrm{nr}})\xrightarrow{\sim}\mathsf{Hom}_{\mathbf{MF}_K(\varphi,N)}(D,\mathsf{B}_{\mathrm{st}}^+)$$

and $\dim_{\mathbf{Q}_p}\left(\mathsf{Hom}_{\mathbf{MF}_K(\varphi,N)}(D,\mathsf{B}_{\mathrm{st}}^+)\right)=\dim_{K_0}(D)$ *so D is admissible.*

The map is constructed as the composite of injective maps

$$\mathsf{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M},\mathfrak{S}^{\mathrm{nr}})\to\mathsf{Hom}_{\mathcal{O},\varphi}(\mathcal{M},\mathsf{B}_{\mathrm{cris}}^+)\to\mathsf{Hom}_{\mathcal{O},\mathsf{Fil},\varphi}(\mathcal{D}_0,\mathsf{B}_{\mathrm{cris}}^+)\to\mathsf{Hom}_{\mathsf{Fil},\varphi,N}(D,\mathsf{B}_{\mathrm{st}}^+)$$

where $\mathfrak{M}=\Theta(\mathfrak{M})=\underline{\mathcal{M}}(D)$ and $\mathcal{D}_0=(\mathcal{O}[\ell_u]\otimes_{K_0}D)^{N=0}$. The first map comes form the injection $\mathfrak{S}\hookrightarrow\mathsf{B}_{\mathrm{cris}}^+$ (in which u is mapped to the Teichmüller element $[\widetilde{\pi}]\in\mathsf{B}_{\mathrm{cris}}^+$), that extends into injections $\mathfrak{S}^{\mathrm{nr}}\hookrightarrow\mathsf{B}_{\mathrm{cris}}^+$ and $\mathcal{O}\hookrightarrow\mathsf{B}_{\mathrm{cris}}^+$. The last one is deduced from the inclusion $D\subset\mathcal{O}[\ell_u]\otimes_{K_0}D$ and the isomorphism $\mathcal{O}[\ell_u]\otimes_{\mathcal{O}}\mathsf{B}_{\mathrm{cris}}^+\to\mathsf{B}_{\mathrm{st}}^+$; $\ell_u\mapsto\log[\widetilde{\pi}]$ (*cf.* proposition 3.4). Classical arguments (*cf.* [34, A.1.2 & B.1.8.4]) imply that $\dim_{\mathbf{Q}_p}(\mathsf{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M},\mathfrak{S}^{\mathrm{nr}}))=\dim_{K_0}(D)$. As D is admissible, lemma 3.27 implies that $\dim_{\mathbf{Q}_p}(\mathsf{Hom}_{\mathsf{Fil},\varphi,N}(D,\mathsf{B}_{\mathrm{st}}^+))\leq\dim_{K_0}(D)$, so the injectivity of the map shows that this inequality is an equality, and that the map is an isomorphism.

4. The comparison theorem

Theorem 4.1 (cf. [61, Theorem 0.2]). — *Let $\mathcal{X}$ be a proper scheme over $\mathcal{O}_K$ with semi-stable reduction, $X_{\overline{K}}$ its geometric generic fiber and $Y=\mathcal{X}_k$ its special fiber (endowed with the log-structure induced by that it defines on $\mathcal{X}$, cf. §6.4). Then there exists a canonical and functorial B_{st}-linear isomorphism*

$$\mathsf{B}_{\mathrm{st}}\otimes_{\mathbf{Q}_p}\mathsf{H}_{\mathrm{\acute{e}t}}^m(X_{\overline{K}},\mathbf{Q}_p)\xrightarrow{\sim}\mathsf{B}_{\mathrm{st}}\otimes_W\mathsf{H}_{\mathrm{log\text{-}cris}}^m(Y/W)$$

compatible with Frobenius maps, monodromy operators and filtrations after extending the scalars to B_{dR}. In other words, the p-adic representation $\mathsf{H}_{\mathrm{\acute{e}t}}^m(X_{\overline{K}},\mathbf{Q}_p)$ is semi-stable, and

$$\mathsf{D}_{\mathrm{st}}\left(\mathsf{H}_{\mathrm{\acute{e}t}}^m(X_{\overline{K}},\mathbf{Q}_p)\right)\simeq K_0\otimes_W\mathsf{H}_{\mathrm{log\text{-}cris}}^m(Y/W).$$

This was proved by Kato when $\dim(X) < \frac{p-1}{2}$ (*cf.* [48, Theorem 1.2]), by Tsuji in general, by Niziol and extended by Faltings to the case of non constant coefficients (see the introduction of [62] for more details). Tsuji's proof uses a generalization of Fontaine-Messing's syntomic cohomology (*cf.* [39]), symbols (defined by Bloch and Kato (*cf.* [14]) and relative versions of period rings (introduced by Faltings). Recently, it has been reproved and generalized by various authors:

- by Andreatta-Iovita (*cf.* [3] & [4]), using Faltings' ideas, based on the sheafification of period rings (in some appropriate topos);
- by Beilinson (*cf.* [6] & [7]), using "derived de Rham complexes" and "h-topology";
- by Yamashita and (*cf.* [64]) Yamashita-Yasuda in the case of open varieties;
- by Tan-Tong (*cf.* [58]) for the crystalline comparison theorem, based on ideas of Scholze (*cf.* [54]);
- by Colmez-Niziol (*cf.* [26]), using syntomic cohomology and the cohomology of (φ, Γ)-modules.

4.1. (Log-)crystalline interpretation of $\mathsf{B}_{\mathrm{cris}}$ and B_{st}. — If $n \in \mathbf{N}_{>0}$, put

$$\mathsf{H}^q_{\mathrm{cris}}\big(\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}|W_n\big) = \varinjlim_{L/K} \mathsf{H}^q\big((\mathcal{O}_L/p^n\mathcal{O}_L|W_n)_{\mathrm{cris}}, \mathscr{O}_{L,n}\big)$$

where the limit runs over finite extensions L/K in $\overline{K}$, and where $\mathscr{O}_{L,n}$ is the structure sheaf of $(\mathcal{O}_L/p^n\mathcal{O}_L|W_n)_{\mathrm{cris}}$.

Theorem 4.2 **(*cf.*** [33, Théorème 1]**).** —

$$\varprojlim_n \mathsf{H}^q_{\mathrm{cris}}\big(\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}|W_n\big) \cong \begin{cases} \mathsf{A}_{\mathrm{cris}} & \text{if } q=0;\\ 0 & \text{if } q>0.\end{cases}$$

Proof. — The first isomorphism follows from the fact that both $\mathsf{H}^0_{\mathrm{cris}}\big(\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}|W_n\big)$ and $\mathsf{A}_{\mathrm{cris}}/p^n\,\mathsf{A}_{\mathrm{cris}}$ are universal divided powers thickenings of $\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}$ in the category of W_n-algebras, and that $\mathsf{A}_{\mathrm{cris}}$ is p-adically separated and complete.

To compute $\mathsf{H}^q\big((\mathcal{O}_L/p^n\mathcal{O}_L|W_n)_{\mathrm{cris}}, \mathscr{O}_{L,n}\big)$, one embeds $\mathsf{Spec}(\mathcal{O}_L/p^n\mathcal{O}_L)$ into a smooth W_n-scheme: as $\mathcal{O}_L = W[x]$ is monogenic, we can write $\mathcal{O}_L/p^n\mathcal{O}_L \simeq W_n[X]/(f(X))$ with $f \in W_n[X]$, so that the crystalline cohomology of $\mathcal{O}_L/p^n\mathcal{O}_L$ is that of the de Rham complex with divided powers

$$0 \to D_{L,n} \to D_{L,n} \otimes_{W_n[X]} \Omega^1_{W_n[X]/W_n} \to D_{L,n} \otimes_{W_n[X]} \Omega^2_{W_n[X]/W_n} = 0 \to \cdots$$

where $D_{L,n}$ is the divided powers envelope of $W_n[X]$ with respect to the ideal generated by f. This implies that $\mathsf{H}^q\big((\mathcal{O}_L/p^n\mathcal{O}_L|W_n)_{\mathrm{cris}}, \mathscr{O}_{L,n}\big) = 0$ if $q \geq 2$. Also, if $y \in \overline{K}$ is such that $y^{p^n} = x$, and take the presentation $\mathcal{O}_{L(y)}/p^n\mathcal{O}_{L(y)} \simeq W_n[Y]/(f(Y^{p^n}))$, we have the commutative diagram

$$\begin{array}{ccccc} X & D_{L,n} & \longrightarrow & D_{L,n} \otimes_{W_n[X]} \Omega^1_{W_n[X]/W_n} & \mathrm{d}X \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ Y^{p^n} & D_{L(y),n} & \longrightarrow & D_{L(y),n} \otimes_{W_n[Y]} \Omega^1_{W_n[Y]/W_n} & \mathrm{d}Y^{p^n} = 0 \end{array}$$

which implies that the image of $\mathsf{H}^1\big((\mathcal{O}_L/p^n\mathcal{O}_L|W_n)_{\mathrm{cris}}, \mathscr{O}_{L,n}\big)$ in $\mathsf{H}^1\big((\mathcal{O}_{L(y)}/p^n\mathcal{O}_{L(y)}|W_n)_{\mathrm{cris}}, \mathscr{O}_{L,n}\big)$ is zero, hence $\mathsf{H}^1_{\mathrm{cris}}\big(\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}|W_n\big) = 0$, proving the case $q > 0$. □

Similarly, the ring B_{st} has a strong connection with log-crystalline cohomology. Endow $S = \mathsf{Spec}(\mathcal{O}_K)$ and $\overline{S} = \mathsf{Spec}(\mathcal{O}_{\overline{K}})$ with the canonical log-structures N and $\overline{N}$ respectively(8), and

(8) Note that the log-structure on $\overline{S}$ is not fine but only "integral" (*cf.* [48, 2.4]), which causes no trouble.

$S_n = \mathsf{Spec}(\mathcal{O}_K/p^n\mathcal{O}_K)$ and $\overline{S}_n = \mathsf{Spec}(\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}})$ with the inverse image log-structures N_n and $\overline{N}_n$ for $n \in \mathbf{N}_{>0}$. Following Kato (*cf.* [48, §3]), let

$$h\colon (\overline{S}_n, \overline{N}_n) \to (S_n, N_n)$$

and $h_{\mathrm{cris}}\colon \big((\overline{S}_n, \overline{N}_n)/W_n\big)_{\mathrm{cris}} \to \big((S_n, N_n)/W_n\big)_{\mathrm{cris}}$ the map induced on the associated log-crystalline topoi.

Proposition 4.3 (cf.** [48, Proposition 3.1]**). — *$h_{\mathrm{cris},*}\mathcal{O}_{\overline{S}_n/W_n}$ is a quasi-coherent flat crystal of $\mathcal{O}_{S_n/W_n}$-modules on $\big((S_n, N_n)/W_n\big)_{\mathrm{cris}}$, and for $q > 0$, one has $\mathsf{R}^q\, h_{\mathrm{cris},*}\mathcal{O}_{\overline{S}_n/W_n} = 0$.*

To describe the crystal $\mathscr{F} := h_{\mathrm{cris},*}\mathcal{O}_{\overline{S}_n/W_n}$, we embed (S_n, N_n) in a smooth object: let $Z_n = \mathsf{Spec}(W_n[T])$ endowed with the log-structure $\mathbf{N} \to W_n[T]; 1 \mapsto T$ (which is smooth over W_n), and $i_n\colon S_n \to Z_n$ the closed immersion given by $T \mapsto \pi$. Let $E_n = \mathsf{Spec}(R_n)$ be the PD-envelope of Z_n with respect to i_n. The crystal $\mathscr{F}$ is characterized by its evaluation $P_n := \mathscr{F}(E_n)$, which is an R_n-module with a connection with log poles

$$\nabla\colon P_n \to P_n \operatorname{dlog}(T)$$

(*cf.* theorem 6.6). Put $B_n = \mathsf{H}^0_{\mathrm{cris}}\big(\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}|W_n\big)$.

Proposition 4.4 (cf.** [48, Proposition 3.3]**). — (1) *For all p^n-th root β of π in $\mathcal{O}_{\overline{K}}$ there exists a canonical element $v_\beta \in P_n$ and an PD-isomorphism:*

$$\begin{aligned} B_n\langle V\rangle &\xrightarrow{\sim} P_n \\ V &\mapsto v_\beta - 1. \end{aligned}$$

(2) *∇ is the unique B_n-linear connection such that $\nabla((v_\beta-1)^{[i]}) = (v_\beta-1)^{[i-1]}\operatorname{dlog} T$ for all $i \in \mathbf{Z}_{>0}$.*
(3) *The Frobenius map on Z_n given by $T \mapsto T^p$ induces a Frobenius map $\varphi\colon P_n \to P_n$ that extends the Frobenius on B_n and satisfies $\varphi(v_\beta) = v_\beta^p$.*
(4) *The natural action of G_K on B_n extends to P_n, and $g(v_\beta) = v_{g(\beta)}$ for all $g \in G_K$.*

There exists a unique map[9] $\mathcal{N}\colon P_n \to P_n$ such that $(\forall x \in P_n)\ \nabla(x) = \mathcal{N}(x)\operatorname{dlog}(T)$. This is a B_n-derivation. Also, as $\mathsf{Ker}(P_n \to \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}})$ has divided powers, one can put $u_\beta = \log(v_\beta)$ for β as above. It is transcendental over B_n. Then

Theorem 4.5 (cf.** [48, Theorem 3.7]**). — *For all $i \in \mathbf{N}_{>0}$, one has $\mathsf{Ker}(\mathcal{N}^i) = \bigoplus_{j=0}^{i-1} B_n u_\beta^{[j]}$, so*

$$P_n^{\mathcal{N}\text{-nilp}} := \big\{x \in P_n \,\big|\, (\exists i \in \mathbf{N})\ \mathcal{N}^i(x) = 0\big\} = B_n\langle u_\beta\rangle$$

and there is a $\mathsf{B}^+_{\mathrm{cris}}$-linear isomorphism

$$\mathsf{B}^+_{\mathrm{st}} := \mathsf{B}^+_{\mathrm{cris}}\big[\log[\widetilde{\pi}]\big] \xrightarrow{\sim} \mathbf{Q} \otimes_{\mathbf{Z}} \varprojlim_n P_n^{\mathcal{N}\text{-nilp}}$$

which is compatible with the actions of G_K, φ and $\mathcal{N}$.

Remark 4.6. — The proofs are basically the same as that of theorem 4.2, though more technical because of log-structures.

(9) Denoted by $\mathcal{N}$ instead of N to avoid confusion with log-structures.

4.2. Reduction of the de Rham case to the semi-stable case. — (*cf.* [11] and [62, Appendix]).

Definition 4.7. — (1) Let X be an integral noetherian scheme. An *alteration* of X is a proper and surjective morphism $a\colon X' \to X$ with X' integral, such that there exists a non empty open $U \subset X$ above which a is finite. It is generically étale when one can choose U such that $f^{-1}(U) \to U$ is étale.

(2) Let R be a complete discrete valuation ring and $S = \mathsf{Spec}(R)$. An S-*variety* is a flat, integral, separated S-scheme of finite type.

(3) Let S as above, η (resp. s) its generic (resp. closed) point. An S-variety X is called *strictly semi-stable* when X_η is smooth over η, X_s is reduced, schematic union of its irreducible components $(X_{s,i})_{i\in I}$ and for all $\varnothing \neq J \subset I$, the s-scheme $X_J := \cap_{j\in J} X_{s,j}$ is smooth, of codimension $\#J$ in X.

(4) Let X as above, and $Z \subset X$ a closed subset containing X_s. Let Z_h be the schematic closure of Z_η, so that $Z = Z_h \cup X_s$. The pair (X, Z) is called *strictly semi-stable* if X is strictly semi-stable over S, Z is a strict normal crossing divisor in X and, if $(Z_{h,i})_{i\in I}$ are the irreducible components of Z_h, then for all $J \subset I$, $Z_{h,J} := \cap_{j\in J} Z_{h,j}$ is a union of strictly semi-stable S-varieties.

Theorem 4.8 **(*cf.*** [46, Theorem 6.5]**).** — *Let $f\colon X \to S$ an S-variety and $f^{-1}(s) \subseteq Z \subsetneq X$ a closed subset. There exist a discrete valuation ring R' which is finite over R, an alteration of S-varieties $a\colon X' \to X$ and an open immersion of S'-varieties $j\colon X' \hookrightarrow \overline{X}'$ (where $S' = \mathsf{Spec}(R')$) such that :*

(i) *$\overline{X}'$ is a projective S'-variety whose generic fiber is geometrically irreducible ;*

(ii) *the pair $(\overline{X}', a^{-1}(Z) \cup (\overline{X}' \setminus X'))$ is strictly semi-stable.*

$$\begin{tikzcd} \overline{X}' \arrow[d] & X' \arrow[l, hook', "j"'] \arrow[r, two heads, "a"] \arrow[d] & X \arrow[d, "f"] \\ S' \arrow[r, equal] & S' \arrow[r] & S. \end{tikzcd}$$

de Jong's and Tsuji's theorems imply C_{dR}. Indeed, let X be a $\mathsf{Spec}(\mathcal{O}_K)$-variety, whose generic fiber X_K is proper smooth over K, and $V = \mathsf{H}^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$: let's show that V is potentially semi-stable (hence de Rham). We may make a base change replacing K by a finite extension and assume X geometrically connected. By theorem 4.8, there exists a finite extension K'/K, a strictly semi-stable $\mathsf{Spec}(\mathcal{O}_{K'})$-variety X' and an alteration $a\colon X' \to X$. The map $a_{K'}\colon X'_{K'} \to X_{K'}$ is generically finite and flat, of some degree d. Being a proper morphism between smooth schemes of the same dimension, there is an associated trace map $\mathsf{Tr}_{a_{K'}}\colon \mathbb{R}\, a_{K'*}\, \mathbf{Q}_p \to \mathbf{Q}_p$, which composed with the canonical morphism $\mathbf{Q}_p \to \mathbb{R}\, a_{K'*}\, \mathbf{Q}_p$ is the multiplication by d (*cf.* [42, VII, Theorem 4.1]). This provides a projector of $\mathsf{H}^\bullet_{\text{ét}}(X'_{\overline{K}}, \mathbf{Q}_p) \to \mathsf{H}^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$, that makes $V = \mathsf{H}^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ a direct factor of $V' = \mathsf{H}^\bullet_{\text{ét}}(X'_{\overline{K}}, \mathbf{Q}_p)$, as $G_{K'}$-modules. As V' is a semi-stable representation of $G_{K'}$ by theorem 4.1, so is V. Extra arguments using algebraic cycles imply that $\mathsf{D}_{\mathrm{dR}}(V) \simeq \mathsf{H}^\bullet_{\mathrm{dR}}(X_K/K)$ (*cf.* [62, Appendix]).

5. The p-adic monodromy theorem

Definition 5.1. — $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ is *potentially semi-stable* if there exists a finite extension L/K such that the restriction $V_{|G_L}$ of V to G_L lies in $\mathrm{Rep}_{\mathrm{st}}(G_L)$.

Remark 5.2. — If $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ is potentially semi-stable, and L/K is as above, then $\dim_L \mathsf{D}^L_{\mathrm{dR}}(V) = \dim_{\mathbf{Q}_p}(V)$ where $\mathsf{D}^L_{\mathrm{dR}}(V) = (\mathsf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_L}$. One has $L \otimes_K \mathsf{D}_{\mathrm{dR}}(V) \cong \mathsf{D}^L_{\mathrm{dR}}(V)$ (Hilbert 90), so V is de Rham.

Conversely:

***Theorem 5.3* (The p-adic monodromy theorem).** — *Every de Rham representation of G_K is potentially semi-stable.*

5.1. Indications on the proof. — Using deep connections between p-adic Hodge theory and the theory of p-adic differential equations, Berger proved (*cf.* [8]) that theorem 5.3 is equivalent to a conjecture on p-adic differential equations. The latter was solved shortly after, independently by André, Kedlaya and Mebkhout (*cf.* theorem 5.7). The interested reader is advised to consult Colmez' extensive Bourbaki survey (*cf.* [23]). To avoid technicalities, we assume that $K = K_0$ for the rest of this section.

Interlude on p-adic differential equations. Recall the Robba ring (*i.e.* the ring of analytic functions which converge on some open annulus of outer radius 1, denoted by $\mathscr{R}$ in §3.6) is:

$$\mathscr{R}_K = \Big\{ f = \sum_{i\in\mathbf{Z}} a_i T^i \in K[\![T, T^{-1}]\!] \,\Big|\, f(T) \text{ converges on } r_f \leq |T| < 1 \text{ for some } r_f \in [0,1[\Big\}$$

and its bounded elements subring is

$$\mathscr{E}_K^\dagger = \Big\{ f(T) = \sum_{i\in\mathbf{Z}} a_i T^i \in K[\![T, T^{-1}]\!] \,\Big|\, f(T) \text{ converges}$$
$$\text{and is bounded on } r_f \leq |T| < 1 \text{ for some } r_f \in [0,1[\Big\}.$$

Note that $\mathscr{E}_K^\dagger$ is a (non complete) discrete valuation field with uniformizer p and residue field $k(\!(T)\!)$. Its ring of integers $\mathcal{O}_{\mathscr{E}_K^\dagger}$ is the subring of functions bounded by one. These rings are endowed with the Frobenius map φ given by

$$\varphi(T) = (1+T)^p - 1$$

and acting by σ on coefficients. They also carry an action of $\Gamma_K := \mathsf{Gal}(K_\infty/K)$ (where $K_\infty = \cup_{n\in\mathbf{N}} K(\varepsilon^{(n)})$ is the cyclotomic extension), given by

$$\gamma(T) = (1+T)^{\chi(\gamma)} - 1$$

for all $\gamma \in \Gamma_K$ (where χ denotes the cyclotomic character).

Remark 5.4. — (1) In general (*i.e.* when $K \neq K_0$), the coefficients should be taken in the maximal unramified extension of K_0 in K_∞ and the actions of φ and Γ_K are more complicated.

(2) In this section, the variable is denoted by T whereas is was u in section 3.6. The reason is that in the latter, the variable u was corresponding to $[\widetilde{\pi}]$, whereas here it corresponds to the cyclotomic "variable" $[\varepsilon] - 1$ (here again, this is a bit more complicated when $K \neq K_0$).

(3) In literature (especially [8]), the rings $\mathscr{R}_K$ and $\mathscr{E}_K^\dagger$ are often denoted $\mathbf{B}_{\mathrm{rig},K}^\dagger$ and $\mathbf{B}_K^\dagger$ respectively. These notations, due to Colmez, are quite useful when has to deal with many period rings.

Definition 5.5. — (1) A *∇-module over $\mathscr{R}_K$* is a free $\mathscr{R}$-module M of finite rank, which is endowed with a connection $\nabla\colon M \to M \otimes_{\mathscr{R}_K} \Omega^1_{\mathscr{R}_K/K_0'}$, *i.e.* a map such that $\nabla(f(T)m) = f(T)\nabla(m) + \frac{\mathrm{d}f}{\mathrm{d}T}(T)m \otimes \mathrm{d}T$ (where $\Omega^1_{\mathscr{R}_K/K} = \mathscr{R}_K\, \mathrm{d}T$).

(2) A *(φ, ∇)-module over $\mathscr{R}_K$* is a ∇-module (M, ∇) over $\mathscr{R}_K$ endowed with a φ-semi-linear operator $\Phi\colon M \to M$, which is compatible with the connection, *i.e.* such that the diagram

$$\begin{tikzcd} M \arrow[r, "\nabla"] \arrow[d, "\Phi"'] & M \otimes_{\mathscr{R}_K} \Omega^1_{\mathscr{R}_K/K} \arrow[d, "\Phi\otimes\mathrm{d}\varphi"] \\ M \arrow[r, "\nabla"] & M \otimes_{\mathscr{R}_K} \Omega^1_{\mathscr{R}_K/K} \end{tikzcd}$$

commutes.

(3) A (φ,∇)-module over $\mathscr{R}_K$ is *quasi-unipotent* when tensored by a finite étale extension $\mathscr{R}_L$ of $\mathscr{R}_K$ (corresponding to a finite separable extension of $k(\!(T)\!)$), it admits a filtration by sub-(φ,∇)-modules whose graded pieces are trivial, *i.e.* admit a basis made of horizontal elements.

Remark 5.6. — The definition of quasi-unipotency can be reformulated without filtrations by saying that $M\otimes_{\mathscr{R}_K}\mathscr{R}_L[\log(T)]$ is trivial, *i.e.* admits a basis made of horizontal elements.

Theorem 5.7. — *Every (φ,∇)-module over $\mathscr{R}_K$ is quasi-unipotent.*

This theorem was conjectured by Crew (*cf.* [27, 4.16.2] & [28, §10.1]), motivated by the study of rigid cohomology with coefficients in an overconvergent F-isocrystal (in particular, the theorem implies the finite dimensionality theorem, *cf.* [28] & [50]). It was proved by Tsuzuki in the unit-root case (*cf.* [63]), and in full generality, independently by André (*cf.* [1]), Kedlaya (*cf.* [49]) and Mebkhout (*cf.* [53]). Kedlaya's proof relies on the study of φ-modules on Robba rings, especially theorem 3.31.

From p-adic representations to p-adic differential equations. Let $V\in\operatorname{Rep}_{\mathbf{Q}_p}(G_K)$. If V is de Rham, Berger associates a (φ,∇)-module $\mathbf{N}_{\mathrm{dR}}(V)$ over $\mathscr{R}_K$, and shows that V is (potentially) semi-stable if and only if $\mathbf{N}_{\mathrm{dR}}(V)$ is (quasi-)unipotent, so that the p-adic monodromy theorem follows from theorem 5.7.

This construction uses (φ,Γ)-modules: there is an equivalence of categories

$$\mathsf{D}^\dagger\colon\operatorname{Rep}_{\mathbf{Q}_p}(G_K)\to\mathbf{Mod}^{\text{ét}}_{\mathscr{E}_K^\dagger}(\varphi,\Gamma)$$

where $\mathbf{Mod}^{\text{ét}}_{\mathscr{E}_K^\dagger}(\varphi,\Gamma)$ is the category of *étale (φ,Γ)-modules* over $\mathscr{E}_K^\dagger$, whose objects are finite dimensional $\mathscr{E}_K^\dagger$-vector spaces D endowed with commuting semi-linear actions of Γ_K and a Frobenius endomorphism $\Phi\colon D\to D$, such that there exists a sub-$\mathcal{O}_{\mathscr{E}_K^\dagger}$-module $\mathcal{D}\subset D$ of finite type such that $\Phi(\mathcal{D})\subset\mathcal{D}$ and the linearization $1\otimes\Phi\colon\varphi^*\mathcal{D}:=\mathcal{O}_{\mathscr{E}_{K,\varphi}^\dagger}\otimes_{\mathcal{O}_{\mathscr{E}_K^\dagger}}\mathcal{D}\to\mathcal{D}$ is an isomorphism (this is a refinement, due to Cherbonnier and Colmez, *cf.* [19] & [23, Proposition 5.3], of "classical" (φ,Γ)-module theory of Fontaine, *cf.* [34] & [20, §1]). As in classical Sen's theory (*cf.* [56] & [23, §3.3]), one considers the infinitesimal action of Γ_K on $\mathsf{D}^\dagger(V)$, given by:

$$\partial_0\colon d\mapsto\frac{\log(\gamma)}{\log(\chi(\gamma))}(d)=\lim_{n\to\infty}\frac{\gamma^n-1}{\chi(\gamma^n)-1}(d)$$

for any non torsion element $\gamma\in\Gamma_K$. This defines a differential operator on $\mathsf{D}^\dagger(V)$. An easy computation shows that $\partial_0(1+T)^i=i(1+T)^i\log(1+T)$ for all $i\in\mathbf{Z}$, so that $\partial_0=(1+T)\log(1+T)\frac{\mathrm{d}}{\mathrm{d}T}$ on $\mathscr{E}_K^\dagger$. As $\log(1+T)\in\mathscr{R}_K\setminus\mathscr{E}_K^\dagger$, this infinitesimal action does not make sense on $\mathsf{D}^\dagger(V)$, so one has to extend the scalars to $\mathscr{R}_K$: put

$$\mathsf{D}^\dagger_{\mathrm{rig}}(V)=\mathscr{R}_K\otimes_{\mathscr{E}_K^\dagger}\mathsf{D}^\dagger(V).$$

This is a free $\mathscr{R}_K$-module of rank $\dim_{\mathbf{Q}_p}(V)$ endowed with a Frobenius (because $\mathsf{D}^\dagger(V)$ and $\mathscr{R}_K$ are equipped with Frobenius maps) and a differential operator. However, in general, this does not provide a (φ,∇)-module over $\mathscr{R}_K$, because $\log(1+T)$ is not invertible in $\mathscr{R}_K$ (it has zeros all the $\zeta-1$ with $\zeta\in\mu_{p^\infty}(\overline{K})$): one would like to consider $\partial_V=\frac{1}{\log(1+T)}\partial_0$ instead. Because of the possible poles it might introduce, this is not always possible. Berger shows that it is when V is de Rham.

To do so, one has to study $(\mathsf{D}^\dagger_{\mathrm{rig}}(V),\partial_V)$ around $\varepsilon^{(n)}-1$ (for n large enough), and show that when V is de Rham, there are local solutions. To this end, one uses localization maps constructed by Fontaine in [37] as follows. Assume V is positive, *i.e.* has no non positive Hodge-Tate weights (so that $\mathsf{D}_{\mathrm{dR}}(V)=(\mathsf{B}^+_{\mathrm{dR}}\otimes_{\mathbf{Q}_p}V)^{G_K}$). It is always possible to reduce to this case by an appropriate

Tate twist. Applying Sen's method to $\mathsf{B}_{\mathrm{dR}}^+$-representations, Fontaine constructed a canonical free $K_\infty[\![t]\!]$-module $\mathsf{D}_{\mathrm{dif}}^+(V)\subset(\mathsf{B}_{\mathrm{dR}}^+\otimes_{\mathbf{Q}_p}V)^{H_k}$ of rank $\dim_{\mathbf{Q}_p}(V)$ (where $H_K=\mathsf{Ker}(\chi)\subset G_K$), stable under the residual action of Γ_K, such that

$$\mathsf{B}_{\mathrm{dR}}^+\otimes_{K_\infty[\![t]\!]}\mathsf{D}_{\mathrm{dif}}^+(V)\xrightarrow{\sim}\mathsf{B}_{\mathrm{dR}}^+\otimes_{\mathbf{Q}_p}V$$

(*cf.* [37, Théorème 3.6]). More precisely, there exists $n\in\mathbf{N}$ (depending on V) and a free $K_n[\![t]\!]$-module $\mathsf{D}_{\mathrm{dif},n}^+(V)$ carrying an action of Γ_K, such that $K_\infty[\![t]\!]\otimes_{K_n[\![t]\!]}\mathsf{D}_{\mathrm{dif},n}^+(V)\xrightarrow{\sim}\mathsf{D}_{\mathrm{dif}}^+(V)$ (where $K_n=K(\varepsilon^{(n)})\subset K_\infty$). Here again, the infinitesimal action of Γ_K provides a connection[10] ∇_V on $K_n(\!(t)\!)\otimes_{K_n[\![t]\!]}\mathsf{D}_{\mathrm{dif},n}(V)$. This connection is trivial if and only if V is de Rham, and then $\mathsf{D}_{\mathrm{dif},n}^+(V)=K_n[\![t]\!]\otimes_K\mathsf{D}_{\mathrm{dR}}(V)$ (*cf.* [37, Théorème 3.12]). One cannot map the Robba ring to $\mathsf{B}_{\mathrm{dR}}^+$ (*cf.* [9, IV.3.2] for enlightening heuristic analytic interpretations of period rings), but if

$$\mathscr{R}_{K,n}=\Big\{f=\sum_{i\in\mathbf{Z}}a_iT^i\in K[\![T,T^{-1}]\!]\;\Big|\;f(T)\text{ converges on }p^{-n}\leq|T|<1\Big\}$$

there is a map

$$\iota_n={}''\varphi^{-n}{}''\colon\mathscr{R}_{K,n}\to K_n[\![t]\!]\subset\mathsf{B}_{\mathrm{dR}}^+$$

which corresponds to the localization at $\varepsilon^{(n)}-1$. For n large enough (depending on V), the overconvergent (φ,Γ)-module $\mathsf{D}^\dagger(V)$ is defined over $\mathscr{E}_{K,n}^\dagger=\mathscr{E}^\dagger\cap\mathscr{R}_{K,n}$: there exist a sub-$\mathscr{E}_{K,n}^\dagger$-module $\mathsf{D}_n^\dagger(V)$ of $\mathsf{D}^\dagger(V)$, stable under φ and Γ, such that $\mathsf{D}^\dagger(V)=\mathscr{E}^\dagger\otimes_{\mathscr{E}_{K,n}^\dagger}\mathsf{D}_n^\dagger(V)$. Now there is an isomorphism

$$K_n[\![t]\!]_{\iota_n}\!\otimes_{\mathscr{E}_{K,n}^\dagger}\mathsf{D}_n^\dagger(V)\xrightarrow{\sim}\mathsf{D}_{\mathrm{dif},n}^+(V)$$

compatible with connections on both sides (*cf.* [8, Proposition 5.7]), which provides the localization of $\mathsf{D}_{\mathrm{rig},n}^\dagger(V)=\mathscr{R}_{K,n}\otimes_{\mathscr{E}_{K,n}^\dagger}\mathsf{D}_n^\dagger(V)$ at $\varepsilon^{(n)}-1$, and Fontaine's theorem [37, Théorème 3.12] mentioned above implies the existence of local solutions when V is de Rham. In this case, one puts

$$\mathbf{N}_{\mathrm{dR}}(V)=\mathscr{R}_K\otimes_{\mathscr{R}_{K,n}}\mathbf{N}_{\mathrm{dR},n}(V)$$

where

$$\mathbf{N}_{\mathrm{dR},n}(V)=\big\{x\in\mathsf{D}_{\mathrm{rig},n}^\dagger(V)\,\big|\,(\forall m\geq n),\ \iota_m(x)\in K_m[\![t]\!]\otimes_K\mathsf{D}_{\mathrm{dR}}(V)\big\}.$$

This is a (φ,∇)-module of rank $\dim_{\mathbf{Q}_p}(V)$ over $\mathscr{R}_K$ (*cf.* [8, Théorème 5.20]).

The quasi-unipotent case. Using an enlarged Robba ring, Berger proves that if $V\in\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, one has

$$\mathsf{D}_{\mathrm{st}}(V)=\big(\mathscr{R}_K[\log(T)]\otimes_{\mathscr{E}_K^\dagger}\mathsf{D}^\dagger(V)\big)^{\Gamma_K}$$

where the action Γ_K on $\log(T)$ is given by $\gamma(\log(T))=\log(T)+\log\big(\frac{\gamma(T)}{T}\big)$ (note that $\log\big(\frac{\gamma(T)}{T}\big)\in\mathscr{R}_K$). The Frobenius φ extends semi-linearly to $\mathscr{R}_K[\log(T)]$ by $\varphi(\log(T))=p\log(T)+\log\big(\frac{\varphi(T)}{T^p}\big)$ (note that $\log\big(\frac{\varphi(T)}{T^p}\big)\in\mathscr{R}_K$), and one endows $\mathscr{R}_K[\log(T)]$ with the $\mathscr{R}_K$-linear derivation N given by $N(\log(T))=-\frac{p}{p-1}$.

Remark 5.8. — Recall that here the variable T corresponds to $[\varepsilon]-1$ (*cf.* remark 5.4 (2)), so that $\log(1+T)$ corresponds to $t\in\mathsf{B}_{\mathrm{dR}}^+$, and $\log(T)$ to $\log([\varepsilon]-1)$. Now $\frac{[\varepsilon]-1}{[\varepsilon-1]}$ is overconvergent (*i.e.* corresponds to an element in $\mathscr{E}_K^\dagger$, *cf.* [19, Corollaire II.1.5]), as $\varepsilon-1\in\widetilde{p}^{\frac{p}{p-1}}\mathcal{R}^\times$ (because $v\big((\varepsilon^{(n)}-1)^{p^n}\big)=\frac{p}{p-1}$ for all $n\in\mathbf{N}_{>0}$), the element $\log(T)$ corresponds to $\frac{p}{p-1}\log[\widetilde{p}]$ (modulo an element in $\mathsf{B}_{\mathrm{cris}}$), so that this definition matches with that of section 3.2.

[10] Coming from a logarithmic connection $\mathsf{D}_{\mathrm{dif},n}^+(V)\to\mathsf{D}_{\mathrm{dif},n}^+(V)\otimes\frac{\mathrm{d}t}{t}$, *cf.* [37, Proposition 3.7].

The Frobenius φ and monodromy operator N on $\mathsf{D}_{\rm st}(V)$ are deduced from the corresponding structures (*cf.* [8, Théorème 3.6]). When V is de Rham, this interprets $\mathsf{D}_{\rm st}(V)$ as the set of horizontal sections of $\mathbf{N}_{\rm dR}(V)[\log(T)]$ (and then one has $\mathbf{N}_{\rm dR}(V) = \big(\mathscr{R}_K[\log(T)]\otimes_K \mathsf{D}_{\rm st}(V)\big)^{N=0}$). In particular, V is semi-stable if and only if the (φ,∇)-module $\mathbf{N}_{\rm dR}(V)$ is unipotent (*cf.* remark 5.6). By theorem 5.7, this always holds after a finite extension of K, *i.e.* V is potentially semi-stable.

5.2. Other proofs. — Theorem 5.3 has been reproved by Colmez (*cf.* [24]) using Galois cohomology of some period rings (instead of using p-adic differential equations or (φ,Γ)-modules) to reduce to the case case of weight zero (that corresponds to the isoclinic case, *i.e.* Tsuzuki's theorem in the point of view of Kedlaya), which is the following:

Theorem 5.9. — (Sen, [55, Theorem 1]) *C-admissible representations are potentially unramified (hence potentially semi-stable) representations.*

Fontaine also gave a proof starting from Sen's result, by induction on the Hodge polygon of V (*cf.* [38]).

5.3. Applications. — Using Hilbert 90, a straightforward consequence of the p-adic monodromy theorem is:

Corollary 5.10 (cf. [8, Théorème 6.2]). — *Let $0\to V'\to V\to V''\to 0$ be an exact sequence in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$. Assume V' and V'' are semi-stable, and that V is de Rham. Then V is in fact semi-stable.*

Definition 5.11. — A *filtered (φ,N,G_K)-module over K* is a finite dimensional $K_0^{\rm nr}$-vector space D, endowed with

- a semi-linear action of G_K which is continuous for the discrete topology ;
- a bijective G_K-equivariant σ-semi-linear Frobenius operator $\varphi\colon D\to D$;
- a G_K-equivariant linear monodromy operator $N\colon D\to D$ such that $N\varphi=p\varphi N$;
- a decreasing, separated, exhaustive filtration on $D_K := (\overline{K}\otimes_{K^{\rm nr}} D)^{G_K}$.

Let $\mathbf{MF}_K(\varphi,N,G_K)$ be the category of filtered (φ,N,G_K)-modules over K. An object $D\in\mathbf{MF}_K(\varphi,N,G_K)$ is said *admissible* if $D^{G_L}\in\mathbf{MF}_L^{\mathsf{ad}}(\varphi,N)$ for some (any) L large enough. This defines a subcategory $\mathbf{MF}_K^{\mathsf{ad}}(\varphi,N,G_K)$ of $\mathbf{MF}_K(\varphi,N,G_K)$.

Let $V\in\mathrm{Rep}_{\rm dR}(G_K)$. If L/K is a finite Galois extension such that $V_{|G_L}$ is semi-stable then $\mathsf{D}_{\rm st}^L(V)=(\mathsf{B}_{\rm st}\otimes_{\mathbf{Q}_p}V)^{G_L}\in\mathbf{MF}_L^{\mathsf{ad}}(\varphi,N)$ is endowed with an action of $\mathsf{Gal}(L/K)$ that commutes with φ and N. Put

$$\mathsf{D}_{\rm pst}(V)=\bigcup_{L/K}(\mathsf{B}_{\rm st}\otimes_{\mathbf{Q}_p}V)^{G_L}.$$

For L as above, one has $\mathsf{D}_{\rm pst}(V)=K_0^{\rm nr}\otimes_{L_0}\mathsf{D}_{\rm st}^L(V)$ (by Hilbert 90): this is a finite $K_0^{\rm nr}$-vector space endowed with a discrete action of G_K, and equivariant operators φ and N as above. Furthermore, one has $\overline{K}\otimes_{K_0^{\rm nr}}\mathsf{D}_{\rm pst}(V)\simeq\overline{K}\otimes_{L_0}\mathsf{D}_{\rm st}^L(V)\simeq\overline{K}\otimes_L(L\otimes_{L_0}\mathsf{D}_{\rm st}^L(V))\simeq\overline{K}\otimes_K\mathsf{D}_{\rm dR}(V)$, so $(\overline{K}\otimes_{K_0^{\rm nr}}\mathsf{D}_{\rm pst}(V))^{G_K}\simeq\mathsf{D}_{\rm dR}(V)$. Note that $\mathsf{D}_{\rm pst}(V)^{G_L}=\mathsf{D}_{\rm st}^L(V)\in\mathbf{MF}_L^{\mathsf{ad}}(\varphi,N)$, so that $\mathsf{D}_{\rm pst}(V)\in\mathbf{MF}_K^{\mathsf{ad}}(\varphi,N,G_K)$.

Theorem 5.12. — *The functor $\mathsf{D}_{\rm pst}$ induces an equivalence of tensor categories*

$$\mathsf{D}_{\rm pst}\colon\ \mathrm{Rep}_{\rm dR}(G_K)\xrightarrow{\approx}\mathbf{MF}_K^{\mathsf{ad}}(\varphi,N,G_K).$$

A quasi-inverse is given by

$$D\mapsto\mathsf{V}_{\rm pst}(D):=(\mathsf{B}_{\rm st}\otimes_{K_0^{\rm nr}}D)^{\varphi=1}_{N=0}\cap\mathsf{Fil}^0(\mathsf{B}_{\rm dR}\otimes_K D_K)$$

(where the action of G_K on $\mathsf{B}_{\mathrm{st}} \otimes_{K_0^{\mathrm{nr}}} D$ is the diagonal one).

Proof. — This is the conjunction of theorems 3.28 and 5.3. □

Let X be a proper and smooth variety on K: the étale cohomology $V = \mathsf{H}^\bullet_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ is de Rham hence potentially semi-stable. The preceding theorem shows that one can recover V from $D_K = \mathsf{D}^\bullet_{\mathrm{dR}}(X/K)$ and the "hidden structure" $D = \mathsf{D}_{\mathrm{pst}}(V)$".

6. Appendix: Inputs from log-geometry

6.1. Basic definitions. — Developed by Kato *and al.* following ideas of Fontaine and Illusie, *log-geometry* is an enlargement of algebraic geometry: one can see the category of schemes as a full subcategory of *log-schemes*, which are schemes with an extra structure, namely a *log-structure*, which basically consists in keeping track of (local) equations of divisors. This allows to deal with normal crossing divisors or K-schemes with semi-stable reduction as smooth objects (hence it is also related to desingularization). What follows is a very cursory survey of the basic ideas, and is taken from [47] and [43, §2] (see also the nice survey [45] by Illusie, and [62, §3]).

All monoids will be assumed to be commutative, and morphisms of monoids preserve unit elements. If M is a monoid, M^{gp} denotes the associated group, and M is called *integral* when the map $M \to M^{\mathsf{gp}}$ is injective.

Definition 6.1. — (i) Let X be a scheme. A *pre-logarithmic structure* on X is a sheaf of monoids $\mathcal{M}$ on $X_{\text{ét}}$ and a homomorphism of monoids $\alpha\colon \mathcal{M} \to \mathcal{O}_X$ (for the *multiplicative* law on $\mathcal{O}_X$). A pre-logarithmic structure is *logarithmic structure* (or *log-structure* for short) if moreover the map α induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\sim} \mathcal{O}_X^\times$. The *trivial* log-structure is $\mathcal{O}_X^\times \hookrightarrow \mathcal{O}_X$.

(ii) A morphism between schemes X and Y with (pre-)log structures $(\mathcal{M}, \alpha)$ and $(\mathcal{N}, \beta)$ respectively is a morphism of schemes $f\colon X \to Y$ and a commutative square

$$\begin{tikzcd} f^{-1}\mathcal{N} \arrow[r, "f^{-1}\beta"] \arrow[d] & f^{-1}\mathcal{O}_Y \arrow[d] \\ \mathcal{M} \arrow[r, "\alpha"] & \mathcal{O}_X \end{tikzcd}$$

of arrows in $X_{\text{ét}}$.

(iii) A *log-scheme* $\underline{X} = (X, \mathcal{M})$ is a scheme X endowed with a log-structure[11] $(\mathcal{M}, \alpha)$.

Given a pre-log structure $(\mathcal{M}, \alpha)$ on a scheme X, the *associated* log-structure is the push-out of

$$\mathcal{O}_X^\times \leftarrow \alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{M}$$

in the category of sheaves of monoids, *i.e.*

$$\mathcal{M}^{\mathsf{a}} = (\mathcal{O}_X^\times \oplus \mathcal{M})/\sim \xrightarrow{(\mathrm{incl.}, \alpha)} \mathcal{O}_X$$

where $(u, a) \sim (v, b) \Leftrightarrow$ there locally exists $c, d \in \alpha^{-1}(\mathcal{O}_X^\times)$ such that $\alpha(c)u = \alpha(d)v$ and $ad = bc$.

Definition 6.2. — A log-structure $\mathcal{M}$ on X is called *fine* if étale locally on X, it is isomorphic to the log-structure associated to $M_X \to \mathcal{O}_X$ (which is then called a *chart* of $\mathcal{M}$) for some finitely generated integral monoid M (where M_X denotes the constant sheaf defined by M). One defines the *chart* of a morphism of fine log-schemes in the obvious way.

[11] As often, the map α does not appear in the notation.

Example 6.2. — *Assume X is regular. Let D be a reduced normal crossing divisor*[12] *on X, and $j\colon U := X \setminus D \to X$. Then the inclusion $\mathcal{M} := \mathcal{O}_X \cap j_*\mathcal{O}_U^\times \to \mathcal{O}_X$ is a fine log-structure, locally associated to*

$$\begin{aligned} \mathbf{N}^r &\to \mathcal{O}_X \\ (n_i)_{1\le i\le r} &\mapsto t_i^{n_i} \end{aligned}$$

where $t_i = 0$ are equations of the components of D. A special case is that of $\mathsf{Spec}(R)$ *where R is a discrete valuation ring. The log-structure corresponding to the special fiber (i.e. associated to $\mathbf{N} \to R$; $1 \mapsto \varpi$, where ϖ is a uniformizer of R) is called the* canonical *log-structure on* $\mathsf{Spec}(R)$. *More generally, if $X \to \mathsf{Spec}(R)$ has semi-stable reduction, its special fiber defines a fine log-structure on X.*

Definition 6.3. — Given a morphism of log-schemes $f\colon \underline{X} = (X, \mathcal{M}) \to \underline{Y} = (Y, \mathcal{N})$, the *sheaf of differentials with logarithmic poles relative to f* is the $\mathcal{O}_X$-module $\omega^1_{\underline{X}/\underline{Y}} = \omega^1_{(X,\mathcal{M})/(Y,\mathcal{N})}$ quotient of

$$\Omega^1_{X/Y} \oplus \big(\mathcal{O}_X \otimes_{\mathbf{Z}} \mathcal{M}^{\mathsf{gp}}\big)$$

by the $\mathcal{O}_X$-submodule generated by sections of the form $(\mathrm{d}\alpha(m), 0) - (0, \alpha(m) \otimes m)$ for $m \in \mathcal{M}$ and $(0, 1 \otimes m')$ for $m' \in f^{-1}(\mathcal{N})$. For $m \in \mathcal{M}$, the image of $(0, 1 \otimes m)$ is denoted by $\mathrm{dlog}(m)$. We get a logarithmic de Rham complex by putting $\omega^n_{\underline{X}/\underline{Y}} = \wedge^n \omega^1_{\underline{X}/\underline{Y}}$ and defining the derivation $\mathrm{d}\colon \omega^n_{X/Y} \to \omega^{n+1}_{X/Y}$ by

$$\mathrm{d}\big(\omega \wedge \mathrm{dlog}(m_1) \wedge \cdots \wedge \mathrm{dlog}(m_r)\big) = \mathrm{d}\omega \wedge \mathrm{dlog}(m_1) \wedge \cdots \wedge \mathrm{dlog}(m_r)$$

for $\omega \in \Omega^{n-r}_{X/Y}$ and $m_1, \ldots, m_r \in \mathcal{M}$.

Definition 6.4. — Let $f\colon (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism of schemes with fine log-structures.

(i) f is a *closed immersion* (resp. *exact closed immersion*) if $f\colon X \to Y$ is a closed immersion of schemes and $f^*\mathcal{N} \to \mathcal{M}$ is surjective (resp. bijective), where $f^*\mathcal{N}$ denotes the log-structure associated to the pre-log structure $f^{-1}\mathcal{N} \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

(ii) f is *smooth* (resp. *étale*) if f is locally of finite presentation (as morphism of schemes) and formally smooth (resp. formally étale), *i.e.* for all commutative square

$$\begin{tikzcd} (T', \mathcal{L}') \arrow[r] \arrow[d, "i"'] & (X, \mathcal{M}) \arrow[d, "f"] \\ (T, \mathcal{L}) \arrow[r] \arrow[ur, dotted, "g"] & (Y, \mathcal{N}) \end{tikzcd}$$

where i is an exact closed immersion whose ideal is nilpotent, then étale locally on T, there exists (resp. there exists a unique) morphism $g\colon (T, \mathcal{L}) \to (X, \mathcal{M})$ making the two triangles commute. In that case, the $\mathcal{O}_X$-module $\omega^1_{X/Y}$ is locally free of finite type.

Example 6.3. — *Let R be a discrete valuation ring, k its residue field and $\mathcal{N}$ the canonical log-structure on $Y = \mathsf{Spec}(R)$. Assume $X \to \mathsf{Spec}(R)$ has semi-stable reduction: let $\mathcal{M}$ be the log-structure on X associated to its special fiber X_k (so that $\mathcal{M}$ is the subsheaf of $\mathcal{O}_X$ made of sections that are invertible on X_k). Then the morphisms of log-schemes $(X, \mathcal{M}) \to (Y, \mathcal{N})$ and its special fiber $(X_k, \overline{\mathcal{M}}) \to (\mathsf{Spec}(k), \overline{\mathcal{N}})$ are smooth. For instance, assume that $X = \mathsf{Spec}(A)$ with $A = R[t_0, \ldots, t_n]/(t_0 \cdots t_r - \pi)$ where $0 \le r \le n$ are integers. The equality $t_0 \cdots t_r = \pi$ in $\mathcal{M}^{\mathsf{gr}}$*

(12) *I.e.* for all closed point x of X, the divisor D has equation $t_1 \cdots t_r = 0$ where $t_1, \ldots, t_r$ is part of a regular sequence of parameters of X at x.

translates into $\sum_{j=0}^{r}\operatorname{dlog}(t_j)=\operatorname{dlog}(\pi)=0$ *in* $\omega^1_{\underline{X}/\underline{Y}}$. *Multiplied by* π, *one recovers the relation* $\sum_{j=0}^{r} t_0\cdots t_{j-1}t_{j+1}\cdots t_r\,\mathrm{d}t_j=0$ *in* $\Omega^1_{X/Y}$*: we have*

$$\begin{aligned}\omega^1_{\underline{X}/\underline{Y}} &= \big(\oplus_{i=0}^{n} A\,\mathrm{d}t_i \oplus \oplus_{i=0}^{r} A\operatorname{dlog}(t_i)\big)/\big(\textstyle\sum_{j=0}^{r}\operatorname{dlog}(t_j),\mathrm{d}t_i - t_i\operatorname{dlog}(t_i)\big)_{0\le i\le r}\\ &= \oplus_{i=1}^{r} A\operatorname{dlog}(t_i)\oplus\oplus_{i=r+1}^{n} A\operatorname{dlog}(t_i)\end{aligned}$$

6.4. Log-crystalline cohomology. — The crystalline theory (*cf.* [12]) can be extended to the logarithmic case, *cf.* [47, §5 & 6]. Let $f\colon \underline{X}=(X,\mathcal{M})\to\underline{S}=(S,\mathcal{N})$ be a morphism of schemes with fine log-structures. We assume that p is nilpotent on S. Let $\mathcal{I}\subset\mathcal{O}_S$ be a quasi-coherent ideal and γ a PD-structure on $\mathcal{I}$ that extends to X.

Definition 6.5. — The *crystalline site* $\big(\underline{X}/\underline{S}\big)_{\mathrm{cris}}$ is the site whose objects are diagrams

$$\begin{tikzcd}(U,\mathcal{M}_{|U}) \arrow[r, hook, "i"] \arrow[d, "f"'] & (T,\mathcal{M}_T) \arrow[d] \\ (X,\mathcal{M}) \arrow[r] & (S,\mathcal{N})\end{tikzcd}$$

where U is étale over X, $\mathcal{M}_T$ a fine log-structure on T, and i an exact closed immersion; and the data of a PD-structure δ on the ideal of i, compatible with γ. The morphisms are the obvious ones, and covering families are $\big\{g_\lambda\colon (U_\lambda,T_\lambda,\mathcal{M}_{T_\lambda},i_\lambda,\delta_\lambda)\to(U,T,\mathcal{M}_T,i,\delta)\big\}_\lambda$ such that for all λ, the map $T_\lambda\to T$ is étale, $U_\lambda\simeq T_\lambda\otimes_T U$ and $\{T_\lambda\to T\}_\lambda$ is a covering for the étale topology. The *structure sheaf* is defined by

$$\mathscr{O}_{\underline{X}/\underline{S}}(U,T,\mathcal{M}_T,i,\delta)=\Gamma(T,\mathcal{O}_T)$$

A sheaf of $\mathscr{O}_{\underline{X}/\underline{S}}$-modules $\mathscr{F}$ on $\big(\underline{X}/\underline{S}\big)_{\mathrm{cris}}$ is a *crystal* if the transition maps[13]

$$g^*\mathscr{F}_T\to\mathscr{F}_{T'}$$

are isomorphisms for all $g\colon T'\to T$ in $\big(\underline{X}/\underline{S}\big)_{\mathrm{cris}}$.

Let $i\colon(X,\mathcal{M})\to(X',\mathcal{M}')$ be a closed immersion. The associated PD-envelope

$$(X,\mathcal{M})\to(D,\mathcal{M}_D)\to(X',\mathcal{M}')$$

is defined by the usual universal property. One just has to be careful about the exactness condition on closed immersions. When i is exact, this is simply the usual PD-envelope D of X in X', endowed with the inverse image of $\mathcal{M}'$. A general i admits (étale locally on X) a factorization

$$\begin{tikzcd}(X,\mathcal{M}) \arrow[rr, "i"] \arrow[dr, "i'"'] & & (X',\mathcal{M}') \\ & (X'',\mathcal{M}'') \arrow[ur, "g"'] & \end{tikzcd}$$

with i' an exact closed immersion and g étale. Then D is the PD-envelope of X in X'' and $\mathcal{M}_D$ is the inverse image of $\mathcal{M}'$.

Theorem 6.6 **(*cf.*** [47, Theorem 6.2]**)**. — *Assume there is a diagram of morphisms of schemes with fine log-structures*

$$\begin{tikzcd}\underline{X} \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & \underline{Y} \arrow[dl, "g"] \\ & \underline{S} & \end{tikzcd}$$

[13] Where, as usual, $\mathscr{F}_T$ is the sheaf on $T_{\text{ét}}$ induced by $\mathscr{F}$.

with g smooth and i a closed immersion. Let $\underline{D}=(D,\mathcal{M}_D)$ be the PD-envelope of i. Then the category of crystals on $\big(\underline{X}/\underline{S}\big)_{\mathrm{cris}}$ is equivalent to the category of $\mathcal{O}_D$-modules $\mathcal{E}$ endowed with an integrable connection[14]

$$\nabla\colon \mathcal{E}\to\mathcal{E}\otimes_{\mathcal{O}_Y}\omega^1_{Y/S}$$

such that, for all $x\in D$ with image $y\in Y$, if $(t_i)_{1\leq i\leq n}\in\big(\mathcal{N}^{\mathrm{gp}}_{\bar y}\big)^r$ are such that $\{\mathrm{dlog}(t_i)\}_{1\leq i\leq r}\cup\{\mathrm{d}t_i\}_{r<i\leq n}$ is a basis of $\omega^1_{Y/S,\bar y}$, for all $x\in\mathcal{E}_{\bar x}$, there exist $c_1,\ldots,c_k\in\mathbf{N}$, $(n_{i,j})_{\substack{1\leq i\leq r\\ 1\leq j\leq k}}\in\mathbf{N}^{rk}$ and $n_{r+1},\ldots,n_n\in\mathbf{N}$ such that

$$\Big(\prod_{i=1}^r\prod_{j=1}^k(\nabla_i^{\log}-c_j)^{n_{i,j}}\Big)\Big(\prod_{i=r+1}^n\nabla_i^{n_i}\Big)(x)=0$$

(where $\nabla=\sum\limits_{i=1}^r\nabla_i^{\log}\otimes\mathrm{dlog}(t_i)+\sum\limits_{i=r+1}^n\nabla_i\otimes\mathrm{d}t_i$ on $\mathcal{E}_{\bar x}$).

Remark 6.7. — (1) If $\mathscr{E}$ is a crystal on $\big(\underline{X}/\underline{S}\big)_{\mathrm{cris}}$, the corresponding $\mathcal{O}_D$-module is $\mathscr{E}_D$.
(2) The pair $(\mathcal{E},\nabla)$ provides a de Rham complex $\mathcal{E}\otimes_{\mathcal{O}_Y}\omega^\bullet_{\underline{Y}/\underline{S}}$.

As "usual", log-crystalline cohomology can be computed by de Rham complexes with divided powers (*cf.* [12, Proposition 5.18 & Theorem 7.1]): let

$$u_{\underline{X}/\underline{S}}\colon \big(\underline{X}/\underline{S}\big)_{\mathrm{cris}}\to X_{\text{ét}}$$

be the map of topoi defined by

$$u_{\underline{X}/\underline{S}}(\mathscr{F})(U)=\Gamma\big((\underline{U}/\underline{S})_{\mathrm{cris}},\mathscr{F}\big)$$

where $\underline{U}$ is the scheme U endowed with the log-structure obtained by the pull-back of the log-structure on X.

Theorem 6.8 (cf.** [47, Theorem 6.4]**)). — *Under the hypothesis of theorem 6.6, if $\mathscr{E}$ is a crystal on $\big(\underline{X}/\underline{S}\big)_{\mathrm{cris}}$ and $\mathcal{E}=\mathscr{E}_D$ the corresponding $\mathcal{O}_D$-module with connection, there is a canonical isomorphism*

$$\mathsf{R}\,u_{\underline{X}/\underline{S}\,*}(\mathscr{F})\overset{\sim}{\to}\mathcal{E}_D\otimes_{\mathcal{O}_D}\omega^\bullet_{\underline{Y}/\underline{S}}$$

Recall k is a perfect field of characteristic p. For $n\in\mathbf{Z}_{>0}$, let $\underline{S}_n$ be the scheme $\mathsf{Spec}(\mathsf{W}_n(k))$ endowed with the log-structure associated to $\mathbf{N}\to\mathsf{W}_n(k)$ mapping 1 to 0 (then $\underline{S}_1$ is the *standard log point*). Assume $f\colon\underline{X}\to\underline{S}_1$ is smooth. We put

$$\mathsf{H}^i\big(\underline{X}/\,\mathsf{W}_n(k)\big):=\mathsf{H}^i\big((\underline{X}/\underline{S}_n)_{\mathrm{cris}},\mathcal{O}_{\underline{X}/\underline{S}_n}\big)\quad\text{and}\quad\mathsf{H}^i\big(\underline{X}/W\big):=\varprojlim_n\mathsf{H}^i\big(\underline{X}/\underline{S}_n\big)$$

which are a $\mathsf{W}_n(k)$-module and a W-module respectively. When X is proper over k, the latter is finitely generated (*cf.* [43, §3.2]).

By functoriality, the absolute Frobenius

$$F\colon\big(\underline{X}/\underline{S}_n\big)\to\big(\underline{X}/\underline{S}_n\big)$$

(the absolute Frobenius between the underlying schemes and the p-th power on the monoids) induces a σ-semi-linear Frobenius map

$$\varphi\colon\mathsf{H}^i(\underline{X}/\,\mathsf{W}_n(k))\to\mathsf{H}^i(\underline{X}/\,\mathsf{W}_n(k))\quad\text{and}\quad\varphi\colon\mathsf{H}^i(\underline{X}/W)\to\mathsf{H}^i(\underline{X}/W).$$

[14] *I.e.* an additive map ∇ such that $\nabla(ax)=a\nabla(x)+x\otimes\mathrm{d}a$ for all $a\in\mathcal{O}_D$ and $x\in\mathcal{E}$, and $\nabla^{(1)}\circ\nabla=0$, where

$$\begin{aligned}\nabla^{(n)}\colon\mathscr{M}\otimes\omega^n_{Y/S}&\to\mathscr{M}\otimes\omega^{n+1}_{Y/S}\\ x\otimes\omega&\mapsto\nabla(x)\wedge\omega+x\otimes\mathrm{d}\omega\end{aligned}$$

When X is *of Cartier type*[15] (*cf.* [43, §2.12]), the map φ is an isogeny, *i.e.* $\varphi\otimes\mathbf{Q}$ is an isomorphism (*cf.* [43, Proposition 2.24]).

Another structure on these log-crystalline cohomology spaces is that of a *monodromy operator*. Denote by $\underline{S}_n^{\mathsf{triv}}$ the scheme $\mathsf{Spec}(\mathsf{W}_n(k))$ endowed with the *trivial* log-structure. Then $\mathsf{R}\,f_*\mathcal{O}_{\underline{X}/\underline{S}_n^{\mathsf{triv}}}$ is a crystal in the derived category of $(\underline{S}_1/\underline{S}_n^{\mathsf{triv}})$: let $\mathscr{R}_n$ be its evaluation at $\underline{S}_1\to\underline{S}_n^{\mathsf{triv}}$, then

$$\mathsf{H}^i(\underline{X}/\mathsf{W}_n(k))=\mathsf{H}^i(\mathscr{R}_n)$$

which can be computed by using the closed embedding $i_n\colon\underline{S}_1\hookrightarrow(\mathbf{A}^1_{\mathsf{W}_n(k)},\mathcal{N})$ (where $\mathcal{N}$ is the log-structure given the divisor $(t=0)$). Then $\mathscr{R}_n$ is described by a complex on the divided power envelope of i_n, together with a connection with log poles at $t=0$, the residue of which induces a map

$$N\colon\mathsf{H}^i(\underline{X}/\mathsf{W}_n(k))\to\mathsf{H}^i(\underline{X}/\mathsf{W}_n(k))\quad\text{ and }\quad N\colon\mathsf{H}^i(\underline{X}/W)\to\mathsf{H}^i(\underline{X}/W)$$

on cohomology. One has $N\varphi=p\varphi N$ (*cf.* [43, §3.6]).

An other relation with de Rham cohomology, on the generic fiber this time, is the generalization of Berthelot-Ogus isomorphism (*cf.* [13, Corollary 2.5]). Let $\mathcal{X}$ a proper scheme with semi-stable reduction, endowed with the log-structure defined by its special fiber $Y=\mathcal{X}_k$, and $X=\mathcal{X}_K$ its generic fiber. We endow Y and X with the induced log-structures (here $\mathsf{Spec}(K)$ is endowed with the trivial log-structure). We write $\mathsf{H}^i_{\log\text{-}\mathrm{cris}}(Y/W)$ for $\mathsf{H}^i(\underline{Y}/W)$. The choice[16] of the uniformizer π provides an isomorphism (for a construction in a more general context, see [43, Theorem 5.1])

$$\rho_\pi\colon\mathsf{H}^i_{\log\text{-}\mathrm{cris}}(Y/W)\otimes_W K\xrightarrow{\sim}\mathsf{H}^i_{\mathrm{dR}}(X/K).$$

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