ON A CONJECTURE BY KOLLÁR AND SACCÀ

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Abstract. In this note we study smooth commutative group schemes over curves, whose generic fibre is an abelian variety. We prove a modified version of the conjecture proposed in [KS25].

Contents

Introduction

The goal of this note is prove the following statement for smooth commutative group schemes over smooth curves, conjectured by Kollár and Saccà in [KS25].

Theorem. *(=Corollary 2.5) Let C be a geometrically connected, smooth projective curve over a field k, and A/C a smooth commutative group scheme with connected fibres, whose generic fibre A_η is an abelian variety which has no abelian subvarieties defined over k.*
Let Z_0 be the zero section and $MW(A/C)$ the Mordell-Weil group. Then the following set map

$$\mathrm{MW}(A/C)/\mathrm{tors} \longrightarrow N_1(A)$$

$$Z \mapsto [Z] - [Z_0]$$

is injective.

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We remark that the previous natural map is *not* a group homomorphism and to prove the injectivity-modulo-torsion we introduce a weakened version of numerical equivalence on 1-cycles (see Definition 2.3 and Theorem 2.4).

The key inputs of our approach are

- extension results for cubist $\mathbb{G}_m$-torsors on connected group schemes (see Section 1); and
- the theory of heights developed for the proof of the Lang-Néron theorem (see [Lan83, Chapter 6]).

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1. Extensions of cubist invertible sheaves

Let k be a field and S an irreducible normal scheme over k. In this note we denote by A a smooth commutative group scheme over S. If we assume that the generic fibre A_η is an abelian variety, all line bundles on A_η verify the Theorem of the Cube [Mum70, Corollary 2, page 58]. In particular, it is natural to ask under which conditions on A (and S), a line bundle $\mathcal{L}_\eta$ on A_η can be extended to $\mathcal{L}$ on A still verifying the Theorem of the Cube. It turns out, after the work of Breen [Bre83] (see also [MB85]), that when S is a curve and A/S has connected fibres, a (unique) *cubist* extension exists for any line bundle on A_η.

The notion of G-torsor endowed with a cubist structure (or cubist G-torsor), where G is a commutative group scheme, is rather general and we will introduce here only what is relevant for our work (see [MB85, Ch. I, Definition 2.4.5]).

Definition 1.1. Let S be a scheme, $\mathbb{G}_{m,S}$ the multiplicative group scheme over S and X a commutative S-group scheme. A *cubist structure* on an invertible sheaf $\mathcal{L}$ of X (or $\mathbb{G}_{m,S}$-torsor on X/S) is the data of a section τ of the torsor

$$\theta(\mathcal{L}) \coloneqq m_{1,2,3}^*\mathcal{L} \otimes m_{1,2}^*\mathcal{L}^\vee \otimes m_{1,3}^*\mathcal{L}^\vee \otimes m_{2,3}^*\mathcal{L}^\vee \otimes m_1^*\mathcal{L} \otimes m_2^*\mathcal{L} \otimes m_3^*\mathcal{L}$$

on X^3, where $m_I\colon X^3 \longrightarrow X$ is the sum of projections corresponding to $I \subseteq \{1,2,3\}$. The category of cubist $\mathbb{G}_{m,S}$-torsors on X is denoted by $\mathrm{CUB}(X, \mathbb{G}_{m,S})$.

We recall the classical notion of rigidification for line bundles.

Definition 1.2. Let k be a field, S an irreducible normal scheme over k and A/S a smooth commutative group scheme, with connected fibres.

Let Z_0 be the zero section. An invertible sheaf $\mathcal{L}$ on A is *rigidified* if $\mathcal{L}|_{Z_0} \cong \mathcal{O}_{Z_0}$. The group of rigidified invertible sheaves on A modulo isomorphism is denoted by $\mathrm{Pic}(A)_{\mathrm{rig}}$.

Proposition 1.3. *Let C be a smooth projective curve over k and A/C a smooth commutative group scheme with connected fibres. Assume that the generic fibre A_η is an abelian variety. Then*

(1) the restriction

$$\mathrm{Pic}(A)_{\mathrm{rig}} \longrightarrow \mathrm{Pic}(A_\eta)$$

is a group isomorphism.

(2) For any $\mathcal{L} \in \mathrm{Pic}(A)_{\mathrm{rig}}$ the following holds:

$$\mathcal{L}_\eta \in \mathrm{Pic}^\circ(A_\eta) \iff [-1]^*\mathcal{L} \cong \mathcal{L}^\vee \, (\textit{i.e. } \mathcal{L} \textit{ is odd}).$$

(3) Let $\mathcal{L} \in \mathrm{Pic}(A)$ be a line bundle such that $\mathcal{L}_\eta \in \mathrm{Pic}^\circ(A_\eta)$, and let Z_1 and Z_2 be two sections of A/C, and $Z_3 := Z_1 \oplus Z_2$ in $\mathrm{MW}(A/C)$ with corresponding translation morphisms $\tau_i \colon A \longrightarrow A$, for $i = 1, 2, 3$. Then

$$\tau_1^*\mathcal{L} \otimes \tau_2^*\mathcal{L} \cong \tau_3^*\mathcal{L} \otimes \mathcal{L}$$

in $\mathrm{Pic}(A)$.

Proof. Our hypothesis on the base C guarantee the existence and unicity of cubist extensions, i.e. the restriction functor

$$\text{(A)} \qquad \mathrm{CUB}(A, \mathbb{G}_{m,C}) \longrightarrow \mathrm{CUB}(A_\eta, \mathbb{G}_{m,\eta})$$

is an equivalence of categories (see [MB85, Ch. 2, Theorem 1.1]). Composing with the forgetful functor

$$\mathrm{CUB}(A, \mathbb{G}_{m,C}) \longrightarrow \mathrm{TORSRIG}(A, \mathbb{G}_{m,C}),$$

where $\mathrm{TORSRIG}(A, \mathbb{G}_{m,C})$ is the category of rigidified (i.e. trivialised at the zero section) $\mathbb{G}_{m,C}$-torsors, we obtain (1).

The theory of abelian varieties gives: $\mathcal{L}_\eta \in \mathrm{Pic}^\circ(A_\eta)$ if and only if $[-1]^*\mathcal{L}_\eta \cong \mathcal{L}_\eta^\vee$ (see [Lan83, Ch. 5, Proposition 2.3]). So (1) implies (2).

Let $\mathcal{L} \in \mathrm{Pic}(A)$. Let p_1 and p_2 the two projections from $A \times A$. Restricting to the generic fibre, $\mathcal{L}_\eta \in \mathrm{Pic}^\circ(A_\eta)$ implies that $(p_1 + p_2)^*\mathcal{L}_\eta \cong p_1^*\mathcal{L}_\eta \otimes p_2^*\mathcal{L}_\eta$ on $A_\eta \times A_\eta$, so, again by (A), $p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \cong (p_1 + p_2)^*\mathcal{L} \otimes 0^*\mathcal{L}$. Pulling-back the previous equation via (f, g), where $f, g \colon A \longrightarrow A$ are morphisms, we get

$$f^*\mathcal{L} \otimes g^*\mathcal{L} \cong (f + g)^*\mathcal{L} \otimes 0^*\mathcal{L}.$$

This implies (3). □

2. The conjecture

In [KS25], the authors prove the following rigidity result, motivated by the work [BKV25].

Theorem 2.1. [KS25, Proposition 1] *Let S be a smooth, projective surface over $\mathbb{C}$ such that $\mathrm{Pic}(S) = \mathbb{Z}[H]$, where $|H|$ is basepoint-free, and members of $|H|$ have at worst nodes in codimension 1 on $|H|$. Let $p\colon J(S,H) \longrightarrow |H|$ be the universal compactified Jacobian, $L \subset |H|$ a general line, $J_L := p^{-1}(L)$, and g the genus of the curves in $|H|$. Let $Z \subset J_L$ be a section whose cohomology class is contained in the image of the restriction map*

$$H^{2g}(J(S,H),\mathbb{Z}) \longrightarrow H^{2g}(J_L,\mathbb{Z}).$$

Then Z is the zero section.

The proof reduces to a monodromy argument combined with an injectivity statement for the map

$$\mathrm{MW}(J_L/L) \cong \mathbb{Z}^{r-1} \longrightarrow N_1(J_L) \tag{B}$$
$$Z \mapsto [Z] - [Z_0]$$

where N_1 denotes the group of complete 1-cycles modulo numerical equivalence.

One is induced to consider the previous map in a generalised setting (see [KS25, Conjecture 7]). First, we remark that the map (B) is *not* a group homomorphism.

Example 2.2. Keep the same hypothesis as in Theorem 2.1. Choose any non-zero $Z \in \mathrm{MW}(J_L/L)$ (the $K/\mathbb{C}$-trace of J_L is automatically trivial and $\mathrm{MW}(J_L/L)$ is torsion-free, see [Shi99, Theorem 3]) and any $\mathcal{L} \in \mathrm{Pic}(J_L)$ relatively ample over L, rigidified and even (i.e. such that $[-1]^*\mathcal{L} \cong \mathcal{L}$). We know that $[2]^*\mathcal{L} = \mathcal{L}^{\otimes 4}$. Moreover, we remark that $(\mathcal{L} \cdot Z) = h_{\mathcal{L}}(Z_\eta)$, where $h_{\mathcal{L}}$ is the canonical Néron-Tate height (see [Lan83, Ch. 12, Proposition 3.5]), so $(\mathcal{L} \cdot Z) > 0$. By the projection formula, $([2]^*\mathcal{L} \cdot Z) = (\mathcal{L} \cdot [2]_* Z)$ and we deduce that

$$(\mathcal{L} \cdot [2]_* Z) = 4(\mathcal{L} \cdot Z).$$

Since $(\mathcal{L} \cdot 2Z) = 2(\mathcal{L} \cdot Z) < 4(\mathcal{L} \cdot Z)$, we have proved that

$$2[Z] \neq [[2]_* Z] + [Z_0] \ \text{ in } N_1(J_L).$$

In particular, the map (B) is not a group homomorphism.

It turns out that the previous example explains the only obstruction for (B) to be a homomorphism.

2.1. **The natural map.** Let $p\colon A \longrightarrow C$ be a smooth commutative group scheme over a smooth projective curve C with connected fibres. Consider the natural set map (B) in this general setting:

$$\phi\colon \mathrm{MW}(A/C) \longrightarrow N_1(A) \tag{C}$$
$$Z \mapsto [Z] - [Z_0]$$

This map is quadratic in the following sense: if we define

$$b\colon \mathrm{MW}(A/C) \times \mathrm{MW}(A/C) \longrightarrow N_1(A)$$
$$(W, Z) \mapsto [W \oplus Z] + [Z_0] - [W] - [Z]$$

the Theorem of the Cube implies that b is bilinear. Moreover the map

$$\ell\colon \mathrm{MW}(A/C) \longrightarrow N_1(A)$$
$$Z \mapsto 4[Z] - [Z \oplus Z] - 3[Z_0]$$

is linear, applying Proposition 1.3(3) to the line bundle $\mathcal{L}^{\otimes 4} \otimes [2]^*\mathcal{L}^\vee$. By construction,

$$2\phi(Z) = b(Z, Z) + \ell(Z),$$

for all $Z \in \mathrm{MW}(A/C)$.

In order to obtain a linear map from (C), we define a weakened version of numerical equivalence.

Definition 2.3. Let $p\colon A \longrightarrow C$ be a smooth commutative group scheme with connected fibres. The *generic numerical equivalence* on (complete) 1-cycles is defined as

$$Z \equiv_{\mathrm{gen}} 0 \text{ if } \ (\mathcal{L} \cdot Z) = 0 \text{ for all } \mathcal{L} \in \mathrm{Pic}(A) \text{ such that } \mathcal{L}_\eta \in \mathrm{Pic}^\circ(A_\eta).$$

The group of complete 1-cycles modulo generic numerical equivalence is denoted by $N_{1,\mathrm{gen}}(A)$.

Our main result is the following.

Theorem 2.4. *Let C be a geometrically connected, smooth projective curve over a field k, and A/C a smooth commutative group scheme with connected fibres, whose generic fibre A_η is an abelian variety. Let Z_0 be the zero section. Then the following holds.*

(1) The map

$$\psi\colon \mathrm{MW}(A/C) \longrightarrow N_{1,\mathrm{gen}}(A) \tag{D}$$
$$Z \mapsto [Z] - [Z_0]$$

is a group homomorphism.

(2) Assume that A_η has no abelian subvarieties defined over k, then $\ker\psi = \mathrm{MW}(A/C)_{\mathrm{tors}}$.

Proof. To prove that ψ is a group homomorphism, we follow the first part of the argument from [KS25, §8]. Let $Z_3 = Z_1 \oplus Z_2 \in \mathrm{MW}(A/C)$, we want to show that $\psi(Z_1) + \psi(Z_2) \equiv_{\mathrm{gen}} \psi(Z_3)$, i.e.

$$(\mathcal{L} \cdot Z_1) + (\mathcal{L} \cdot Z_2) = (\mathcal{L} \cdot Z_3) + (\mathcal{L} \cdot Z_0), \tag{E}$$

for any $\mathcal{L} \in \mathrm{Pic}(A)$ such that $\mathcal{L}_\eta \in \mathrm{Pic}^\circ(A_\eta)$.
Let us denote by $\tau_i \colon A \longrightarrow A$ the translation by Z_i, then the projection formula implies that (E) is equivalent to

$$(\tau_1^* \mathcal{L} \otimes \tau_2^* \mathcal{L} \otimes \tau_3^* \mathcal{L}^\vee \otimes \mathcal{L}^\vee \cdot Z_0) = 0,$$

which holds true by Proposition 1.3(3).
To describe the kernel of ψ, we recall once again that $(\mathcal{L} \cdot Z) = h_{\mathcal{L}}(Z_\eta)$, for all $\mathcal{L} \in \mathrm{Pic}(A)_{\mathrm{rig}}$, where $h_{\mathcal{L}}$ is the canonical Néron-Tate height (see [Lan83, Ch. 12, Proposition 3.5]). Let $Z \in \mathrm{MW}(A/C)$ verifying $(\mathcal{L} \cdot Z) = 0$ for all $\mathcal{L} \in \mathrm{Pic}(A)_{\mathrm{rig}}$ such that $\mathcal{L}_\eta \in \mathrm{Pic}^\circ(A_\eta)$; then [Lan83, Ch. 6, Theorem 5.4.2)] implies that $Z \in \mathrm{MW}(A/C)_{\mathrm{tors}}$, since the K/k-trace vanishes by the hypothesis (we assumed that A_η has no abelian subvarieties defined over k). □

Corollary 2.5. *Let A/C be as in Theorem 2.4 and assume that A_η has no abelian subvarieties defined over k. Then the set map*

$$\mathrm{MW}(A/C)/\mathrm{tors} \longrightarrow N_1(A)$$
$$Z \mapsto [Z] - [Z_0]$$

is injective.

Proof. Theorem 2.4 implies that, for any $Z_1, Z_2 \in \mathrm{MW}(A/C)$,

$$[Z_1] = [Z_2] \text{ in } N_1(A) \iff Z_1 \ominus Z_2 \in \mathrm{MW}(A/C)_{\mathrm{tors}}.$$

The statement follows. □

Remark 2.6. The previous corollary is precisely the injectivity-modulo-torsion statement from [KS25, Conjecture 7].

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