THE DISTRIBUTION OF THE MAXIMUM OF PARTIAL SUMS OF KLOOSTERMAN SUMS AND OTHER TRACE FUNCTIONS

PASCAL AUTISSIER, DANTE BONOLIS, AND YOUNESS LAMZOURI

Abstract. In this paper, we investigate the distribution of the maximum of partial sums of families of m-periodic complex valued functions satisfying certain conditions. We obtain precise uniform estimates for the distribution function of this maximum in a near optimal range. Our results apply to partial sums of Kloosterman sums and other families of ℓ-adic trace functions, and are as strong as those obtained by Bober, Goldmakher, Granville and Koukoulopoulos for character sums. In particular, we improve on the recent work of the third author for Birch sums. However, unlike character sums, we are able to construct families of m-periodic complex valued functions which satisfy our conditions, but for which the Pólya-Vinogradov inequality is sharp.

1. Introduction

Let $m \geq 2$ be an integer, and $\varphi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}$ a complex valued function which we extend to an m-periodic function $\varphi : \mathbb{Z} \to \mathbb{C}$. An important problem in analytic number theory is to obtain non-trivial estimates for the quantity

$$\mathcal{M}(\varphi) := \max_{x<m} \left| \sum_{0 \leq n \leq x} \varphi(n) \right|.$$

The special case where $\varphi = \chi$ is a Dirichlet character modulo m has been extensively studied over the last century, going back to the classical inequality proved by Pólya and Vinogradov in 1918:

$$\mathcal{M}(\chi) \ll \sqrt{m} \log m.$$

A straightforward generalization of this bound for a general m-periodic complex valued function φ gives

$$\mathcal{M}(\varphi) \ll ||\widehat{\varphi}||_\infty \sqrt{m} \log m, \tag{1.1}$$

where $\widehat{\varphi} : \mathbb{Z} \to \mathbb{C}$ is the normalized discrete Fourier transform of φ, defined by

$$\widehat{\varphi}(h) = \frac{1}{\sqrt{m}} \sum_{n \pmod m} \varphi(n) e_m(hn),$$

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where here and throughout we let $e(z) := \exp(2\pi i z)$, and $e_m(z) := e(z/m)$ is the standard additive character modulo m. To see this, consider the discrete Plancherel formula

$$\sum_{0\le n\le x} \varphi(n) = \sum_{-m/2<h\le m/2} \overline{\gamma_m(h;x)}\widehat{\varphi}(h), \tag{1.2}$$

where

$$\gamma_m(h;x) := \frac{1}{\sqrt{m}} \sum_{0\le n\le x} e_m(nh)$$

are the Fourier coefficients modulo m of the characteristic function of the interval $[0,x]$. The Pólya-Vinogradov bound (1.1) follows by using the elementary estimate (see for example page 1501 of [16])

$$\frac{1}{\sqrt{m}}\gamma_m(h;x) = \frac{e_m(xh)-1}{2\pi i h} + O\left(\frac{1}{m}\right), \tag{1.3}$$

which holds uniformly for $1 \le |h| \le m/2$.

We shall only consider those φ for which the Fourier transform $\widehat{\varphi}$ is uniformly bounded (this includes primitive Dirichlet characters), which in view of the Pólya-Vinogradov bound (1.1) gives

$$\mathcal{M}(\varphi) \ll \sqrt{m}\log m. \tag{1.4}$$

In the case of character sums, Montgomery and Vaughan [19] proved that this bound is not optimal conditionally on the generalized Riemann hypothesis GRH. Indeed, they showed that assuming GRH we have

$$\mathcal{M}(\chi) \ll \sqrt{m}\log\log m,$$

for all non-principal Dirichlet characters $\chi \pmod m$. This last bound is in fact optimal in view of an old result of Paley [20] who showed that $\mathcal{M}(\chi_m) \gg \sqrt{m}\log\log m$ for infinitely many m, where χ_m is the quadratic character modulo m.

Recently, Bober, Goldmakher, Granville and Koukoulopoulos [2] investigated the distribution of $\mathcal{M}(\chi)$ over non-principal characters χ modulo a large prime q. If we denote by $\Phi_{\text{char}}(V)$ the proportion of non-principal characters χ mod q for which $\mathcal{M}(\chi)/\sqrt{q} > V$, then the main result of [2] states that for $1 \le V \le C_0 \log\log q - C$ (where C is an absolute constant), one has

$$\Phi_{\text{char}}(V) = \exp\left(-\frac{e^{V/C_0+O(1)}}{V}\right), \tag{1.5}$$

where $C_0 = e^{\gamma}/\pi$, and γ is the Euler-Mascheroni constant.

Building on the work of Kowalski and Sawin [16], Lamzouri [17] investigated a similar question for the partial sums of certain exponential sums. For a prime $p \ge 3$ the

Birch sum associated to $a \in \mathbb{F}_p$ is the following normalized complete cubic exponential sum

$$\mathrm{Bi}_p(a) := \frac{1}{\sqrt{p}} \sum_{n \in \mathbb{F}_p} e_p(n^3 + an).$$

These sums were first considered by Birch [1] who conjectured that $\mathrm{Bi}_p(a)$ becomes equidistributed according to the Sato-Tate measure as a varies in $\mathbb{F}_p^\times$ and $p \to \infty$. This conjecture was subsequently proved by Livné in [18]. Let $\varphi_a(n) = e_p(n^3 + an)$ and define

$$\Phi_{\mathrm{Bi}}(V) = \frac{1}{p-1} \left| \left\{ a \in \mathbb{F}_p^\times : \frac{\mathcal{M}(\varphi_a)}{\sqrt{p}} > V \right\} \right|.$$

Lamzouri [17] proved that for V in the range $1 \le V \le (2/\pi) \log\log p - 2 \log\log\log p$, we have

$$\exp\left(-\exp\left(\frac{\pi}{2}V + O(1)\right)\right) \le \Phi_{\mathrm{Bi}}(V) \le \exp\left(-\exp\left(\left(\frac{\pi}{2} - \delta\right) V + O(1)\right)\right) \tag{1.6}$$

where $\delta = \frac{4\pi - \pi^2}{2\pi + 8} = 0.18880....$ He also conjectured that the lower bound corresponds to the true order of magnitude for $\Phi_{\mathrm{Bi}}(V)$. The techniques are different in this setting, due to the lack of multiplicativity for these exponential sums. Indeed, in the case of character sums, Bober, Goldmakher, Granville and Koukoulopoulos [2] exploit the relation with L-functions and smooth numbers, while ingredients from algebraic geometry and notably Deligne's equidistribution theorem play a central role in [17].

Lamzouri also showed that the lower bound in (1.6) holds for the maximum of partial sums of Kloosterman sums. The normalized classical Kloosterman sums are defined by

$$\mathrm{Kl}_p(a, b) := \frac{1}{\sqrt{p}} \sum_{n \in \mathbb{F}_p^\times} e_p(an + b\overline{n}),$$

where $\overline{n}$ denotes the multiplicative inverse of n modulo p. Similarly to Birch sums, Katz [13] proved that $\mathrm{Kl}_p(a, 1)$ becomes equidistributed according to the Sato-Tate measure as a varies in $\mathbb{F}_p^\times$ and $p \to \infty$. Let $\varphi_{(a,b)}(n) = e_p(an + b\overline{n})$. The method of [17] allows one to prove that in the range $1 \le V \le (2/\pi) \log\log p - 2 \log\log\log p$ we have

$$\Phi_{\mathrm{Kl}}(V) := \frac{1}{(p-1)^2} \left| \left\{ (a, b) \in \mathbb{F}_p^\times \times \mathbb{F}_p^\times : \frac{\mathcal{M}(\varphi_{(a,b)})}{\sqrt{p}} > V \right\} \right| \ge \exp\left(-\exp\left(\frac{\pi}{2}V + O(1)\right)\right).$$

However, the argument is not strong enough to yield an upper bound for the distribution function $\Phi_{\mathrm{Kl}}(V)$ in this case, since it relies on strong bounds for short sums of exponential sums, which are not currently known for Kloosterman sums.

In this paper, we prove Lamzouri's conjecture for the maximum of partial sums of Birch and Kloosterman sums, obtaining estimates for their distribution functions that are as strong as (1.5) for character sums. In particular, we relax the condition on short sums of exponential sums, requiring only bounds for these sums on average.

We also obtain analogous results for families of periodic functions which satisfy certain hypotheses (see Theorem 1.2 below). A corollary of our main theorem is the following result.

Corollary 1.1. *Let p be a large prime. There exists a constant C such that for all real numbers $1 \leq V \leq (2/\pi)(\log\log p - 2\log\log\log p - C)$ we have*

$$\Phi_{\mathrm{Kl}}(V) = \exp\left(-\exp\left(\frac{\pi}{2}V + O(1)\right)\right).$$

The same estimate also holds for $\Phi_{\mathrm{Bi}}(V)$.

More generally, we shall consider families of periodic functions $\mathcal{F} = \{\varphi_a\}_{a\in\Omega_m}$, where Ω_m is a non-empty finite set, and for each $a \in \Omega_m$, $\varphi_a : \mathbb{Z} \to \mathbb{C}$ is m-periodic and its Fourier transform $\widehat{\varphi}_a$ is real-valued and uniformly bounded. For a positive real number V, we define

$$\Phi_{\mathcal{F}}(V) := \frac{1}{|\Omega_m|}\left|\left\{a \in \Omega_m : \frac{\mathcal{M}(\varphi_a)}{\sqrt{m}} > V\right\}\right|.$$

We will obtain precise uniform estimates for this distribution function, assuming that our family $\mathcal{F}$ satisfies certain hypotheses, which are mainly related to the distribution of the Fourier transform $\widehat{\varphi}_a$. Such assumptions will be verified by several important functions in analytic number theory, which arise naturally in applications and originate in the deep work of Deligne and others from algebraic geometry. These functions correspond to certain *Frobenius trace functions* modulo m, and their analytic properties have been investigated by several authors, and notably in a series of recent works by Fouvry, Kowalski, and Michel [7], [8], [9], [10], Fouvry, Kowalski, Michel, Raju, Rivat, and Soundararajan [11], Kowalski and Sawin [16], and Perret-Gentil [21]. In particular, these include the families of trace functions $\mathcal{F}_{\mathrm{Bi}} = \{e_p(n^3 + an)\}_{a\in\mathbb{F}_p^\times}$ and $\mathcal{F}_{\mathrm{Kl}} = \{e_p(an + b\overline{n})\}_{(a,b)\in\mathbb{F}_p^\times\times\mathbb{F}_p^\times}$, which give rise to partial sums of Birch and Kloosterman sums respectively. More specifically, let $\mathcal{F} = \{\varphi_a\}_{a\in\Omega_m}$ be a family of m-periodic complex valued functions, and consider the following assumptions:

Assumption 1. Uniform boundedness

We have $\max_{a\in\Omega_m} ||\varphi_a||_\infty \ll 1$, where the implied constant is independent of m.

Assumption 2. Support of the Fourier transform

There exists an absolute constant $N > 0$ such that for all $a \in \Omega_m$ and $h \in \mathbb{Z}/m\mathbb{Z}$ we have $\widehat{\varphi_a}(h) \in [-N, N]$.

Assumption 3. Joint distribution of the Fourier transform

There exists a sequence of I.I.D. random variables $\{\mathbb{X}(h)\}_{h\in\mathbb{Z}^*}$ supported on $[-N, N]$, and absolute constants $\eta \geq 1/2$ and $C_1 > 1$, such that for all positive integers $k \leq \log m/\log\log m$, and all k-uples $(h_1, \ldots, h_k) \in (-m/2, m/2]^k$ with $h_i \neq 0$ for $i = 1, ..., k$

we have

$$\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\widehat{\varphi_a}(h_1)\cdots\widehat{\varphi_a}(h_k)=\mathbb{E}\left(\mathbb{X}(h_1)\ldots\mathbb{X}(h_k)\right)+O\left(\frac{C_1^k}{m^\eta}\right).$$

Furthermore, if we let $\mathbb{X}$ be a random variable with the same distribution as the $\mathbb{X}(h)$, then $\mathbb{X}$ verifies the following conditions:

3a. There exists a positive constant A such that for all $\varepsilon\in(0,1]$ we have $\mathbb{P}(\mathbb{X}>N-\varepsilon)\gg\varepsilon^A$, and $\mathbb{P}(\mathbb{X}<-N+\varepsilon)\gg\varepsilon^A$.

3b. For all integers $\ell\geq 0$ we have $\mathbb{E}\left(\mathbb{X}^{2\ell+1}\right)=0$.

Assumption 4. Strong bounds for short sums on average

There exist absolute constants $\alpha\geq 1$, and $0<\delta<1/2$ such that for any interval I of length $|I|\leq m^{1/2+\delta}$, one has

$$\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\left|\frac{1}{\sqrt{m}}\sum_{n\in I}\varphi_a(n)\right|^\alpha\ll m^{-1/2-\delta}.$$

Our main result is the following theorem.

Theorem 1.2. *Let m be large, and $\mathcal{F}=\{\varphi_a\}_{a\in\Omega_m}$ be a family of m-periodic complex valued functions satisfying one of the following subsets of the above assumptions:*

A. *Assumption 2 and Assumption 3 with $\eta>1$.*

B. *Assumptions 1, 2, and Assumption 3 with $1/2<\eta\leq 1$.*

C. *Assumptions 1, 2, 4, and Assumption 3 with $\eta=1/2$.*

Then there exists a constant $B=B(A)$ such that for all real numbers $1\leq V\leq(N/\pi)(\log\log m-2\log\log\log m-B)$ we have

$$\Phi_{\mathcal{F}}(V)=\exp\left(-\exp\left(\frac{\pi}{N}V+O(1)\right)\right).$$

Remark 1.3. Case C) is the most interesting and difficult case of Theorem 1.2. In particular, all the examples of trace functions we consider (including Kloosterman sums, see Corollaries 1.11, 1.12 and 1.13) fall into this case. For these examples, the saving of $\sqrt{m}$ in the error term of Assumption 3 follows from Deligne's equidistribution theorem.

Remark 1.4. Assumption 4 was first considered by Kowalski and Sawin [16] but for a different purpose. The authors of [16] investigated *Birch and Kloosterman paths*, which are the polygonal paths formed by linearly interpolating the partial sums of Birch and Kloosterman sums. They used Assumption 4 to establish a weak-compactness property known as *tightness*, which was necessary in order to show that the processes obtained from Birch and Kloosterman paths converge in law (in the Banach space $C[0,1]$) to a random Fourier series (which is the series inside the absolute value in (1.11) below). In our case, we found a new argument that allows us to use Assumption 4 (which holds for Kloosterman sums) instead of strong point wise bounds for short sums of exponential sums, which were needed in [17].

Remark 1.5. One can wonder whether a condition on the size of Ω_m is necessary to prove Theorem 1.2. In fact, such a condition is implicitly contained in Assumptions 2 and 3. More specifically, we show in Lemma 7.3 below that if $\mathcal{F} = \{\varphi_a\}_{a\in\Omega_m}$ satisfies these assumptions, then we must have $|\Omega_m| \gg m$.

One should note that the implicit upper bound in Theorem 1.2 holds in the slightly larger range $1 \leq V \leq (N/\pi)(\log\log m - \log\log\log m - B')$ for some constant B' that depends at most on the parameters in the assumptions of Theorem 1.2. Moreover, our proof of the implicit lower bound gives a much more precise estimate. In this case only Assumptions 2 and 3 are needed.

Theorem 1.6. *Let m be large, and $\mathcal{F} = \{\varphi_a\}_{a\in\Omega_m}$ be a family of m-periodic complex valued functions satisfying Assumptions 2 and 3 above. For all real numbers $1 \leq V \leq (N/\pi)(\log\log m - 2\log\log\log m - B)$ we have*

$$\Phi_{\mathcal{F}}(V) \geq \exp\left(-A_0 \exp\left(\frac{\pi}{N}V\right)\left(1 + O\left(Ve^{-\pi V/(2N)}\right)\right)\right)$$

where

$$A_0 = \frac{N}{2}\exp\left(-\gamma - 1 - \frac{1}{2N}\int_{-\infty}^{\infty}\frac{f_{\mathbb{X}}(u)}{u^2}du\right), \quad B = \log A_0 + 9, \tag{1.7}$$

γ is the Euler-Mascheroni constant, and $f_{\mathbb{X}} : \mathbb{R} \to \mathbb{R}$ is defined by

$$f_{\mathbb{X}}(t) := \begin{cases} \log\mathbb{E}(e^{t\mathbb{X}}) & \text{if } |t| < 1, \\ \log\mathbb{E}(e^{t\mathbb{X}}) - N|t| & \text{if } |t| \geq 1, \end{cases} \tag{1.8}$$

where $\mathbb{X}$ is a random variable with the same distribution as the $\{\mathbb{X}(h)\}_{h\in\mathbb{Z}^*}$ in Assumption 3 above.

As an application of Theorem 1.6 (more specifically of Theorem 7.1 which is stronger), we exhibit large values of partial sums in families of periodic functions $\{\varphi_a\}_{a\in\Omega_m}$ satisfying Assumptions 2 and 3. This was obtained by Lamzouri [17] for Birch and Kloosterman sums, and independently by Bonolis [3] for more general trace functions (though with a smaller constant).

Corollary 1.7. *Let m be large, and $\mathcal{F} = \{\varphi_a\}_{a\in\Omega_m}$ be a family of m-periodic complex valued functions satisfying Assumptions 2 and 3 above. There exist at least $|\Omega_m|^{1-1/\log\log m}$ elements $a \in \Omega_m$ such that*

$$\left|\sum_{0\leq n\leq m/2}\varphi_a(n)\right| \geq \left(\frac{N}{\pi} + o(1)\right)\sqrt{m}\log\log m. \tag{1.9}$$

Given a family $\mathcal{F} = \{\varphi_a\}_{a\in\Omega_m}$ of m-periodic complex valued functions satisfying the assumptions in Theorem 1.2, a natural question to ask is which of the bounds (1.4) and (1.9) is optimal (up to a constant). Note that if $\Phi_{\mathcal{F}}(V) \neq 0$ then $\Phi_{\mathcal{F}}(V) \geq 1/|\Omega_m|$. Using this observation, a simple heuristic argument suggests that if Theorem 1.2 were to be valid in the whole viable range, then we would have

$$\max_{a\in\Omega_m} \mathcal{M}(\varphi_a) \leq \left(\frac{N}{\pi} + o(1)\right) \sqrt{m} \log\log |\Omega_m|.$$

In particular, if $|\Omega_m| \ll m^B$ with an absolute constant $B > 0$ (which is the case in all the families we consider), this heuristic argument suggests that

$$\max_{a\in\Omega_m} \mathcal{M}(\varphi_a) \ll \sqrt{m} \log\log m, \tag{1.10}$$

a bound similar to the one proved by Montgomery and Vaughan for character sums under the assumption of the Generalized Riemann Hypothesis. Surprisingly, we show that unlike this case (in which multiplicativity plays a central role), the above heuristic argument is false for certain families of m-periodic complex valued functions satisfying the assumptions in case A) of Theorem 1.2 (namely Assumption 2, and Assumption 3 with $\eta > 1$). More precisely, we construct such a family $\mathcal{F} = \{\varphi_a\}_{a\in\Omega_m}$ with $|\Omega_m| \asymp m^3$, for which the Pólya-Vinogradov inequality (1.4) is sharp (up to the value of the implicit constant). This suggests the existence of a transition in the behavior of the distribution function $\Phi_{\mathcal{F}}(V)$ near the maximal values. It also confirms the common belief in analytic number theory that the Pólya-Vinogradov inequality, though simple to derive, is extremely difficult to improve.

Proposition 1.8. *Let m be large. There exists a family $\mathcal{F} = \{\varphi_a\}_{a\in\Omega_m}$ of m-periodic complex valued functions satisfying Assumption 2 with $N = 1$ and Assumption 3 with $\eta = 4/3$, such that $|\Omega_m| \asymp m^3$ and*

$$\max_{a\in\Omega_m} \mathcal{M}(\varphi_a) \geq \frac{1}{\pi}\sqrt{m} \log m + O(\sqrt{m}).$$

Remark 1.9. The family we construct in Proposition 1.8 does not satisfy Assumption 1. In fact one has $\max_{a\in\Omega_m} ||\varphi_a||_\infty \gg \sqrt{m}$ for this family. One therefore wonders whether a similar result to Proposition 1.8 holds for certain families of m-periodic complex valued functions satisfying the assumptions in case C) of Theorem 1.2, which is the case of most interest. Unfortunately, we were unable to construct such families. However, it seems plausible that in this case there are less fluctuations in the partial sums of φ_a, and that a bound similar to (1.10) holds.

In [16], Kowalski and Sawin showed that

$$\lim_{p\to\infty} \Phi_{\mathrm{Kl}}(V) = \lim_{p\to\infty} \Phi_{\mathrm{Bi}}(V) = \mathbb{P}(\mathbb{M}_{\mathrm{st}} > V),$$

for any fixed V for which $\mathbb{P}(\mathbb{M}_{\text{st}} > V)$ is continuous, where

$$\mathbb{M}_{\text{st}} = \max_{\alpha\in[0,1)} \left|\alpha\mathbb{Y}(0) + \sum_{h\neq 0} \frac{e(\alpha h)-1}{2\pi i h}\mathbb{Y}(h)\right|, \tag{1.11}$$

and $\{\mathbb{Y}(h)\}_{h\in\mathbb{Z}}$ is a sequence of independent random variables with Sato-Tate distributions on $[-2,2]$. A straightforward generalization of their argument shows that if m is large and $\mathcal{F} = \{\varphi_a\}_{a\in\Omega_m}$ is a family of m-periodic complex valued functions satisfying Assumptions 1, 2, 3, and 4, then for $V \geq 1$ fixed we have

$$\lim_{m\to\infty} \Phi_{\mathcal{F}}(V) = \mathbb{P}(\mathbb{M}_{\mathbb{X}} > V),$$

where

$$\mathbb{M}_{\mathbb{X}} = \max_{\alpha\in[0,1)} \left|\alpha\mathbb{X}(0) + \sum_{h\neq 0} \frac{e(\alpha h)-1}{2\pi i h}\mathbb{X}(h)\right|,$$

and $\{\mathbb{X}(h)\}_{h\in\mathbb{Z}}$ is a sequence of I.I.D. random variables supported on $[-N,N]$ and satisfying Assumptions 3a and 3b above. Combining this result with Theorem 1.2 leads to the following estimate for the large deviations of the random model $\mathbb{M}_{\mathbb{X}}$, which improves on the estimates of Lamzouri [17] and Kowalski-Sawin [16] for the large deviations of $\mathbb{M}_{\text{st}}$.

Corollary 1.10. *Let $\{\mathbb{X}(h)\}_{h\in\mathbb{Z}}$ be a sequence of I.I.D. random variables supported on $[-N,N]$ and satisfying Assumptions 3a and 3b above. For all $V \geq 1$ we have*

$$\mathbb{P}(\mathbb{M}_{\mathbb{X}} > V) = \exp\left(-\exp\left(\frac{\pi}{N}V + O(1)\right)\right).$$

We exhibit several examples of families of exponential sums that satisfy the assumptions in part C) of Theorem 1.2, namely Assumptions 1, 2, 4, and Assumption 3 with $\eta = 1/2$. These correspond to families of ℓ-adic trace functions which satisfy several conditions, and notably that their arithmetic and geometric monodromy groups are both equal to $\mathrm{Sp}_{2r}(\mathbb{C})$, for a certain integer $r \geq 1$. We shall describe these families in details in section 9. In particular, we obtain the following applications of Theorem 1.2. In all of these examples, $m = p$ is a large prime, and $\Omega_p = \mathbb{F}_p^\times$ or $\Omega_p = \mathbb{F}_p^\times \times \mathbb{F}_p^\times$. The first corollary concerns generalizations of Birch sums.

Corollary 1.11. *Let $g \in \mathbb{Z}[t]$ be an odd polynomial of degree $2r+1$, such that $r \geq 1$. Let $\mathcal{F}_1 = \{\varphi_a\}_{a\in\mathbb{F}_p^\times}$ where $\varphi_a(n) = e_p(an + g(n))$. There exists a constant B_1 such that for all real numbers $1 \leq V \leq (2r/\pi)(\log\log p - 2\log\log\log p - B_1)$ we have*

$$\Phi_{\mathcal{F}_1}(V) = \exp\left(-\exp\left(\frac{\pi}{2r}V + O(1)\right)\right).$$

The next application concerns generalizations of the classical Kloosterman sums.

Corollary 1.12. *Let $r \geq 1$ be an odd integer, and $\mathcal{F}_2 = \{\varphi_{(a,b)}\}_{(a,b)\in\mathbb{F}_p^\times\times\mathbb{F}_p^\times}$ where $\varphi_{(a,b)}(n) = e_p(bn + (a\overline{n})^r)$. There exists a constant B_2 such that for all real numbers $1 \leq V \leq ((r+1)/\pi)(\log\log p - 2\log\log\log p - B_2)$ we have*

$$\Phi_{\mathcal{F}_2}(V) = \exp\left(-\exp\left(\frac{\pi}{r+1}V + O(1)\right)\right).$$

Finally our last application concerns additive twists of hyper-Kloosterman sums. Recall that for an integer $r \geq 2$, the r-th hyper-Kloosterman sum on $\mathbb{F}_p$ is defined for $n \in \mathbb{F}_p^\times$ by

$$\mathrm{Kl}_r(n;p) = \frac{(-1)^{r-1}}{p^{(r-1)/2}} \sum_{\substack{y_1,\dots,y_r\in\mathbb{F}_p^\times \\ y_1\cdot\dots\cdot y_r = n}} e_p\left(y_1 + \cdots + y_r\right).$$

Corollary 1.13. *Let $r \geq 3$ be an odd integer, and $\mathcal{F}_4 = \{\varphi_{(a,b)}\}_{(a,b)\in\mathbb{F}_p^\times\times\mathbb{F}_p^\times}$ where $\varphi_{a,b}(n) = \mathrm{Kl}_r(\overline{an};p)e_p(bn)$. There exists a constant B_4 such that for all real numbers $1 \leq V \leq ((r+1)/\pi)(\log\log p - 2\log\log\log p - B_4)$ we have*

$$\Phi_{\mathcal{F}_4}(V) = \exp\left(-\exp\left(\frac{\pi}{r+1}V + O(1)\right)\right).$$

Our method also works in the case where the Fourier transforms $\widehat{\varphi_a}$ are complex valued, but yields weaker estimates for $\Phi_{\mathcal{F}}$ in this case. This corresponds for example to certain families of ℓ-adic trace functions whose monodromy group is $\mathrm{SL}_N(\mathbb{C})$ for some integer $N \geq 3$ (since in the case $N = 2$ we have $\mathrm{SL}_2(\mathbb{C}) = \mathrm{Sp}_2(\mathbb{C})$). In this case we need to change Assumption 3 to include all mixed moments of $\widehat{\varphi_a}(h_1), \dots, \widehat{\varphi_a}(h_k)$ and their complex conjugates. We also assume that the $\{\mathbb{X}(h)\}_{h\in\mathbb{Z}^*}$ are supported inside the disk $\{z \in \mathbb{C} : |z| \leq N\}$, and replace $\mathbb{X}$ by $\mathrm{Re}\mathbb{X}$ in Assumption 3a. Then, given a family $\mathcal{F} = \{\varphi_a\}_{a\in\Omega_m}$ of m-periodic complex valued functions such that $|\widehat{\varphi_a}(h)| \leq N$, and $\mathcal{F}$ verifies (the new) Assumption 3 with $\eta > 1$, or Assumptions 1 and 3 with $1/2 < \eta \leq 1$, or Assumptions 1, 4, and 3 with $\eta = 1/2$, we can prove that in the range $1 \leq V \leq (N/\pi)(\log\log m - 2\log\log\log m - B)$ we have [1]

$$\exp\left(-\exp\left(\frac{\pi}{N}V + O(1)\right)\right) \leq \Phi_{\mathcal{F}}(V) \leq \exp\left(-\exp\left(\frac{\pi^2}{4N}V + O(1)\right)\right).$$

[1]In the case of families of trace functions whose monodromy groups are equal to $\mathrm{SL}_N(\mathbb{C})$ for some odd integer $N \geq 3$, we obtain a slightly bigger constant in the leading order term of the lower bound for $\Phi_{\mathcal{F}}(V)$, since the condition $\mathbb{P}(\mathrm{Re}\mathbb{X} < -N + \varepsilon) \gg \varepsilon^A$ in Assumption 3a is not satisfied in this case.

2. Proof of the upper bound in Theorem 1.2: Main ideas and key ingredients

Let $\{\varphi_a\}_{a\in\Omega_m}$ be a family of m-periodic complex valued functions satisfying Assumptions 2 and 3. Recall that

$$\mathcal{M}(\varphi_a) = \max_{0\le x<m}\left|\sum_{0\le n\le x}\varphi_a(n)\right|.$$

Using the discrete Plancherel formula (1.2) and the estimate (1.3) we obtain

$$\frac{\mathcal{M}(\varphi_a)}{\sqrt{m}} = \frac{1}{2\pi}\max_{1\le j\le m}\Big|\sum_{1\le |h|<m/2}\frac{e_m(jh)-1}{h}\widehat{\varphi_a}(h)\Big| + O(1).$$

Similarly as in [17], we shall treat the Fourier transforms $\widehat{\varphi_a}(h)$ for small h as *random* values in $[-N,N]$. This yields

$$\frac{\mathcal{M}(\varphi_a)}{\sqrt{m}} \le \frac{N}{2\pi}\mathcal{G}(H) + \frac{1}{2\pi}\max_{1\le j\le m}\left|\sum_{H<|h|<m/2}\frac{e_m(jh)-1}{h}\widehat{\varphi_a}(h)\right| + O(1), \tag{2.1}$$

where H is a positive integer and

$$\mathcal{G}(H) := \max_{\alpha\in[0,1)}\max_{(y_{-H},\dots y_{-1},y_1,\dots,y_H)\in[-1,1]^{2H}}\left|\sum_{1\le|h|\le H}\frac{e(\alpha h)-1}{h}y_h\right|.$$

One has the trivial bounds

$$2\log H + O(1) \le \mathcal{G}(H) \le 4\log H + O(1), \tag{2.2}$$

where the upper bound follows from the trivial inequality $|e(\alpha h)-1|\le 2$, and the lower bound follows by taking $\alpha = 1/2$, $y_h=-1$ if $h>0$ and $y_h=1$ if $h<0$. Using Fourier analytic techniques, the third author showed in [17] that

$$\mathcal{G}(H) \le \left(1+\frac{4}{\pi}\right)\log H + O(1),$$

and conjectured that the lower bound of (2.2) is closer to the true order of magnitude of $\mathcal{G}(H)$. In section 4 we shall prove a stronger form of this conjecture.

Theorem 2.1. *Let H be a positive integer. Then, we have*

$$\mathcal{G}(H) = 2\log H + 2\log 2 + 2\gamma + O\left(\frac{1}{H}\right).$$

In order to prove the upper bound in Theorem 1.2, it remains to show that for large H, and for "most" $a\in\Omega_m$, the maximum of the sum $|\sum_{H<|h|<m/2}\frac{e_m(jh)-1}{h}\widehat{\varphi_a}(h)|$ is "small". To this end we prove the following result in section 6.

Theorem 2.2. *Let m be large, and k be an integer such that $10^5N^2 < k \leq (\log m)/(50 \log\log m)$. Let $\{\varphi_a\}_{a\in\Omega_m}$ be a family of m-periodic complex valued functions satisfying Assumptions 2 and 3. Let S be a non-empty subset of $[0, 1)$, and put $y = 10^5N^2k$. Then we have*

$$\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\max_{\alpha\in S}\left|\sum_{y\leq|h|<m/2}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k} \ll e^{-2k} + \frac{|S|(4C_1\log m)^{8k}}{m^{\eta}}.$$

In the case where $\{\varphi_a\}_{a\in\Omega_m}$ satisfies Assumption 3 with $\eta > 1$, we can deduce the upper bound of Theorem 1.2 from Theorems 2.1 and 2.2.

Proof of the upper bound in case A) of Theorem 1.2. Let $k \leq (\eta-1)\log m/(50\log\log m)$ be a large positive integer to be chosen, and put $H = 10^5N^2k$. First, combining equation (2.1) with Theorem 2.1 we deduce that

$$\frac{\mathcal{M}(\varphi_a)}{\sqrt{m}} \leq \frac{N}{\pi}\log k + \frac{1}{2\pi}\max_{1\leq j\leq m}\left|\sum_{H\leq|h|<m/2}\frac{e_m(jh)-1}{h}\widehat{\varphi_a}(h)\right| + C_0,$$

for some positive constant C_0. Since the result trivially holds when V is small, we might assume that V is sufficiently large and choose $k = [C_2\exp(\pi V/N)]$, where $C_2 = \exp(-\frac{\pi}{N}(C_0 + \frac{1}{2\pi}))$. Therefore, appealing to Theorem 2.2 with $S = \{j/m : 1 \leq j \leq m\}$ we obtain

$$\begin{aligned}\Phi_{\mathcal{F}}(V) &\leq \frac{1}{|\Omega_m|}\left|\left\{a\in\Omega_m : \max_{1\leq j\leq m}\left|\sum_{H\leq|h|<m/2}\frac{e_m(jh)-1}{h}\widehat{\varphi_a}(h)\right| \geq 1\right\}\right| \\ &\leq \frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\max_{1\leq j\leq m}\left|\sum_{H\leq|h|<m/2}\frac{e_m(jh)-1}{h}\widehat{\varphi_a}(h)\right|^{2k} \\ &\ll e^{-2k} + (4\log m)^{10k}m^{1-\eta} \ll \exp\left(-C_2\exp\left(\frac{\pi}{N}V\right)\right),\end{aligned} \tag{2.3}$$

as desired.

□

If $\{\varphi_a\}_{a\in\Omega_m}$ satisfies Assumption 3 with $\eta \leq 1$ (which corresponds to cases B) and C) of Theorem 1.2), then the above argument no longer works since $|\{j/m : 1 \leq j \leq m\}| = m$ is too big. To overcome this problem, we shall suppose that our family satisfies Assumption 1, and use it to reduce the number of points $j \leq m$ where the maximum of $\left|\sum_{0\leq n\leq j}\varphi_a(n)\right|$ can occur. Let $J \leq \sqrt{m}$ be a parameter to be chosen, and split the interval $[0, m]$ into J intervals $I_j := [x_j, x_{j+1}]$ where for each $j = 0, ..., J$ we put

$$x_j := \frac{j}{J}m.$$

We first consider case B) of Theorem 1.2 since it is easier.

Proof of the upper bound in case B) of Theorem 1.2. We choose $J = \lfloor \sqrt{m} \rfloor$. For $a \in \Omega_m$, let r_a be an integer in the interval $[0, m)$ such that

$$\mathcal{M}(\varphi_a) = \Big| \sum_{0 \le n \le r_a} \varphi_a(n) \Big|.$$

Then there exists $0 \le j \le J-1$ such that $r_a \in [x_j, x_{j+1}]$, and hence

$$\begin{aligned} \Big| \frac{1}{\sqrt{m}} \sum_{0 \le n \le r_a} \varphi_a(n) \Big| &\le \Big| \frac{1}{\sqrt{m}} \sum_{0 \le n \le x_j} \varphi_a(n) \Big| + \Big| \frac{1}{\sqrt{m}} \sum_{x_j < n \le r_a} \varphi_a(n) \Big| \\ &\le \Big| \frac{1}{\sqrt{m}} \sum_{0 \le n \le x_j} \varphi_a(n) \Big| + O(1), \end{aligned}$$

since $\max_{a \in \Omega_m} ||\varphi_a|| \ll 1$ and $|r_a - x_j| \le m/J \ll \sqrt{m}$. This implies that

$$\frac{\mathcal{M}(\varphi_a)}{\sqrt{m}} = \max_{1 \le j \le J-1} \Big| \frac{1}{\sqrt{m}} \sum_{0 \le n \le x_j} \varphi_a(n) \Big| + O(1).$$

We now use the same argument leading up to (2.3) with the same choices of $k \le (\eta - 1/2) \log m / (30 \log\log m)$ and H, but with $S = \{x_j/m : 1 \le j \le J-1\}$ (and perhaps a different choice for the constant C_0). This gives

$$\begin{aligned} \Phi_{\mathcal{F}}(V) &\le \frac{1}{|\Omega_m|} \sum_{a \in \Omega_m} \max_{1 \le j \le J-1} \left| \sum_{H \le |h| < m/2} \frac{e_m(x_j h) - 1}{h} \widehat{\varphi_a}(h) \right|^{2k} \\ &\ll e^{-2k} + (4 \log m)^{10k} m^{1/2-\eta} \ll \exp\left(-C_2 \exp\left(\frac{\pi}{N} V \right) \right). \end{aligned}$$

□

The above argument fails if $\{\varphi_a\}_{a \in \Omega_m}$ satisfies Assumption 3 with $\eta = 1/2$, which is the most interesting case of Theorem 1.2. In this case, to reduce the number of points of S further (below $m^{1/2-\varepsilon}$ for some ε), we need power saving bounds for short sums $\sum_{x \le n \le x+h} \varphi_a(n)$ in *the Pólya-Vinogradov range*, which corresponds to h being of size around $\sqrt{m}$. Unfortunately, such bounds are only known in very few cases (for example they are known for Birch sums but not for Kloosterman sums). To overcome this problem, we use Assumption 4 to obtain strong uniform bounds for these shorts sums on average. Let α and δ be as in Assumption 4. As before we will split the interval $[0, m]$ into J intervals $I_j := [x_j, x_{j+1}]$ where $x_j := \frac{j}{J} m$, and where we now choose $J = \lfloor m^{1/2-\delta/5} \rfloor$. We shall prove the following result in section 3.

Theorem 2.3. *Let m be large and $J = \lfloor m^{1/2-\delta/5} \rfloor$. Let $\{\varphi_a\}_{a \in \Omega_m}$ be a family of m-periodic complex valued functions satisfying Assumptions 1 and 4. There exists a set*

$\mathcal{E}_m \subset \Omega_m$ *with* $|\mathcal{E}_m| \le m^{-\delta/10}|\Omega_m|$ *such that for all* $a \in \Omega_m \setminus \mathcal{E}_m$ *we have*

$$\mathcal{M}(\varphi_a) = \max_{1\le j\le J-1}\Big|\sum_{0\le n\le x_j}\varphi_a(n)\Big| + O\left(m^{1/2-\delta/(8\alpha)}\right).$$

We end this section by deducing the upper bound in case C) of Theorem 1.2 from Theorems 2.1, 2.2 and 2.3.

Proof of the upper bound in case C) of Theorem 1.2. Let $\mathcal{E}_m$ be the exceptional set in Theorem 2.3, and $a \in \Omega_m \setminus \mathcal{E}_m$. Combining this result with the discrete Plancherel formula (1.2) and the estimate (1.3) we obtain

$$\frac{\mathcal{M}(\varphi_a)}{\sqrt{m}} = \frac{1}{2\pi}\max_{1\le j\le J-1}\Big|\sum_{1\le|h|<m/2}\frac{e_m(x_jh)-1}{h}\widehat{\varphi_a}(h)\Big| + O(1).$$

Let $k \le \delta(\log m)/(200\log\log m)$ be a large positive integer to be chosen, and put $H = 10^5N^2k$. First, combining equation (2.1) with Theorem 2.1, we deduce that if $a \in \Omega_m \setminus \mathcal{E}_m$ we have

$$\frac{\mathcal{M}(\varphi_a)}{\sqrt{m}} \le \frac{N}{\pi}\log k + \frac{1}{2\pi}\max_{1\le j\le J-1}\left|\sum_{H\le|h|<m/2}\frac{e_m(x_jh)-1}{h}\widehat{\varphi_a}(h)\right| + C_0,$$

for some positive constant C_0. Repeating the same argument as before with the same choice of k gives

$$\begin{aligned}\Phi_{\mathcal{F}}(V) &\le \frac{1}{|\Omega_m|}\left|\left\{a\in\Omega_m\setminus\mathcal{E}_m : \max_{1\le j\le J-1}\left|\sum_{H\le|h|<m/2}\frac{e_m(x_jh)-1}{h}\widehat{\varphi_a}(h)\right|\ge 1\right\}\right| + O\left(\frac{|\mathcal{E}_m|}{|\Omega_m|}\right)\\ &\le \frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\max_{1\le j\le J-1}\left|\sum_{H\le|h|<m/2}\frac{e_m(x_jh)-1}{h}\widehat{\varphi_a}(h)\right|^{2k} + O\left(m^{-\delta/10}\right)\\ &\ll e^{-2k} + (4\log m)^{10k}m^{-\delta/10} \ll \exp\left(-C_2\exp\left(\frac{\pi}{N}V\right)\right).\end{aligned}$$

$\square$

3. Controlling short sums of trace functions: Proof of Theorem 2.3

In order to prove Theorem 2.3, we will use Assumptions 1 and 4 to obtain a non-trivial upper bound for the α-th moment of the *maximum* over intervals I (with length up to a certain parameter L) of the short sum $\sum_{n\in I}\varphi_a(n)$.

Lemma 3.1. *Let* m *be large, and* $\{\varphi_a\}_{a\in\Omega_m}$ *be a family of* m*-periodic complex valued functions satisfying Assumptions 1 and 4. For any real number* $1 \le L \le m^{1/2+\delta/2}$ *we have*

$$\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\max_{|I|\le L}\left|\frac{1}{\sqrt{m}}\sum_{n\in I}\varphi_a(n)\right|^{\alpha} \ll Lm^{-1/2-\delta/2} + m^{-\delta/4},$$

where the maximum is taken over all intervals $I = [x, y] \subset [0, m]$ with $|I| \leq L$.

Proof. The intervals $I = [x, y]$ with $0 \leq x < y \leq m$, and $|I| = y - x \leq L$ can be parametrized by the set of points in the region of the plane delimited by the trapezoid $0 \leq x < y \leq m$ and $y \leq x + L$. We denote by T_L this region, i.e.

$$T_L := \{(x, y) \in \mathbb{R}^2 : 0 \leq x < y \leq m, \quad y \leq x + L\}.$$

Let $0 < B \leq \sqrt{m}$ be a parameter to be chosen, and for any $k, \ell \in \mathbb{N}$ we define $S_{k,\ell,B} := [kB, (k+1)B) \times [\ell B, (\ell+1)B)$. The set of squares given by

$$\{S_{k,\ell,B} \mid S_{k,\ell,B} \cap T_L \neq \emptyset\}$$

is a disjoint cover of T_L, and moreover

$$\begin{aligned} \mathcal{N}_{L,B} := |\{S_{k,\ell,B} \mid S_{k,\ell,B} \cap T_L \neq \emptyset\}| &\ll \frac{A(T_L)}{B^2} + \frac{m}{B} \\ &\ll \frac{mL}{B^2} + \frac{m}{B}, \end{aligned} \tag{3.1}$$

where $A(D)$ denotes the area of D. For any $a \in \Omega_m$ let us denote by $I_a = [x_a, y_a]$ an interval with $|I_a| \leq L$ such that

$$\Big|\frac{1}{\sqrt{m}} \sum_{n \in I_a} \varphi_a(n)\Big| = \max_{|I| \leq L} \Big|\frac{1}{\sqrt{m}} \sum_{n \in I} \varphi_a(n)\Big|.$$

Then there exists $k_a, \ell_a \in \mathbb{N}$ such that $(x_a, y_a) \in S_{k_a,\ell_a,B}$. Hence

$$\begin{aligned} \frac{1}{\sqrt{m}} \sum_{n \in I_a} \varphi_a(n) &= \frac{1}{\sqrt{m}} \sum_{k_a B \leq n \leq \ell_a B} \varphi_a(n) + O\Big(\frac{x_a - k_a B + y_a - \ell_a B}{\sqrt{m}}\Big) \\ &= \frac{1}{\sqrt{m}} \sum_{k_a B \leq n \leq \ell_a B} \varphi_a(n) + O\Big(\frac{B}{\sqrt{m}}\Big), \end{aligned}$$

by Assumption 1. Using this estimate together with the elementary inequality $|x+y|^\alpha \leq (2\max(|x|, |y|))^\alpha \leq 2^\alpha(|x|^\alpha + |y|^\alpha)$ we get

$$\begin{aligned} \frac{1}{|\Omega_m|} \sum_{a \in \Omega_m} \max_{|I| \leq L} \Big|\frac{1}{\sqrt{m}} \sum_{n \in I} \varphi_a(n)\Big|^\alpha &= \frac{1}{|\Omega_m|} \sum_{a \in \Omega_m} \Big|\frac{1}{\sqrt{m}} \sum_{k_a B \leq n \leq \ell_a B} \varphi_a(n) + O\Big(\frac{B}{\sqrt{m}}\Big)\Big|^\alpha \\ &\ll \frac{1}{|\Omega_m|} \sum_{a \in \Omega_m} \Big|\frac{1}{\sqrt{m}} \sum_{k_a B \leq n \leq \ell_a B} \varphi_a(n)\Big|^\alpha + \frac{B^\alpha}{m^{\alpha/2}}. \end{aligned}$$

Furthermore, observe that

$$\begin{aligned}\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\Big|\frac{1}{\sqrt{m}}\sum_{k_aB\le n\le \ell_aB}\varphi_a(n)\Big|^\alpha &\le \frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\sum_{\substack{k,\ell\in\mathbb{N}:\\ S_{k,\ell,B}\cap T_L\neq\emptyset}}\Big|\frac{1}{\sqrt{m}}\sum_{kB\le n\le \ell B}\varphi_a(n)\Big|^\alpha\\ &= \sum_{\substack{k,\ell\in\mathbb{N}:\\ S_{k,\ell,B}\cap T_L\neq\emptyset}}\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\Big|\frac{1}{\sqrt{m}}\sum_{kB\le n\le \ell B}\varphi_a(n)\Big|^\alpha\\ &\ll m^{-1/2-\delta}\mathcal{N}_{L,B}.\end{aligned}$$

by Assumption 4, since $B\le m^{1/2}$. Therefore, we deduce from (3.1) that

$$\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\max_{|I|\le L}\Big|\frac{1}{\sqrt{m}}\sum_{n\in I}\varphi_a(n)\Big|^\alpha \ll \frac{Lm^{1/2-\delta}}{B^2}+\frac{m^{1/2-\delta}}{B}+\frac{B^\alpha}{m^{\alpha/2}}.$$

Choosing $B=m^{1/2-\delta/4}$ gives the result. □

Proof of Theorem 2.3. It only suffices to prove the implicit upper bound, since one trivially has

$$\mathcal{M}(\varphi_a)\ge \max_{1\le j\le J-1}\Big|\sum_{0\le n\le x_j}\varphi_a(n)\Big|.$$

Let $L=m^{1/2+\delta/4}$ and define $\mathcal{E}_m$ to be the set of elements $a\in\Omega_m$ such that

$$\max_{|I|\le L}\Big|\frac{1}{\sqrt{m}}\sum_{n\in I}\varphi_a(n)\Big|>m^{-\delta/(8\alpha)}.$$

Then, it follows from Lemma 3.1 that

$$\frac{|\mathcal{E}_m|}{|\Omega_m|}\le\frac{m^{\frac{\delta}{8}}}{|\Omega_m|}\sum_{a\in\Omega_m}\max_{|I|\le L}\Big|\frac{1}{\sqrt{m}}\sum_{n\in I}\varphi_a(n)\Big|^\alpha\ll m^{-\delta/8}.$$

Moreover, for $a\in\Omega_m$, let r_a be an integer in the interval $[0,m)$ such that

$$\mathcal{M}(\varphi_a)=\Big|\sum_{0\le n\le r_a}\varphi_a(n)\Big|.$$

Then there exists $0\le j\le J-1$ such that $r_a\in[x_j,x_{j+1}]$, and hence

$$\Big|\frac{1}{\sqrt{m}}\sum_{0\le n\le r_a}\varphi_a(n)\Big|\le\Big|\frac{1}{\sqrt{m}}\sum_{0\le n\le x_j}\varphi_a(n)\Big|+\Big|\frac{1}{\sqrt{m}}\sum_{x_j<n\le r_a}\varphi_a(n)\Big|.$$

Recall that $J=\lfloor m^{1/2-\delta/5}\rfloor$, and hence $r_a-x_j\le x_{j+1}-x_j=m/J\le L$ if m is large enough. Therefore, we deduce that if $a\in\Omega_m\setminus\mathcal{E}_m$ then

$$\Big|\frac{1}{\sqrt{m}}\sum_{x_j<n\le r_a}\varphi_a(n)\Big|\le m^{-\delta/(8\alpha)}.$$

This implies

$$\frac{\mathcal{M}(\varphi_a)}{\sqrt{m}} \leq \max_{1\leq j\leq J-1} \left|\frac{1}{\sqrt{m}} \sum_{0\leq n\leq x_j} \varphi_a(n)\right| + O\left(m^{-\delta/(8\alpha)}\right),$$

for all $a \in \Omega_m \setminus \mathcal{E}_m$, completing the proof. □

4. An asymptotic estimate for the maximum of a random sum: Proof of Theorem 2.1

Recall that

$$\mathcal{G}(H) = \max_{\alpha\in[0,1]} \max_{(x_h)_{1\leq|h|\leq H}\in[-1,1]^{2H}} \left|\sum_{1\leq|h|\leq H} \frac{e(\alpha h)-1}{h} x_h\right|.$$

We shall deduce Theorem 2.1 from the following result, which is an exact formula for $\mathcal{G}(H)$ when H is odd.

Proposition 4.1. *If H is an odd positive integer, then*

$$\mathcal{G}(H) = 2\sum_{h=1}^{H} \frac{1-(-1)^h}{h}.$$

To prove this result, we need the following lemma.

Lemma 4.2. *Let α be a real number. If $H \geq 1$ is odd, then*

$$\sum_{h=1}^{H} \frac{\sin^2(\pi\alpha h)}{h} \leq \sum_{h=1}^{H} \frac{1-(-1)^h}{2h}.$$

Proof. Since $\sin^2(\pi(1-\alpha)h) = \sin^2(\pi\alpha h)$ we may assume that $\alpha \in [0, 1/2]$. Let $g : [0, 1/2] \to \mathbb{R}$ be defined by

$$g(t) := \frac{\cos(2\pi tH)}{4H} + \sum_{h=1}^{H} \frac{\sin^2(\pi th)}{h} = \frac{\cos(2\pi tH)}{4H} + \sum_{h=1}^{H} \frac{1-\cos(2\pi th)}{2h}.$$

Since g is differentiable and

$$\begin{aligned} g'(t) &= -\frac{\pi}{2}\sin(2\pi tH) + \pi\sum_{h=1}^{H}\sin(2\pi th) \\ &= \pi\left(-\sin(\pi tH)\cos(\pi tH) + \frac{\sin(\pi tH)\sin(\pi t(H+1))}{\sin(\pi t)}\right) = \pi\frac{\sin^2(\pi tH)\cos(\pi t)}{\sin(\pi t)} \geq 0, \end{aligned}$$

we deduce that g is increasing on $[0, 1/2]$. This implies that for all $\alpha \in [0, 1/2]$ we have

$$\sum_{h=1}^{H} \frac{\sin^2(\pi\alpha h)}{h} \leq g(1/2) - \frac{\cos(2\pi\alpha H)}{4H} = \sum_{h=1}^{H} \frac{1-(-1)^h}{2h} - \frac{\cos(2\pi\alpha H)+1}{4H} \leq \sum_{h=1}^{H} \frac{1-(-1)^h}{2h},$$

as desired. □

Proof of Proposition 4.1. The lower bound follows easily by taking $\alpha = 1/2$, $x_h = -1$ if $h > 0$ and $x_h = 1$ if $h < 0$. Let us now show the upper bound. Let $\alpha \in \mathbb{R}$ and $(x_h)_{1\leq |h|\leq H} \in [-1,1]^{2H}$. Put

$$S = \sum_{1\leq |h|\leq H} \frac{e(\alpha h)-1}{h} x_h \quad \text{and} \quad S_0 = \sum_{h=1}^{H} \frac{\sin^2(\pi\alpha h)}{h}.$$

On one hand, we have the relation

$$\mathrm{Im}(S) = 2\sum_{h=1}^{H} \sin(\pi\alpha h)\cos(\pi\alpha h)\frac{x_h + x_{-h}}{h}.$$

Moreover, using the Cauchy-Schwarz inequality we get

$$\mathrm{Im}(S)^2 \leq 4\Big(\sum_{h=1}^{H} \frac{\sin^2(\pi\alpha h)}{h}|x_h + x_{-h}|\Big)\Big(\sum_{h=1}^{H} \frac{\cos^2(\pi\alpha h)}{h}|x_h + x_{-h}|\Big).$$

On the other hand, we have

$$\mathrm{Re}(S) = 2\sum_{h=1}^{H} \frac{\sin^2(\pi\alpha h)}{h}(x_{-h} - x_h).$$

Observe that $|x_h - x_{-h}| + |x_h + x_{-h}| \leq 2$. This implies the upper bound

$$|\mathrm{Re}(S)| \leq 4S_0 - 2\sum_{h=1}^{H} \frac{\sin^2(\pi\alpha h)}{h}|x_h + x_{-h}|.$$

We are now ready to estimate $|S|^2 = \mathrm{Re}(S)^2 + \mathrm{Im}(S)^2$. We infer

$$\begin{aligned}|S|^2 &\leq 16S_0^2 - 16S_0\sum_{h=1}^{H} \frac{\sin^2(\pi\alpha h)}{h}|x_h + x_{-h}| + 4\Big(\sum_{h=1}^{H} \frac{\sin^2(\pi\alpha h)}{h}|x_h + x_{-h}|\Big)\sum_{h=1}^{H}\frac{|x_h + x_{-h}|}{h}\\ &\leq 16S_0^2 + 8\Big(\sum_{h=1}^{H} \frac{\sin^2(\pi\alpha h)}{h}|x_h + x_{-h}|\Big)\Big(\sum_{h=1}^{H}\frac{1}{h} - 2S_0\Big).\end{aligned}$$

To finish the proof, let us study two cases:

i) If $2S_0 \geq \sum_{h=1}^{H}\frac{1}{h}$, then $|S|^2 \leq 16S_0^2$ and we conclude by applying Lemma 4.2.

ii) If $2S_0 \leq \sum_{h=1}^{H}\frac{1}{h}$, then

$$|S|^2 \leq 16S_0^2 + 16S_0\Big(\sum_{h=1}^{H}\frac{1}{h} - 2S_0\Big) \leq 4\Big(\sum_{h=1}^{H}\frac{1}{h}\Big)^2 \leq \Big(2\sum_{h=1}^{H}\frac{1-(-1)^h}{h}\Big)^2.$$

Whence the result. $\square$

We end this section by deducing Theorem 2.1.

Proof of Theorem 2.1. If $H \geq 1$ is odd, the desired asymptotic follows directly from Proposition 4.1, so it only remains to prove the result when H is even. To this end, we observe that if $k \geq 1$ is an integer, then we have

$$\mathcal{G}(k+1) = \max_{\alpha\in[0,1]} \max_{(x_h)_{1\leq|h|\leq k+1}\in[-1,1]^{2k+2}} \left|\sum_{1\leq|h|\leq k+1} \frac{e(\alpha h)-1}{h} x_h\right| \geq \mathcal{G}(k),$$

which follows by taking $x_{k+1} = x_{-k-1} = 0$. Hence Theorem 2.1 follows in this case from Proposition 4.1, together with the inequality $\mathcal{G}(H-1) \leq \mathcal{G}(H) \leq \mathcal{G}(H+1)$. □

Remark 4.3. Using the same method of proof as in Proposition 4.1, we can in fact obtain the following exact formula for $\mathcal{G}(H)$ when H is even

$$\mathcal{G}(H) = 2\sum_{h=1}^{H} \frac{1}{h}\Big(1-(-1)^h \cos\frac{\pi h}{H+1}\Big).$$

5. Investigating the probabilistic random model

Let $\{\mathbb{X}(h)\}_{h\in\mathbb{Z}^*}$ be a sequence of independent random variables supported on $[-N, N]$ and satisfying Assumptions 3a and 3b above. In this section we shall study the moments and the moment generating function of the sum of random variables $\sum_{y\leq|h|<z} c(h)\mathbb{X}(h)$, where $c(h)$ are certain complex numbers such that $c(h) \ll 1/|h|$ for $|h| \geq 1$.

By Deligne's equidistribution theorem and the work of Katz [13], it follows that the random variables in Assumption 3 for the famillies of ℓ-adic trace functions we consider correspond to the trace functions on the compact classical groups USp_{2r}. At the end of this section we will show that these random variables satisfy Assumptions 3a and 3b.

5.1. The moments of $\sum_{y\leq|h|<z} c(h)\mathbb{X}(h)$.

The purpose of this section is to prove the following lemma.

Lemma 5.1. *Let $\mathbb{X}(h)$ be a sequence of I.I.D. random variables satisfying Assumption 3b above. Let $\{c(h)\}_{h\in\mathbb{Z}^*}$ be a sequence of complex numbers such that $|c(h)| \leq c_0/|h|$, where c_0 is a positive constant. Let $1 \leq y < z$ be real numbers. Then, for all integers $k \geq 1$ we have*

$$\mathbb{E}\left(\left|\sum_{y\leq|h|<z} c(h)\mathbb{X}(h)\right|^k\right) \leq \left(\frac{8(c_0N)^2k}{y}\right)^{k/2}. \tag{5.1}$$

Moreover, if $k > y$ then

$$\mathbb{E}\left(\left|\sum_{y\leq |h|<z} c(h)\mathbb{X}(h)\right|^k\right) \leq (10c_0 N\log k)^k. \tag{5.2}$$

Proof. We first prove (5.1) when $k = 2n$ is even. Expanding the moments we obtain

$$\mathbb{E}\left(\left|\sum_{y\leq |h|<z} c(h)\mathbb{X}(h)\right|^{2n}\right) \leq c_0^{2n} \sum_{y\leq |h_1|,\dots,|h_{2n}|<z} \frac{|\mathbb{E}\left(\mathbb{X}(h_1)\cdots\mathbb{X}(h_{2n})\right)|}{|h_1\cdots h_{2n}|}. \tag{5.3}$$

By the independence of the $\{\mathbb{X}(h)\}$ and Assumption 3b, we get

$$\begin{aligned} &\sum_{y\leq |h_1|,\dots,|h_{2n}|<z} \frac{|\mathbb{E}\left(\mathbb{X}(h_1)\cdots\mathbb{X}(h_{2n})\right)|}{|h_1\cdots h_{2n}|} \\ &= \sum_{\ell=1}^{2n} \sum_{\substack{j_1<\cdots<j_\ell\\ y\leq |j_1|,\dots,|j_\ell|<z}} \sum_{\substack{n_1,\dots,n_\ell\geq 1\\ n_1+\cdots+n_\ell=2n}} \binom{2n}{n_1,\dots,n_\ell} \frac{|\mathbb{E}(\mathbb{X}(j_1)^{n_1})|\cdots|\mathbb{E}(\mathbb{X}(j_\ell)^{n_\ell})|}{|j_1^{n_1}\cdots j_\ell^{n_\ell}|} \\ &\leq N^{2n} \sum_{\ell=1}^{n} \sum_{\substack{j_1<\cdots<j_\ell\\ y\leq |j_1|,\dots,|j_\ell|<z}} \sum_{\substack{r_1,\dots,r_\ell\geq 1\\ r_1+\cdots+r_\ell=n}} \binom{2n}{2r_1,\dots,2r_\ell} \frac{1}{j_1^{2r_1}\cdots j_\ell^{2r_\ell}}, \end{aligned} \tag{5.4}$$

where we have used that $\mathbb{E}(\mathbb{X}(h)^m) = 0$ if m is odd, and $|\mathbb{E}(\mathbb{X}(h)^m)| \leq N^m$ if m is even, since $|\mathbb{X}(h)| \leq N$ for all h. Furthermore, observe that

$$\binom{2n}{2r_1,\dots,2r_\ell} \leq \frac{2n!}{n!}\binom{n}{r_1,\dots,r_\ell} \leq (2n)^n\binom{n}{r_1,\dots,r_\ell}.$$

Inserting this bound in (5.4) gives

$$\sum_{y\leq |h_1|,\dots,|h_{2n}|<z} \frac{|\mathbb{E}\left(\mathbb{X}(h_1)\cdots\mathbb{X}(h_{2n})\right)|}{|h_1\cdots h_{2n}|} \leq (2N^2 n)^n \left(\sum_{y\leq |j|<z} \frac{1}{j^2}\right)^n.$$

Therefore, in view of (5.3) and the elementary inequality $\sum_{y\leq |j|<z} 1/j^2 \leq 4/y$ we deduce that

$$\mathbb{E}\left(\left|\sum_{y\leq |h|<z} c(h)\mathbb{X}(h)\right|^{2n}\right) \leq \left(\frac{8(c_0N)^2 n}{y}\right)^n. \tag{5.5}$$

We now establish (5.1) when k is odd. By the Cauchy-Schwarz inequality and (5.5) we have

$$\mathbb{E}\left(\left|\sum_{y\leq |h|<z} c(h)\mathbb{X}(h)\right|^{k}\right) \leq \mathbb{E}\left(\left|\sum_{y\leq |h|<z} c(h)\mathbb{X}(h)\right|^{2k}\right)^{1/2} \leq \left(\frac{8(c_0N)^2 k}{y}\right)^{k/2},$$

as desired.

We now prove (5.2). By (5.1) and Minkowski's inequality we have

$$\mathbb{E}\left(\left|\sum_{y\le|h|<z} c(h)\mathbb{X}(h)\right|^k\right)^{1/k} \le \mathbb{E}\left(\left|\sum_{y\le|h|<k} c(h)\mathbb{X}(h)\right|^k\right)^{1/k} + \mathbb{E}\left(\left|\sum_{k\le|h|<z} c(h)\mathbb{X}(h)\right|^k\right)^{1/k}$$
$$\le c_0N\sum_{y\le|h|<k}\frac{1}{|h|} + \sqrt{8}c_0N \le 10c_0N\log k.$$

This completes the proof. $\square$

5.2. **The moment generating function of a sum involving the $\mathbb{X}(h)$.** In this section we shall estimate the moment generating function of the sum of random variables $\sum_{\substack{-m/2<h\le m/2\\ h\neq 0}}\gamma_m(h)\mathbb{X}(h)$, where the $\gamma_m(h)$ are defined by

$$\gamma_m(h) =: -\frac{1}{m}\mathrm{Im}\sum_{0\le n\le m/2} e_m(nh).$$

This is in fact the probabilistic random model corresponding to the imaginary part of the partial sum $\sum_{0\le n\le x}\varphi_a(n)$ when $x = m/2$. Indeed, by (1.2) we have

$$\frac{1}{\sqrt{m}}\mathrm{Im}\sum_{0\le n\le m/2}\varphi_a(n) = \sum_{\substack{-m/2<h\le m/2\\ h\neq 0}}\gamma_m(h)\widehat{\varphi_a}(h), \tag{5.6}$$

since $\gamma_m(0) = 0$. We prove the following proposition, which generalizes Proposition 3.2 of [17], and will be used to prove the lower bound of Theorem 1.2 in section 7.

Proposition 5.2. *Let m be a large integer and $2 \le s \le (\log m)^2$ be a real number. Then we have*

$$\mathbb{E}\left(\exp\left(s\cdot\sum_{\substack{-m/2<h\le m/2\\ h\neq 0}}\gamma_m(h)\mathbb{X}(h)\right)\right) = \exp\left(\frac{N}{\pi}s\log s + B_0 s + O\left(\log^2 s\right)\right),$$

where

$$B_0 = \frac{N}{\pi}\left(\gamma + \log 2 - \log\pi + \frac{1}{2N}\int_{-\infty}^{\infty}\frac{f_{\mathbb{X}}(u)}{u^2}du\right).$$

To prove this result we need the following lemma.

Lemma 5.3. *Let $\mathbb{X}$ be a random variable with values in $[-N, N]$, such that $\mathbb{E}(\mathbb{X}) = 0$ and $\mathbb{X}$ satisfies Assumption 3a. Let $f_{\mathbb{X}}$ be the function defined in* (1.8). *Then we have the following estimates*

$$f_{\mathbb{X}}(t) \ll \begin{cases} t^2 & \text{if } |t| < 1,\\ \log(2|t|) & \text{if } |t| \ge 1,\end{cases} \tag{5.7}$$

and

$$f_{\mathbb{X}}'(t) \ll \begin{cases} |t| & \text{if } |t| < 1, \\ \dfrac{\log(2|t|)}{|t|} & \text{if } |t| > 1. \end{cases} \tag{5.8}$$

Proof of Lemma 5.3. We start by proving (5.7). If $|t| \leq 1$, we use the Taylor expansion $\mathbb{E}(e^{t\mathbb{X}}) = \mathbb{E}(1 + t\mathbb{X} + O(t^2\mathbb{X}^2)) = 1 + O(t^2)$ since $\mathbb{E}(\mathbb{X}) = 0$ and $|\mathbb{X}| \leq N$. This implies the desired estimate for $f_{\mathbb{X}}(t)$ when $|t| \leq 1$.

We now suppose that $|t| > 1$. We will only prove the result when $t > 1$, since the proof in the case $t < -1$ is similar. Let $\varepsilon > 0$ be a parameter to be chosen. Then we have

$$\mathbb{P}(\mathbb{X} > N - \varepsilon)e^{t(N-\varepsilon)} \leq \mathbb{E}(e^{t\mathbb{X}}) \leq e^{tN}.$$

Choosing $\varepsilon = 1/(2t)$ and using Assumption 3a we obtain

$$\frac{e^{tN}}{(2t)^A} \ll \mathbb{E}(e^{t\mathbb{X}}) \leq e^{tN}, \tag{5.9}$$

from which the desired estimate for $f_{\mathbb{X}}(t)$ follows in this case.

Next, we establish (5.8). Note that $f_{\mathbb{X}}$ is differentiable on $\mathbb{R} \setminus \{-1, 1\}$ and we have

$$f_{\mathbb{X}}'(t) := \begin{cases} \dfrac{\mathbb{E}(\mathbb{X}e^{t\mathbb{X}})}{\mathbb{E}(e^{t\mathbb{X}})} & \text{if } |t| < 1, \\ \dfrac{\mathbb{E}(\mathbb{X}e^{t\mathbb{X}})}{\mathbb{E}(e^{t\mathbb{X}})} - N & \text{if } t > 1, \\ \dfrac{\mathbb{E}(\mathbb{X}e^{t\mathbb{X}})}{\mathbb{E}(e^{t\mathbb{X}})} + N & \text{if } t < -1. \end{cases} \tag{5.10}$$

As before, in the case $|t| < 1$ the estimate of $f_{\mathbb{X}}'(t)$ follows from the Taylor expansions $\mathbb{E}(e^{t\mathbb{X}}) = 1 + O(t^2)$ and $\mathbb{E}(\mathbb{X}e^{t\mathbb{X}}) = \mathbb{E}(\mathbb{X} + t\mathbb{X}^2 + O(t^2|\mathbb{X}^3|)) = t\mathbb{E}(\mathbb{X}^2) + O(t^2)$.

We now suppose that $t > 1$, and let $\delta > 0$ be a parameter to be chosen. Let $\mathcal{A}$ be the event $\mathbb{X} > N - \delta$, and $\mathcal{A}^c$ be its complement. Then we have

$$\mathbb{E}(\mathbb{X}e^{t\mathbb{X}}) = \mathbb{E}(\mathbf{1}_{\mathcal{A}} \cdot \mathbb{X}e^{t\mathbb{X}}) + \mathbb{E}(\mathbf{1}_{\mathcal{A}^c} \cdot \mathbb{X}e^{t\mathbb{X}}) \geq (N - \delta)\mathbb{E}(\mathbf{1}_{\mathcal{A}} \cdot e^{t\mathbb{X}}) + O(e^{t(N-\delta)}).$$

where $\mathbf{1}_{\mathcal{B}}$ denotes the indicator function of an event $\mathcal{B}$. Hence, using that $\mathbb{E}(\mathbf{1}_{\mathcal{A}^c} \cdot e^{t\mathbb{X}}) \leq e^{t(N-\delta)}$ we deduce

$$\mathbb{E}(\mathbb{X}e^{t\mathbb{X}}) \geq (N - \delta)\mathbb{E}(e^{t\mathbb{X}}) + O(e^{t(N-\delta)}). \tag{5.11}$$

We choose $\delta = (A+1)(\log 2t)/t$. Then, it follows from (5.9) that

$$e^{t(N-\delta)} = \frac{e^{tN}}{(2t)^{A+1}} \ll \frac{\mathbb{E}(e^{t\mathbb{X}})}{t}.$$

Inserting this estimate in (5.11), and using the bound $\mathbb{X} \leq N$ gives

$$N - C\frac{\log(2t)}{t} \leq \frac{\mathbb{E}(\mathbb{X}e^{t\mathbb{X}})}{\mathbb{E}(e^{t\mathbb{X}})} \leq N,$$

for some positive constant C which depends only on A. This implies the desired estimate for $f'_{\mathbb{X}}$ in this case. The proof in the case $t < -1$ follows along the same lines.

$\square$

Proof of Proposition 5.2. First, note that for $-m/2 < h \leq m/2$ with $h \neq 0$ we have

$$|\gamma_m(h)| = \left|\mathrm{Im}\left(\frac{e_m\left(h\left(\lfloor\frac{m}{2}\rfloor+1\right)\right)-1}{m\left(e_m(h)-1\right)}\right)\right| \leq \frac{1}{m|\sin(\pi h/m)|} \leq \frac{1}{2|h|}, \tag{5.12}$$

since $\sin(\pi\alpha) \geq 2\alpha$ for $0 \leq \alpha \leq 1/2$. Furthermore, it follows from (1.3) that

$$\gamma_m(h) = \mathrm{Im}\left(\frac{1-e^{\pi i h}}{2\pi i h}\right) + O\left(\frac{1}{m}\right) = \begin{cases} O(\frac{1}{m}) & \text{if } h \text{ is even,} \\ -\frac{1}{\pi h} + O(\frac{1}{m}) & \text{if } h \text{ is odd.} \end{cases} \tag{5.13}$$

By the independence of the $\mathbb{X}(h)$ we have

$$\log \mathbb{E}\left(\exp\left(s \cdot \sum_{\substack{-m/2<h\leq m/2\\ h\neq 0}} \gamma_m(h)\mathbb{X}(h)\right)\right) = \sum_{\substack{-m/2<h\leq m/2\\ h\neq 0}} \log \mathbb{E}\big(\exp\left(s\cdot\gamma_m(h)\mathbb{X}(h)\right)\big).$$

Using the estimate (5.13) and Lemma 5.3 we obtain

$$\sum_{\substack{-m/2<h\leq m/2\\ h\neq 0 \text{ is even}}} \log \mathbb{E}\big(\exp\left(s\cdot\gamma_m(h)\mathbb{X}(h)\right)\big) \ll \sum_{\substack{-m/2<h\leq m/2\\ h\neq 0 \text{ is even}}} \frac{s^2}{m^2} \ll \frac{(\log m)^4}{m}.$$

We now restrict ourselves to the case $h = 2k+1$ is odd. First, it follows from (5.12) and Lemma 5.3 that

$$\sum_{|k|>s^2} \log \mathbb{E}\big(\exp\left(s\cdot\gamma_m(2k+1)\mathbb{X}(2k+1)\right)\big) \ll \sum_{|k|>s^2} \frac{s^2}{k^2} \ll 1.$$

Moreover, when $|k| \leq s^2$ we use (5.13) to get

$$\log \mathbb{E}\big(\exp\left(s\cdot\gamma_m(2k+1)\mathbb{X}(2k+1)\right)\big) = \log \mathbb{E}\left(\exp\left(-\frac{s}{(2k+1)\pi}\mathbb{X}(2k+1)\right)\right) + O\left(\frac{s}{m}\right).$$

Combining these estimates, and using Lemma 5.3 we obtain

$$\begin{aligned} &\log \mathbb{E}\left(\exp\left(s \cdot \sum_{\substack{-m/2<h\leq m/2\\ h\neq 0}} \gamma_m(h)\mathbb{X}(h)\right)\right) \\ &= \frac{2N}{\pi} s \sum_{1\leq 2k+1\leq s/\pi} \frac{1}{2k+1} + \sum_{-s^2\leq k\leq s^2} f_{\mathbb{X}}\left(-\frac{s}{(2k+1)\pi}\right) + O(1). \end{aligned} \tag{5.14}$$

Next, we observe that

$$\sum_{1\leq 2k+1\leq s/\pi} \frac{1}{2k+1} = \frac{1}{2}\sum_{1\leq k\leq s/2\pi} \frac{1}{k} + \log 2 + O\left(\frac{1}{s}\right) = \frac{\log s}{2} + \frac{1}{2}\left(\gamma + \log 2 - \log \pi\right) + O\left(\frac{1}{s}\right).$$

Furthermore, by partial summation and Lemma 5.3 we get

$$\sum_{-s^2\le k\le s^2} f_{\mathbb{X}}\left(-\frac{s}{(2k+1)\pi}\right) = \int_{-s^2}^{-1} f_{\mathbb{X}}\left(-\frac{s}{(2u+1)\pi}\right) du + \int_{0}^{s^2} f_{\mathbb{X}}\left(-\frac{s}{(2u+1)\pi}\right) du + O(\log^2 s).$$

Finally, making the change of variables $v = -s/((2u+1)\pi)$, the main term on the right hand side of this estimate becomes

$$\frac{s}{2\pi}\int_{-s/\pi}^{-s/((2s^2+1)\pi)} \frac{f_{\mathbb{X}}(v)}{v^2}dv + \frac{s}{2\pi}\int_{s/((2s^2-1)\pi)}^{s/\pi} \frac{f_{\mathbb{X}}(v)}{v^2}dv = \frac{s}{2\pi}\int_{-\infty}^{\infty} \frac{f_{\mathbb{X}}(v)}{v^2}dv + O(\log s),$$

by Lemma 5.3. Inserting these estimates in (5.14) completes the proof. □

5.3. **The distribution of the trace function in the classical group** USp_{2n}**.** Fix a positive integer n and put $N = 2n$. Let us endow the unitary symplectic group $G = \mathrm{USp}_{2n}$ with its Haar measure μ, and consider the function $\mathbb{X} : G \to [-N, N]$ that maps M to $\mathrm{Tr}\, M$. Assumption 3b is easy to check for $\mathbb{X}$. Indeed, let ℓ be an odd positive integer. Observing that $-I_{2n} \in \mathrm{USp}_{2n}$ we get

$$\int_G (\mathrm{Tr}\, M)^\ell \mathrm{d}\mu(M) = \int_G (\mathrm{Tr}(-M))^\ell \mathrm{d}\mu(M) = -\int_G (\mathrm{Tr}\, M)^\ell \mathrm{d}\mu(M),$$

whence $\mathbb{E}(\mathbb{X}^\ell) = 0$. In the same way, we have $\mathbb{P}(\mathbb{X} > N - \varepsilon) = \mathbb{P}(\mathbb{X} < -N + \varepsilon)$ for every $\varepsilon > 0$. Let us now verify Assumption 3a.

Lemma 5.4. *There exists a positive real number c_n such that for every $\varepsilon \in (0, 2]$, one has $\mathbb{P}(\mathbb{X} > N - \varepsilon) \ge c_n \varepsilon^{n(2n+1)/2}$.*

Proof. Put

$$\mathcal{A} = \Big\{\underline{\theta} = (\theta_1, \cdots, \theta_n) \in [0, \pi]^n \;\Big|\; \sum_{h=1}^n 2\cos\theta_h > N - \varepsilon\Big\} = \Big\{\underline{\theta} \in [0, \pi]^n \;\Big|\; \sum_{h=1}^n \sin^2\frac{\theta_h}{2} < \frac{\varepsilon}{4}\Big\}$$

and $\mathcal{B} = \{\underline{t} \in \mathbb{R}_+^n \mid t_1^2 + \cdots + t_n^2 < 1\}$. The Weyl integration formula (see for example page 117 of [4]) gives

$$\mathbb{P}(\mathbb{X} > 2n - \varepsilon) = \frac{2^{n^2}}{n!\pi^n}\int_{\mathcal{A}} \prod_{1\le j<k\le n} (\cos\theta_k - \cos\theta_j)^2 \prod_{h=1}^n \sin^2\theta_h \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n.$$

Let us remark that if $\underline{\theta} \in \mathcal{A}$ then

$$\prod_{h=1}^n \cos^2\frac{\theta_h}{2} = \prod_{h=1}^n\Big(1 - \sin^2\frac{\theta_h}{2}\Big) \ge 1 - \sum_{h=1}^n \sin^2\frac{\theta_h}{2} > 1 - \frac{\varepsilon}{4}.$$

We infer that

$$\begin{aligned}\mathbb{P}(\mathbb{X} > N - \varepsilon) &= \frac{2^{n(2n+1)}}{n!\pi^n} \int_{\mathcal{A}} \prod_{j<k} \Big(\sin^2 \frac{\theta_j}{2} - \sin^2 \frac{\theta_k}{2}\Big)^2 \prod_{h=1}^{n} \sin^2 \frac{\theta_h}{2} \cos^2 \frac{\theta_h}{2} \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \\ &\geq \frac{2^{n(2n+1)}}{n!\pi^n} \sqrt{1 - \frac{\varepsilon}{4}} \int_{\mathcal{A}} \prod_{j<k} \Big(\sin^2 \frac{\theta_j}{2} - \sin^2 \frac{\theta_k}{2}\Big)^2 \prod_{h=1}^{n} \sin^2 \frac{\theta_h}{2} \cos \frac{\theta_h}{2} \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \\ &= \frac{2^n \varepsilon^{n(2n+1)/2}}{n!\pi^n} \sqrt{1 - \frac{\varepsilon}{4}} \int_{\mathcal{B}} \prod_{1 \leq j < k \leq n} (t_j^2 - t_k^2)^2 \prod_{h=1}^{n} t_h^2 \mathrm{d}t_1 \cdots \mathrm{d}t_n,\end{aligned}$$

where we use the change of variables $t_h = \dfrac{2}{\sqrt{\varepsilon}} \sin \dfrac{\theta_h}{2}$. Whence the result. □

6. Completing the proof of the upper bound in Theorem 1.2: Proof of Theorem 2.2

In this section, we assume that $\mathcal{F} = \{\varphi_a\}_{a \in \Omega_m}$ is a family of m-periodic complex valued functions satisfying Assumptions 2 and 3. We start by proving the following lemma which follows from combining Assumption 3 with Lemma 5.1.

Lemma 6.1. *Let m be a large positive integer, and $1 \leq y < z \leq m/2$ be real numbers. Let $\{\varphi_a\}_{a \in \Omega_m}$ be a family of m-periodic complex valued functions satisfying Assumptions 2 and 3. Let $\{c(h)\}_{h \in \mathbb{Z}^*}$ be a sequence of complex numbers such that $|c(h)| \leq c_0/|h|$, where c_0 is a positive constant. Then, for all positive integers $k \leq (\log m)/(5 \log\log m)$ we have*

$$\frac{1}{|\Omega_m|} \sum_{a \in \Omega_m} \left| \sum_{y \leq |h| < z} c(h) \widehat{\varphi_a}(h) \right|^{2k} \ll \left(\frac{16 (c_0 N)^2 k}{y} \right)^k + \frac{(4 C_1 c_0 \log m)^{2k}}{m^\eta}.$$

Proof. Expanding the moments and using Assumptions 2 and 3, we obtain

$$\begin{aligned}&\frac{1}{|\Omega_m|} \sum_{a \in \Omega_m} \left| \sum_{y \leq |h| < z} c(h) \widehat{\varphi_a}(h) \right|^{2k} \\ &= \sum_{\substack{y \leq |h_1|, \dots, |h_k| < z \\ y \leq |r_1|, \dots, |r_k| < z}} c(h_1) \cdots c(h_k) \overline{c(r_1) \cdots c(r_k)} \, \frac{1}{|\Omega_m|} \sum_{a \in \Omega_m} \prod_{u=1}^{k} \widehat{\varphi_a}(h_u) \prod_{v=1}^{k} \widehat{\varphi_a}(r_v) \\ &= \sum_{\substack{y \leq |h_1|, \dots, |h_k| < z \\ y \leq |r_1|, \dots, |r_k| < z}} c(h_1) \cdots c(h_k) \overline{c(r_1) \cdots c(r_k)} \, \mathbb{E} \left(\prod_{u=1}^{k} \mathbb{X}(h_u) \prod_{v=1}^{k} \mathbb{X}(r_v) \right) + E_k(y, z),\end{aligned}$$

where the error term satisfies

$$E_k(y,z) \ll \frac{C_1^{2k}}{m^\eta}\Big(\sum_{y\le |h|<z} |c(h)|\Big)^{2k} \le \frac{C_1^{2k}}{m^\eta}\Big(\sum_{y\le |h|<z} \frac{c_0}{|h|}\Big)^{2k} \ll \frac{(4C_1c_0\log m)^{2k}}{m^\eta},$$

by Assumption 3. The result follows upon noting that

$$\begin{aligned}&\sum_{\substack{y\le |h_1|,\dots,|h_k|<z\\ y\le |r_1|,\dots,|r_k|<z}} c(h_1)\cdots c(h_k)\overline{c(r_1)\cdots c(r_k)}\ \mathbb{E}\left(\prod_{u=1}^{k}\mathbb{X}(h_u)\prod_{v=1}^{k}\mathbb{X}(r_v)\right)\\ &= \mathbb{E}\left(\left|\sum_{y\le |h|<z} c(h)\mathbb{X}(h)\right|^{2k}\right) \le \left(\frac{16(c_0N)^2k}{y}\right)^k,\end{aligned}$$

by Lemma 5.1. □

We will deduce Theorem 2.2 from the following results, which generalize Propositions 6.1 and 6.2 of [17]. In both results we assume that $\{\varphi_a\}_{a\in\Omega_m}$ is a family of m-periodic complex valued functions satisfying Assumptions 2 and 3.

Proposition 6.2. *Let m be large, and k be an integer such that $10^5N^2 < k \le (\log m)/(5\log\log m)$. Let S be a non-empty subset of $[0,1)$, and put $y = 10^5N^2k$. Then we have*

$$\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\max_{\alpha\in S}\left|\sum_{y\le |h|<k^2}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k} \ll e^{-4k}.$$

Proposition 6.3. *Let m be a large positive integer, and k be an integer such that $3\le k\le (\log m)/(50\log\log m)$. Let S be a non-empty finite subset of $[0,1)$. Then we have*

$$\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\max_{\alpha\in S}\left|\sum_{k^2\le |h|<m/2}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k} \ll e^{-4k} + \frac{|S|(2C_1\log m)^{8k}}{m^\eta}.$$

We start by proving Proposition 6.2, as its proof is simpler since the inner sum over $|h|$ is short.

Proof of Proposition 6.2. Let $\mathcal{A}_k = \{b/k^4 : 1\le b\le k^4\}$. Then for all $\alpha\in S$, there exists $\beta_\alpha\in\mathcal{A}_k$ such that $|\alpha-\beta_\alpha|\le 1/k^4$. In this case we have $e(\alpha h) = e(\beta_\alpha h) + O(h/k^4)$, and hence

$$\sum_{y\le |h|<k^2}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h) = \sum_{y\le |h|<k^2}\frac{e(\beta_\alpha h)-1}{h}\widehat{\varphi_a}(h) + O\left(\frac{1}{k^2}\right).$$

Therefore, using the elementary inequality $|x+y|^{2k} \le 2^{2k}(|x|^{2k}+|y|^{2k})$ we deduce that

(6.1)
$$\max_{\alpha\in S}\left|\sum_{y\le|h|<k^2}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k} \le 2^{2k}\max_{\alpha\in\mathcal{A}_k}\left|\sum_{y\le|h|<k^2}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}+\left(\frac{c_1}{k^2}\right)^{2k}$$
$$\le 2^{2k}\sum_{\alpha\in\mathcal{A}_k}\left|\sum_{y\le|h|<k^2}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}+\left(\frac{c_1}{k^2}\right)^{2k},$$

for some positive constant c_1. Thus, it follows from Lemma 6.1 that in this case we have

$$\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\max_{\alpha\in S}\left|\sum_{y\le|h|<k^2}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}$$
$$\le 2^{2k}\sum_{\alpha\in\mathcal{A}_k}\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\left|\sum_{y\le|h|<k^2}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}+\left(\frac{c_1}{k^2}\right)^{2k} \tag{6.2}$$
$$\ll k^4 2^{2k}\left(\left(\frac{64N^2k}{y}\right)^k+\frac{(8C_1\log m)^{2k}}{m^\eta}\right)+\left(\frac{c_1}{k^2}\right)^{2k}\ll e^{-4k},$$

which completes the proof. □

Proof of Proposition 6.3. Since the inner sum over $|h|$ is long in this case, we shall split it into dyadic intervals. Let $J_1=\lfloor\log(k^2)/\log 2\rfloor$ and $J_2=\lfloor\log(m/2)/\log 2\rfloor$. We define $z_{J_1}:=k^2$, $z_{J_2+1}:=m/2$, and $z_j:=2^j$ for $J_1+1\le j\le J_2$. Then, using Hölder's inequality we obtain

(6.3)
$$\left|\sum_{k^2\le|h|<m/2}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}=\left|\sum_{J_1\le j\le J_2}\frac{1}{j^2}\cdot\left(j^2\sum_{z_j\le|h|<z_{j+1}}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right)\right|^{2k}$$
$$\le\left(\sum_{J_1\le j\le J_2}\frac{1}{j^{4k/(2k-1)}}\right)^{2k-1}\left(\sum_{J_1\le j\le J_2}j^{4k}\left|\sum_{z_j\le|h|<z_{j+1}}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}\right)$$
$$\le\left(\frac{c_2}{\log k}\right)^{2k+1}\sum_{J_1\le j\le J_2}j^{4k}\left|\sum_{z_j\le|h|<z_{j+1}}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k},$$

for some constant $c_2 > 0$. Therefore, this reduces the problem to bounding the corresponding moments over each dyadic interval $[z_j, z_{j+1}]$, namely

$$\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\max_{\alpha\in S}\left|\sum_{z_j\le|h|<z_{j+1}}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}.$$

We shall consider two cases, depending on whether j is large in terms of $|S|$. First, if $4^j \ge |S|$ then by Lemma 6.1 we have

(6.4)

$$\begin{aligned}&\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\max_{\alpha\in S}\left|\sum_{z_j\le|h|<z_{j+1}}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}\\ &\le\sum_{\alpha\in S}\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\left|\sum_{z_j\le|h|<z_{j+1}}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}\ll 4^j\left(\frac{64N^2k}{2^j}\right)^k+\frac{|S|(8C_1\log m)^{2k}}{m^\eta}.\end{aligned}$$

since $z_j \ge 2^j$ for $J_1 \le j \le J_2$. We now suppose that $4^j < |S|$, and let $\mathcal{B}_j = \{b/4^j : 1 \le b \le 4^j\}$. Then for all $\alpha \in S$ there exists $\beta_\alpha \in \mathcal{B}_j$ such that $|\alpha - \beta_\alpha| \le 1/4^j$. In this case we have $e(\alpha h) = e(\beta_\alpha h) + O(h/4^j)$, and hence we obtain

$$\sum_{z_j\le|h|<z_{j+1}}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)=\sum_{z_j\le|h|<z_{j+1}}\frac{e(\beta_\alpha h)-1}{h}\widehat{\varphi_a}(h)+O\left(\frac{1}{2^j}\right),$$

since $z_{j+1} \asymp z_j \asymp 2^j$. Therefore, similarly to (6.1) we derive

$$\begin{aligned}&\max_{\alpha\in S}\left|\sum_{z_j\le|h|<z_{j+1}}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}\\ &\le 2^{2k}\max_{\alpha\in\mathcal{B}_j}\left|\sum_{z_j\le|h|<z_{j+1}}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}+\left(\frac{c_3}{2^j}\right)^{2k},\end{aligned}$$

for some positive constant c_3. Thus, appealing to Lemma 6.1 we get

$$\begin{aligned}&\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\max_{\alpha\in S}\left|\sum_{z_j\le|h|<z_{j+1}}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}\\ &\le 2^{2k}\sum_{\alpha\in\mathcal{B}_j}\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\left|\sum_{z_j\le|h|<z_{j+1}}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}+\left(\frac{c_3}{2^j}\right)^{2k}\\ &\ll 4^j\left(\frac{2^8N^2k}{2^j}\right)^k+\frac{|S|(16C_1\log m)^{2k}}{m^\eta},\end{aligned}\tag{6.5}$$

since $|\mathcal{B}_j| = 4^j < |S|$. Combining (6.4) and (6.5) we deduce that in all cases we have

$$\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\max_{\alpha\in S}\left|\sum_{z_j\leq|h|<z_{j+1}}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k} \ll 4^j\left(\frac{2^8N^2k}{2^j}\right)^k+\frac{|S|(16C_1\log m)^{2k}}{m^\eta}.$$

Inserting this bound in (6.3) gives

$$\begin{aligned}&\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\max_{\alpha\in S}\left|\sum_{k^2\leq|h|<m/2}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}\\ &\ll\left(\frac{c_4}{\log k}\right)^{2k+1}k^k\sum_{J_1\leq j\leq J_2}4^j\left(\frac{j^4}{2^j}\right)^k+\frac{|S|(2C_1\log m)^{8k}}{m^\eta}\\ &\ll e^{-4k}+\frac{|S|(2C_1\log m)^{8k}}{m^\eta},\end{aligned}\tag{6.6}$$

for some positive constant c_4, where the last estimate follows since $j^4\leq 2^{j/4}$ for j large enough, and $2^{J_1}\asymp k^2$. This completes the proof. □

Finally, we deduce Theorem 2.2.

Proof of Theorem 2.2. By Minkowski's inequality we have

$$\begin{aligned}&\left(\sum_{a\in\Omega_m}\max_{\alpha\in S}\left|\sum_{y\leq|h|<m/2}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}\right)^{1/2k}\\ &\leq\left(\sum_{a\in\Omega_m}\max_{\alpha\in S}\left|\sum_{y\leq|h|<k^2}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}\right)^{1/2k}\\ &\qquad+\left(\sum_{a\in\Omega_m}\max_{\alpha\in S}\left|\sum_{k^2\leq|h|<m/2}\frac{e(\alpha h)-1}{h}\widehat{\varphi_a}(h)\right|^{2k}\right)^{1/2k}.\end{aligned}$$

The result follows upon using Propositions 6.2 and 6.3. □

7. Proof of Theorem 1.6

In this section we shall investigate the distribution of the partial sums $\sum_{0\leq n\leq x}\varphi_a(n)$ in the special case $x=m/2$, where $\mathcal{F}=\{\varphi_a\}_{a\in\Omega_m}$ is a family of m-periodic complex valued functions satisfying Assumptions 2 and 3. For a real number t, we define

$$\Psi_{\mathcal{F}}(t):=\frac{1}{|\Omega_m|}\left|\left\{a\in\Omega_m:\frac{1}{\sqrt{m}}\mathrm{Im}\sum_{0\leq n\leq m/2}\varphi_a(n)>t\right\}\right|.\tag{7.1}$$

We will prove the following result from which Theorem 1.6 follows.

Theorem 7.1. *Let m be large and $\mathcal{F} = \{\varphi_a\}_{a\in\Omega_m}$ be a family of m-periodic complex valued functions satisfying Assumptions 2 and 3. Uniformly for V in the range $1 \leq V \leq \frac{N}{\pi}(\log\log m - 2\log\log\log m - B)$ we have*

$$\Psi_{\mathcal{F}}(V) = \exp\left(-A_0 \exp\left(\frac{\pi}{N}V\right)\left(1 + O\left(Ve^{-\pi V/(2N)}\right)\right)\right).$$

Furthermore, the same estimate holds for the proportion of $a \in \Omega_m$ such that $\frac{1}{\sqrt{m}}\mathrm{Im}\sum_{0\leq n\leq m/2}\varphi_a(n) < -V$, in the same range of V.

Recall from (5.6) that

$$\frac{1}{\sqrt{m}}\mathrm{Im}\sum_{0\leq n\leq m/2}\varphi_a(n) = \sum_{\substack{-m/2<h\leq m/2\\ h\neq 0}}\gamma_m(h)\widehat{\varphi_a}(h).$$

In order to prove Theorem 7.1, we will show that the moment generating function of the sum $\sum_{\substack{-m/2<h\leq m/2\\ h\neq 0}}\gamma_m(h)\widehat{\varphi_a}(h)$ (after removing a "small" set of "bad" points a) is very close to the moment generating function of the probabilistic random model $\sum_{\substack{-m/2<h\leq m/2\\ h\neq 0}}\gamma_m(h)\mathbb{X}(h)$, which we already estimated in Proposition 5.2.

Proposition 7.2. *Let m be large. There exists a set $\mathcal{E}_m \subset \Omega_m$ with cardinality $|\mathcal{E}_m| \leq m^{-1/10}|\Omega_m|$ such that for all complex numbers s with $N|s| \leq (\log m)/(50\log\log m)^2$ we have*

$$\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m\setminus\mathcal{E}_m}\exp\left(s\cdot\sum_{\substack{-m/2<h\leq m/2\\ h\neq 0}}\gamma_m(h)\widehat{\varphi_a}(h)\right) = \mathbb{E}\left(\exp\left(s\cdot\sum_{\substack{-m/2<h\leq m/2\\ h\neq 0}}\gamma_m(h)\mathbb{X}(h)\right)\right) + O\left(\exp\left(-\frac{\log m}{20\log\log m}\right)\right).$$

Proof. Let $\mathcal{E}_m$ be the set of $a \in \Omega_m$ such that

$$\left|\sum_{\substack{-m/2<h\leq m/2\\ h\neq 0}}\gamma_m(h)\widehat{\varphi_a}(h)\right| \geq 4N\log\log m.$$

Using Assumption 2 together with the bound (5.12) we get

$$\left|\sum_{\substack{-m/2<h\leq m/2\\ h\neq 0}}\gamma_m(h)\widehat{\varphi_a}(h)\right| \leq 3N\log\log m + \left|\sum_{(\log m)^2<|h|<m/2}\gamma_m(h)\widehat{\varphi_a}(h)\right|,$$

if m is sufficiently large. Therefore, it follows from Lemma 6.1 that for $r = \lfloor \log m/(10\log\log m)\rfloor$ we have

$$
\begin{aligned}
|\mathcal{E}_m| &\le \Bigg|\Bigg\{a\in\Omega_m : \Big|\sum_{(\log m)^2<|h|<m/2}\gamma_m(h)\widehat{\varphi_a}(h)\Big| \ge N\log\log m\Bigg\}\Bigg| \\
&\le (N\log\log m)^{-2r}\sum_{a\in\Omega_m}\left|\sum_{(\log m)^2<|h|<m/2}\gamma_m(h)\widehat{\varphi_a}(h)\right|^{2r} \qquad (7.2)\\
&\ll m^{-1/10}|\Omega_m|.
\end{aligned}
$$

Let $L = \lfloor \log m/(20\log\log m)\rfloor$. Then we have

$$
\begin{aligned}
&\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m\setminus\mathcal{E}_m}\exp\left(s\cdot\sum_{\substack{-m/2<h\le m/2\\ h\neq 0}}\gamma_m(h)\widehat{\varphi_a}(h)\right) \qquad (7.3)\\
&=\sum_{k=0}^{L}\frac{s^k}{k!}\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m\setminus\mathcal{E}_m}\left(\sum_{\substack{-m/2<h\le m/2\\ h\neq 0}}\gamma_m(h)\widehat{\varphi_a}(h)\right)^k + E_1
\end{aligned}
$$

where

$$
E_1 \ll \sum_{k>L}\frac{|s|^k}{k!}(4N\log\log m)^k \le \sum_{k>L}\left(\frac{15N|s|\log\log m}{L}\right)^k \ll e^{-L}
$$

by Stirling's formula and our assumption on s. Furthermore, note that

$$
\sum_{\substack{-m/2<h\le m/2\\ h\neq 0}}|\gamma_m(h)\widehat{\varphi_a}(h)| \le \frac{N}{2}\sum_{1\le|h|\le m/2}\frac{1}{|h|} \le 3N\log m,
$$

if m is sufficiently large. Therefore, it follows from equation (7.2) that for all integers $0\le k\le L$ we have

$$
\begin{aligned}
&\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m\setminus\mathcal{E}_m}\left(\sum_{\substack{-m/2<h\le m/2\\ h\neq 0}}\gamma_m(h)\widehat{\varphi_a}(h)\right)^k \\
&=\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\left(\sum_{\substack{-m/2<h\le m/2\\ h\neq 0}}\gamma_m(h)\widehat{\varphi_a}(h)\right)^k + O\left(m^{-1/10}(3N\log m)^k\right) \qquad (7.4)\\
&=\mathbb{E}\left(\Bigg(\sum_{\substack{-m/2<h\le m/2\\ h\neq 0}}\gamma_m(h)\mathbb{X}(h)\Bigg)^k\right) + O\left(m^{-1/25}\right),
\end{aligned}
$$

where the last equality follows from expanding the moments and using Assumption 3 as in the proof of Lemma 6.1.

Furthermore, it follows from equation (5.12), Lemma 5.1 and Stirling's formula that

$$\sum_{k>L}\frac{|s|^k}{k!}\mathbb{E}\left(\Big|\sum_{\substack{-m/2<h\le m/2\\ h\neq 0}}\gamma_m(h)\mathbb{X}(h)\Big|^k\right)\ll\sum_{k>L}\left(\frac{15N|s|\log k}{k}\right)^k\ll\sum_{k>L}\left(\frac{15N|s|\log L}{L}\right)^k\ll e^{-L}.$$

Finally, combining this bound with (7.3) and (7.4), we derive

$$\begin{aligned}
&\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m\setminus\mathcal{E}_m}\exp\left(s\cdot\sum_{\substack{-m/2<h\le m/2\\ h\neq 0}}\gamma_m(h)\widehat{\varphi_a}(h)\right)\\
&=\sum_{k=0}^{L}\frac{s^k}{k!}\mathbb{E}\left(\Big(\sum_{\substack{-m/2<h\le m/2\\ h\neq 0}}\gamma_m(h)\mathbb{X}(h)\Big)^k\right)+O\left(e^{-L}+m^{-1/25}e^{|s|}\right)\\
&=\mathbb{E}\left(\exp\left(s\cdot\sum_{\substack{-m/2<h\le m/2\\ h\neq 0}}\gamma_m(h)\mathbb{X}(h)\right)\right)+O\left(e^{-L}\right),
\end{aligned}$$

as desired. □

Using the saddle-point method and Propositions 5.2 and 7.2, we prove Theorem 7.1.

Proof Theorem 7.1. Let $\mathcal{E}_m$ be the set in the statement of Proposition 7.2, and $\widetilde{\Psi_{\mathcal{F}}}(t)$ be the proportion of $a\in\Omega_m\setminus\mathcal{E}_m$ such that $\sum_{\substack{-m/2<h\le m/2\\ h\neq 0}}\gamma_m(h)\widehat{\varphi_a}(h)>t$. Then, it follows from equation (5.6) and Proposition 7.2 that

$$\Psi_{\mathcal{F}}(t)=\widetilde{\Psi_{\mathcal{F}}}(t)+O\left(m^{-1/10}\right).$$

Furthermore, it follows from Propositions 7.2 and 5.2 that for all positive real numbers s such that $2N\le Ns\le(\log m)/(50\log\log m)^2$ we have

$$\begin{aligned}
\int_{-\infty}^{\infty}e^{st}\widetilde{\Psi_{\mathcal{F}}}(t)dt&=\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m\setminus\mathcal{E}_m}\int_{-\infty}^{\sum_{\substack{-m/2<h\le m/2\\ h\neq 0}}\gamma_m(h)\widehat{\varphi_a}(h)}e^{st}dt\\
&=\frac{1}{s|\Omega_m|}\sum_{a\in\Omega_m\setminus\mathcal{E}_m}\exp\left(s\cdot\sum_{\substack{-m/2<h\le m/2\\ h\neq 0}}\gamma_m(h)\widehat{\varphi_a}(h)\right)\qquad(7.5)\\
&=\exp\left(\frac{N}{\pi}s\log s+B_0s+O(\log^2 s)\right).
\end{aligned}$$

The result trivially holds if V is small, so we might assume that V is a sufficiently large real number such that $V \leq (N/\pi)(\log\log m - 2\log\log\log m - B)$, where $B = \log(N/\pi) + 8 - B_0\pi/N$. We shall choose s (the saddle point) such that

$$\left(\frac{N}{\pi}s\log s + B_0 s - sV\right)' = 0 \Longleftrightarrow s = \exp\left(\frac{\pi}{N}V - \frac{\pi}{N}B_0 - 1\right). \tag{7.6}$$

Let $0 < \varepsilon < 1$ be a small parameter to be chosen, and put $S = se^{\varepsilon}$. Then, it follows from (7.5) that

$$\begin{aligned}\int_{V+N\varepsilon/\pi}^{\infty} e^{st}\widetilde{\Psi_{\mathcal{F}}}(t)dt &\leq \exp\left(s(1-e^{\varepsilon})(V+N\varepsilon/\pi)\right)\int_{V+N\varepsilon/\pi}^{\infty} e^{St}\widetilde{\Psi_{\mathcal{F}}}(t)dt \\ &\leq \exp\left(s(1-e^{\varepsilon})(V+N\varepsilon/\pi) + \frac{N}{\pi}se^{\varepsilon}\log s + \frac{N}{\pi}se^{\varepsilon}\varepsilon + B_0 se^{\varepsilon} + O(\log^2 s)\right) \\ &= \exp\left(\frac{N}{\pi}s\log s + B_0 s + \frac{N}{\pi}s(1+\varepsilon-e^{\varepsilon}) + O(\log^2 s)\right).\end{aligned}$$

Therefore, choosing $\varepsilon = C_0(\log s)/\sqrt{s}$ for a suitably large constant C_0 and using (7.5) we obtain

$$\int_{V+N\varepsilon/\pi}^{\infty} e^{st}\widetilde{\Psi_{\mathcal{F}}}(t)dt \leq e^{-V^2}\int_{-\infty}^{\infty} e^{st}\widetilde{\Psi_{\mathcal{F}}}(t)dt.$$

A similar argument shows that

$$\int_{-\infty}^{V-N\varepsilon/\pi} e^{st}\widetilde{\Psi_{\mathcal{F}}}(t)dt \leq e^{-V^2}\int_{-\infty}^{\infty} e^{st}\widetilde{\Psi_{\mathcal{F}}}(t)dt.$$

Combining these bounds with (7.5) gives

$$\int_{V-N\varepsilon/\pi}^{V+N\varepsilon/\pi} e^{st}\widetilde{\Psi_{\mathcal{F}}}(t)dt = \exp\left(\frac{N}{\pi}s\log s + B_0 s + O(\log^2 s)\right). \tag{7.7}$$

Furthermore, since $\widetilde{\Psi_{\mathcal{F}}}(t)$ is non-increasing as a function of t we can bound the above integral as follows

$$e^{sV+O(s\varepsilon)}\widetilde{\Psi_{\mathcal{F}}}(V+N\varepsilon/\pi) \leq \int_{V-N\varepsilon/\pi}^{V+N\varepsilon/\pi} e^{st}\widetilde{\Psi_{\mathcal{F}}}(t)dt \leq e^{sV+O(s\varepsilon)}\widetilde{\Psi_{\mathcal{F}}}(V-N\varepsilon/\pi).$$

Inserting these bounds in (7.7) and using the definition of s in terms of V, we obtain

$$\widetilde{\Psi_{\mathcal{F}}}(V+N\varepsilon/\pi) \leq \exp\left(-\frac{N}{\pi}\exp\left(\frac{\pi}{N}V - \frac{\pi}{N}B_0 - 1\right)\left(1+O(\varepsilon)\right)\right) \leq \widetilde{\Psi_{\mathcal{F}}}(V-N\varepsilon/\pi),$$

and thus

$$\widetilde{\Psi_{\mathcal{F}}}(V) = \exp\left(-\frac{N}{\pi}\exp\left(\frac{\pi}{N}V - \frac{\pi}{N}B_0 - 1\right)\left(1+O\left(Ve^{-\pi V/(2N)}\right)\right)\right),$$

as desired.

$\square$

We end this section by proving Corollary 1.7. By Theorem 7.1, it follows that there are $\gg |\Omega_m| m^{-1/\log\log m}$ elements $a \in \Omega_m$ such that

$$\left| \sum_{0 \leq n \leq m/2} \varphi_a(n) \right| \geq \left(\frac{N}{\pi} + o(1) \right) \sqrt{m} \log\log m.$$

Hence, in order to deduce Corollary 1.7 it suffices to show that $|\Omega_m|$ is larger than a multiple of m.

Lemma 7.3. *Let $\mathcal{F} = \{\varphi_a\}_{a \in \Omega_m}$ be a family of m-periodic complex valued functions satisfying Assumptions 2 and 3. Then, we must have $|\Omega_m| \gg m$.*

Proof. Set $J = (-m/2, m/2] \cap \mathbb{Z} \smallsetminus \{0\}$. Let us recall the following elementary result: if M is a symmetric real matrix, then $(\operatorname{Tr} M)^2 \leq (\mathrm{rk} M) \operatorname{Tr}(M^2)$; one sees this by applying the Cauchy-Schwarz inequality to the non-zero eigenvalues of M. We use it with $M = [b_{hj}]_{h,j} = {}^{\mathrm{t}}LL$ where $L = [\widehat{\varphi_a}(h)]_{a,h}$ (here the index a is in Ω_m and h, j are in J). Putting $\beta = \mathbb{E}(\mathbb{X}^2)$, Assumption 3 with $k = 2$ gives $b_{hh}/|\Omega_m| = \beta + O(1/\sqrt{m})$ for every $h \in J$ and $b_{hj}/|\Omega_m| = O(1/\sqrt{m})$ for every distinct h, j. We deduce that

$$|\Omega_m|^2(\beta^2 m^2 + O(m^{3/2})) = (\operatorname{Tr} M)^2 \leq (\mathrm{rk} M) \operatorname{Tr}(M^2) \ll m |\Omega_m|^2 \mathrm{rk} M.$$

Whence $m \ll \mathrm{rk} M \leq \mathrm{rk} L \leq |\Omega_m|$. □

8. An example with very large partial sums

In this section we shall prove Proposition 1.8. Let $m \geq 7$ be an integer. Put $r = \lfloor 3 \log m / \log 2 \rfloor$ and $P = \sum_{k=1}^{r} X^{2k-1}$. Take a finite field Ω_m with 2^r elements and $\psi : \Omega_m \to \{-1, 1\}$ a non-trivial additive character. By Weil's theorem one has

$$\Big| \sum_{a \in \Omega_m} \psi(P(a)) \Big| \leq (2r-2)2^{r/2}, \text{ so } |\{a \in \Omega_m \mid \psi(P(a)) = 1\}| \geq \frac{2^r}{2} - (r-1)2^{r/2} > \frac{m}{2}$$

and the same is true for $|\{a \in \Omega_m \mid \psi(P(a)) = -1\}|$. Putting $J = (-m/2, m/2] \cap \mathbb{Z}$, we can therefore choose distinct elements $(\alpha_h)_{h \in J}$ of $\Omega_m^\times$ such that $\psi(P(\alpha_h)) = 1$ if $h \geq 1$ and $\psi(P(\alpha_h)) = -1$ if $h \leq 0$. For every $a \in \Omega_m$, we define φ_a in such a way that $\forall h \in J \; \widehat{\varphi_a}(h) = \psi(P(\alpha_h a))$, that is, we put

$$\forall n \in \mathbb{Z} \quad \varphi_a(n) = \frac{1}{\sqrt{m}} \sum_{h \in J} \psi(P(\alpha_h a)) e_m(-nh).$$

Let $\{h_1, \cdots, h_k\}$ be a non-empty subset of J with at most r elements. By Vandermonde's formula, the polynomial $P(\alpha_{h_1} X) + \cdots + P(\alpha_{h_k} X)$ has at least one non-zero coefficient and has odd degree. Applying Weil's theorem, we obtain

$$\Big|\sum_{a\in\Omega_m}\widehat{\varphi_a}(h_1)\cdots\widehat{\varphi_a}(h_k)\Big|\leq(2r-2)2^{r/2}\ll m^{3/2}\log m.$$

This implies Assumption 3 with any $1<\eta<3/2$. Indeed, take a sequence $(\mathbb{X}(h))_{h\in\mathbb{Z}}$ of I.I.D. random variables such that $\mathbb{P}(\mathbb{X}(h)=1)=\mathbb{P}(\mathbb{X}(h)=-1)=1/2$. For every positive integer $k\leq 3\log m/\log 2$ and every $(h_1,\dots,h_k)\in J^k$, one has

$$\frac{1}{|\Omega_m|}\sum_{a\in\Omega_m}\widehat{\varphi_a}(h_1)\cdots\widehat{\varphi_a}(h_k)=\mathbb{E}(\mathbb{X}(h_1)\cdots\mathbb{X}(h_k))+O\Big(\frac{\log m}{m^{3/2}}\Big)$$

(one can in fact prove this estimate for all positive $k\leq 2r$). Thus, we deduce that our family satisfies Assumption 2 with $N=1$, and Assumption 3 with $\eta=4/3$.

To conclude the proof of Proposition 1.8, let us look at $a=1$: we have $\widehat{\varphi_1}(h)=1$ if $h\geq 1$ and $\widehat{\varphi_1}(h)=-1$ if $h\leq 0$. Using the estimate (1.3) we deduce

$$\mathcal{M}(\varphi_1)\geq\Big|\sum_{0\leq n\leq m/2}\varphi_1(n)\Big|=\Big|\sqrt{m}\sum_{1\leq n\leq m/2}\frac{(-1)^n-1}{i\pi n}+O(\sqrt{m})\Big|=\frac{\sqrt{m}}{\pi}\log m+O(\sqrt{m}).$$

9. Applications to families of ℓ-adic trace functions: Proof of Corollaries 1.11, 1.12 and 1.13

In this section we recall some notions of the formalism of ℓ-adic trace functions and list some examples of families of functions for which we can apply Theorem 1.2 and Theorem 1.6. For a general introduction on this subject we refer the reader to [7]. Basic statements and references can also be found in [8]. In the following $p,\ell>2$ are distinct prime numbers and $i:\overline{\mathbb{Q}}_\ell\hookrightarrow\mathbb{C}$ is a fixed isomorphism. Let $\mathcal{F}$ be a middle-extension ℓ-adic sheaf on $\overline{\mathbb{P}}^1_{\mathbb{F}_p}$ pure of weight 0. For any $x\in\overline{\mathbb{P}}^1_{\mathbb{F}_p}(\mathbb{F}_{p^n})$ one defines

$$t_{\mathcal{F},n}(x):=i(\mathrm{Tr}(\mathrm{Fr}_{p^n}|\mathcal{F}_{\overline{x}})),$$

where Fr is the geometric Frobenius automorphism of $\mathbb{F}_{p^n}$ and $\mathcal{F}_{\overline{x}}$ is the stalk of $\mathcal{F}$ at a geometric point $\overline{x}$ over x. The function $t_{\mathcal{F},n}$ is called *the trace function attached to $\mathcal{F}$ over $\mathbb{F}_{p^n}$*. If there is not ambiguity, we denote by $t_{\mathcal{F}}$ the trace function $t_{\mathcal{F},1}$.

9.1. **Conductor and Fourier transform.**

Definition 9.1 ([10] pp. $4-6$)**.** Let $\mathcal{F}$ be a middle-extension ℓ-adic sheaf on $\overline{\mathbb{P}}^1_{\mathbb{F}_p}$. The *conductor of $\mathcal{F}$* is defined as

$$c(\mathcal{F}):=\mathrm{Rank}(\mathcal{F})+|\,\mathrm{Sing}(\mathcal{F})|+\sum_x\mathrm{Swan}_x(\mathcal{F}),$$

where

i) $\mathrm{Rank}(\mathcal{F}):=\dim\mathcal{F}_x$, for any x where $\mathcal{F}$ is lisse.
ii) $\mathrm{Sing}(\mathcal{F}):=\{x\in\overline{\mathbb{P}}^1_{\mathbb{F}_p}:\mathcal{F}\text{ is not lisse at }x\}$.

iii) For any $x \in \overline{\mathbb{P}}^1_{\mathbb{F}_p}$, $\mathrm{Swan}_x(\mathcal{F})$ is the *Swan conductor of $\mathcal{F}$ at x* (see [13][Chapter 1] for the definition of the Swan conductor).

Let $\mathcal{F}$ be a middle-extension ℓ-adic sheaf and let $t_{\mathcal{F},n}$ be the trace function attached to $\mathcal{F}$. We recall that the normalized Fourier transform of $t_{\mathcal{F},n}$

$$\mathrm{FT}(t_{\mathcal{F},n})(x) := -\frac{1}{\sqrt{p^n}} \sum_{y \in \mathbb{F}_{p^n}} t_{\mathcal{F},n}(y) e_p(\mathrm{Tr}_{\mathbb{F}_{p^n}/\mathbb{F}_p}(xy))$$

is still a trace function. Indeed, one proves the following Theorem ([13, Chapter 5, 8],[8, Proposition 8.2]):

Theorem 9.2 (Fourier sheaves)**.** *Let $\mathcal{F}$ be a middle-extension ℓ-adic sheaf over $\overline{\mathbb{P}}^1_{\mathbb{F}_p}$, such that $\mathcal{F}$ does not contain any Artin-Schreier sheaf $\mathcal{L}_{e_p(aT)}$ in its Jordan-Hölder decomposition (a sheaf with this property is called Fourier sheaf). Then there exists a middle-extension ℓ-adic sheaf* $\mathrm{FT}(\mathcal{F})$ *over $\overline{\mathbb{P}}^1_{\mathbb{F}_p}$, such that*

$$t_{\mathrm{FT}(\mathcal{F}),n}(x) = -\frac{1}{p^{n/2}} \sum_{y \in \mathbb{F}_{p^n}} t_{\mathcal{F},n}(y) e_p(\mathrm{Tr}_{\mathbb{F}_{p^n}/\mathbb{F}_p}(xy)),$$

for any $n \geq 1$ and any $x \in \mathbb{F}_{p^n}$. Moreover one has

i) If $\mathcal{F}$ is geometrically irreducible, pure of weight 0 *then the same holds for* $\mathrm{FT}(\mathcal{F})$. *Moreover* $\mathrm{FT}(\mathcal{F})$ *is a Fourier sheaf with the property*

$$\mathrm{FT}(\mathrm{FT}(\mathcal{F})) = [\times(-1)]^*\mathcal{F},$$

ii) $\mathrm{Rank}(\mathrm{FT}(\mathcal{F})) \leq \Big(\sum_x \mathrm{Swan}_x(\mathcal{F}) + |\mathrm{Sing}(\mathcal{F})| \cdot \mathrm{Rank}(\mathcal{F})\Big)$,

iii) $|\mathrm{Sing}(\mathrm{FT}(\mathcal{F}))| \leq 2 + \mathrm{Rank}(\mathcal{F})$,

iv) $\mathrm{Swan}_x(\mathrm{FT}(\mathcal{F})) \leq c(\mathcal{F})$.

In particular one gets $c(\mathrm{FT}(\mathcal{F})) \leq 10c(\mathcal{F})^2$.

The main examples of trace functions we should have in mind are

(*i*) For any $f, g \in \mathbb{F}_p(T)$, and any multiplicative character χ on $\mathbb{F}_p^\times$ the function $x \mapsto e(f(x)/p)\chi(g(x))$: this is the trace function attached to the Artin-Schreier sheaf $\mathcal{L}_{e_p(f(T))\chi(g(T))}$.

(*ii*) The r-th hyper-Kloosterman sums: the map

$$x \mapsto \mathrm{Kl}_r(x;p) = \frac{(-1)^{r-1}}{p^{(r-1)/2}} \sum_{\substack{y_1,\dots,y_r \in \mathbb{F}_p^\times \\ y_1 \cdot \dots \cdot y_r = x}} e\Big(\frac{y_1 + \cdots + y_r}{p}\Big)$$

can be seen as the trace function attached to the *Kloosterman sheaf* $\mathcal{K}\ell_r$ (see [13] for the definition of such sheaf and for its basic properties).

9.2. **2-parameter families.**

Definition 9.3. Let $p, \ell > 2$ be prime numbers with $p \neq \ell$ and let $r \geq 2$ be an integer. A middle-extension ℓ-adic sheaf, $\mathcal{K}$, is said of Sp_{2r}-type (respectively of SL_r-type, SO_r-type) if

i) $\mathcal{K}$ is pure of weight 0,
ii) one has $G_{\mathcal{K}}^{\mathrm{arith}} = G_{\mathcal{K}}^{\mathrm{geom}}$ (see [13][Chapter 3] for the definition of the monodromy groups) and $G_{\mathcal{K}}^{\mathrm{geom}} = \mathrm{Sp}_{2r}(\mathbb{C})$ (respectively $G_{\mathcal{K}}^{\mathrm{geom}} = \mathrm{SL}_r(\mathbb{C})$, $G_{\mathcal{K}}^{\mathrm{geom}} = \mathrm{SO}_r(\mathbb{C})$).

Definition 9.4. Let $p, \ell > 2$ be prime numbers and let $r \geq 2$ be an integer. A family $\{\mathcal{F}_{a,b}\}_{(a,b)\in\mathbb{F}_p^\times\times\mathbb{F}_p^\times}$ is said to be a 2*-parameter family of* Sp_{2r}*-type (respectively of* SL_r*-type,* SO_r*-type)* if the following conditions are satisfied:

i) for any $a, b \in \mathbb{F}_p^\times$, $\mathcal{F}_{a,b}$ is a Fourier, irreducible middle-extension ℓ-adic sheaf on $\mathbb{A}^1_{\mathbb{F}_p}$ pointwise pure of weight 0.
ii) there exists $C \geq 1$ such that

$$c(\mathcal{F}_{a,b}) \leq C,$$

for any p and $a, b \in \mathbb{F}_p^\times$. We call the smallest C with this property the *conductor of the family* and we denote it by $C_{\mathfrak{F}}$.
iii) for any $a \in \mathbb{F}_p^\times$ the ℓ-adic sheaf $\mathcal{K}_{a,1} = \mathrm{FT}(\mathcal{F}_{a,1})$ is of Sp_{2r}-type (respectively of SL_r-type, SO_r-type).
iv) for any $a, b, z \in \mathbb{F}_p^\times$, there exists $\gamma_z \in \mathrm{PGL}_2(\mathbb{F}_p)$ such that

$$t_{\mathcal{K}_{a,b}}(z) = t_{\mathcal{K}_{a,1}}(\gamma_z(b)).$$

Moreover $\gamma_z \neq \gamma_{z'}$ if $z \neq z'$.
v) for any $a, z_1, z_2 \in \mathbb{F}_p^\times$, if $z_1 \neq z_2$ one has

$$[\gamma_{z_1}\gamma_{z_2}^{-1}]^*\mathcal{K}_{a,1} \neq \mathcal{K}_{a,1} \otimes \mathcal{L},$$

for any ℓ-adic sheaf, $\mathcal{L}$, of rank 1.
vi) there exists $\delta, \alpha > 0$ such that for any interval I of length $|I| \leq p^{1/2+\delta}$, one has

$$\frac{1}{p^2}\sum_{(a,b)\in\mathbb{F}_p^\times\times\mathbb{F}_p^\times}\Big|\frac{1}{\sqrt{p}}\sum_{n\in I} t_{a,b}(n)\Big|^\alpha \ll p^{-1/2-\delta}.$$

Definition 9.5. Let $p, \ell > 2$ be prime numbers and let $r \geq 2$ be an integer. A family $\{\mathcal{F}_\beta\}_{\beta\in\mathbb{F}_p^\times}$ is said to be a 1*-parameter family of* Sp_{2r}*-type (respectively of* SL_r*-type,* SO_r*-type)* if the family $\{\mathcal{G}_{a,\beta}\}_{(a,\beta)\in\mathbb{F}_p^\times\times\mathbb{F}_p^\times}$ is a 2-parameter family of Sp_{2r}-type (respectively of SL_r-type, SO_r-type), where $\mathcal{G}_{a,\beta} = \mathcal{F}_\beta$ for any $a, \beta \in \mathbb{F}_p^\times$.

Proposition 9.6. *Let $r \geq 1$ be an integer. Let $\{\mathcal{F}_{a,b}\}_{(a,b)\in\mathbb{F}_p^\times\times\mathbb{F}_p^\times}$ be a 2-parameter family of Sp_{2r}-type. Then Theorem 1.2 and Theorem 1.6 hold for $\{t_{\mathcal{F}_{a,b}}\}_{(a,b)\in\mathbb{F}_p^\times\times\mathbb{F}_p^\times}$. The same is true for 1-parameter families of Sp_{2r}-type.*

Proof. It is enough to show that the set $\{t_{\mathcal{F}_{a,b}}\}_{(a,b)\in\mathbb{F}_p^\times\times\mathbb{F}_p^\times}$ satisfies Assumptions 1, 2, 3 and 4. We start showing that this family satisfies Assumption 1: for any a, b one has that $\|t_{\mathcal{F}_{a,b}}\|_\infty \leq \mathrm{Rank}(\mathcal{F}_{a,b}) \leq c(\mathcal{F}_{a,b}) \leq C_{\mathfrak{F}}$ thanks to [5][Lemma 1.8.1]. We continue checking Assumption 2. For any $a, b \in \mathbb{F}_p^\times$ and any $x \in \overline{\mathbb{P}}^1_{\mathbb{F}_p}(\mathbb{F}_p)$, one has that the conjugacy class of $(\mathrm{Fr}_{p^n}|(\mathcal{K}_{a,b})_{\overline{x}})$ intersects USp_{2r}. It follows that $t_{\mathcal{K}_{a,b}}(x) = \mathrm{Tr}(\mathrm{Fr}_{p^n}|(\mathcal{K}_{a,b})_{\overline{x}}) \in [-r, r]$, since the trace of any element in USp_{2r} is supported in $[-r, r]$. Let us check Assumption 3. Let $(h_1, ..., h_k) \in (-p/2, p/2]^k$ with $h_i \neq 0$ for $i = 1, ..., k$ and consider

$$\frac{1}{(p-1)^2}\sum_{a\in\mathbb{F}_p^\times}\sum_{b\in\mathbb{F}_p^\times} t_{\mathcal{K}_{a,b}}(h_1)\cdots t_{\mathcal{K}_{a,b}}(h_k). \tag{9.1}$$

We know that for any $a, b, h_i \in \mathbb{F}_p^\times$, $t_{\mathcal{K}_{a,b}}(h_i) = t_{\mathcal{K}_{a,1}}(\gamma_{h_i}(b))$ for some $\gamma_{h_i} \in \mathrm{PGL}_2(\mathbb{F}_p)$ ((iv) in Definition 9.4). Thus we can rewrite the equation above as

$$\frac{1}{(p-1)^2}\sum_{a\in\mathbb{F}_p^\times}\sum_{b\in\mathbb{F}_p^\times} t_{\mathcal{K}_{a,1}}(\gamma_{h_1}(b))\cdots t_{\mathcal{K}_{a,1}}(\gamma_{h_k}(b)).$$

Thanks to the property (v) in the definition of a 2-parameter family, we can argue as in [21][4.2.1] getting

$$(-1)^k\sum_{b\in\mathbb{F}_p^\times} t_{\mathcal{K}_{a,1}}(\gamma_{h_1}(b))\cdots t_{\mathcal{K}_{a,1}}(\gamma_{h_k}(b)) = \mathbb{E}(\mathbb{X}(h_1)...\mathbb{X}(h_k))(p-1) + O(c(\mathcal{H})\sqrt{p}),$$

where $\mathcal{H} = [\gamma_{h_1}]^*\mathcal{K}_{a,1} \otimes \cdots \otimes [\gamma_{h_k}]^*\mathcal{K}_{a,1}$ and the $\mathbb{X}(h_i)$'s are independent random variables uniformly distributed with respect to the Haar measure on USp_{2r} which satisfy Assumptions 3a (Lemma 5.4) and 3b. Let us bound $c(\mathcal{H})$. Recall that

$$c(\mathcal{H}) = \mathrm{Rank}(\mathcal{H}) + |\mathrm{Sing}(\mathcal{H})| + \sum_x \mathrm{Swan}_x(\mathcal{H}).$$

One has that $\mathrm{Rank}(\mathcal{H}) = \prod_i \mathrm{Rank}([\gamma_{h_i}]^*\mathcal{K}_{a,1}) = \mathrm{Rank}(\mathcal{K}_{a,1})^k$ and that $|\mathrm{Sing}(\mathcal{H})| \leq \sum_i |\mathrm{Sing}([\gamma_{h_i}]^*\mathcal{K}_{a,1})| \leq k|\mathrm{Sing}(\mathcal{K}_{a,1})|$. On the other hand, [13][Lemma 1.3] implies that

$$\mathrm{Swan}_x(\mathcal{H}) \leq \mathrm{Rank}(\mathcal{H})\cdot\Big(\sum_{i=1}^k \mathrm{Swan}_x([\gamma_{h_i}]^*\mathcal{K}_{a,1})\Big) \leq \mathrm{Rank}(\mathcal{K}_{a,1})^k k c(\mathcal{K}_{a,1}).$$

Thus we have that

$$\begin{aligned} c(\mathcal{H}) &\leq \mathrm{Rank}(\mathcal{K}_{a,1})^k + k|\mathrm{Sing}(\mathcal{K}_{a,1})| + k|\mathrm{Sing}(\mathcal{K}_{a,1})|\cdot\mathrm{Rank}(\mathcal{K}_{a,1})^k k c(\mathcal{K}_{a,1}) \\ &\leq c(\mathcal{K}_{a,1})^k + k c(\mathcal{K}_{a,1}) + k^2 c(\mathcal{K}_{a,1})^{k+2} \\ &\ll C_{\mathfrak{F}}^{8k}, \end{aligned}$$

where in the last step we used property (ii) in the definition of 2-parameter family together with Theorem 9.2. Thus (9.1) becomes

$$\frac{(-1)^k}{(p-1)^2}\sum_{a\in\mathbb{F}_p^\times}\sum_{b\in\mathbb{F}_p^\times} t_{\mathcal{K}_{a,b}}(h_1)\cdots t_{\mathcal{K}_{a,b}}(h_k) = \mathbb{E}\left(\mathbb{X}(h_1)\ldots\mathbb{X}(h_k)\right) + O\left(\frac{C_{\mathfrak{F}}^{8k}}{\sqrt{p}}\right),$$

as we wanted. Finally, Assumption 4 simply follows from the definition of a 2-parameter family (property (vi)).

□

9.3. **Examples of 1-parameter families.**

9.3.1. *Exponential sums I.* Let $g\in\mathbb{Z}[t]$ be an odd polynomial of degree $2r+1$, such that $r\geq 1$. For p large enough

$$\{\mathcal{L}_{e_p(\beta T+g(T))}\}_{\beta\in\mathbb{F}_p^\times}$$

is a 1-parameter family of Sp_{2r}-type.

$i)$ For any $\beta\in\mathbb{F}_p^\times$, the Artin-Schreier sheaf $\mathcal{L}_{e_p(\beta T+g(T))}$ is a Fourier, irreducible middle-extension ℓ-adic sheaf on $\overline{\mathbb{P}}^1_{\mathbb{F}_p}$ pointwise pure of weight 0. Moreover its trace function is $t_{\mathcal{F}_\beta}: x\mapsto e_p(\beta x+g(x))$.

$ii)$ One has that $\mathrm{Sing}(\mathcal{L}_{e_p(\beta T+g(T))})=\{\infty\}$ for any $\beta\in\mathbb{F}_p^\times$. Moreover, if $p>2r+1$ one has that $\mathrm{Swan}_\infty(\mathcal{L}_{e_p(\beta T+g(T))})=\deg g=2r+1$. Thus $c(\mathcal{L}_{e_p(\beta T+g(T))})=2r+3$ for any $\beta\in\mathbb{F}_p^\times$.

$iii)$ The sheaf $\mathcal{K}_1$ is such that $G^{\mathrm{arith}}_{\mathcal{K}_1}=G^{\mathrm{geom}}_{\mathcal{K}_1}=\mathrm{Sp}_{2r}(\mathbb{C})$. This is done in [14, Sp-example (2)].

$iv)$ Let $\beta\in\mathbb{F}_p^\times$. By definition of the Fourier transform we have

$$\begin{aligned} t_{\mathcal{K}_\beta}(z) &= -\frac{1}{\sqrt{p}}\sum_{x\in\mathbb{F}_p} e_p(x+g(x)+(\beta+z-1)x) \\ &= t_{\mathcal{K}_1}(\beta+z-1) \end{aligned}$$

thus $t_{\mathcal{K}_\beta}(z)=t_{\mathcal{K}_1}(\gamma_z(\beta))$ for $\gamma_z:=\begin{pmatrix}1 & z-1\\ 0 & 1\end{pmatrix}$ as we wanted.

$v)$ This is done in [21][Proposition 7.5].

$vi)$ By Weyl's method (see for example [12][Lemma 20.3]), there exists $\eta>0$ such that

$$\Big|\frac{1}{\sqrt{p}}\sum_{n\in I} e_p(n\beta+g(n))\Big| \ll p^{-\eta}$$

for any interval I of length $|I| \leq p^{1/2+\eta}$. Moreover η and the implied constant depend only on $\deg g$. Thus for any α we get

$$\frac{1}{p-1} \sum_{\beta \in \mathbb{F}_p^\times} \Big| \frac{1}{\sqrt{p}} \sum_{n \in I} e_p(n\beta + g(n)) \Big|^\alpha \ll p^{-\alpha\eta}.$$

Choosing a suitable $\alpha > 1$, property (vi) in Definition 9.4 is satisfied.

9.4. **Examples of 2-parameter families.** In this section we present some examples of families of Sp_{2r}-type. In the following, if $\mathcal{F}_{a,b}$ is a sheaf we denote $\mathcal{K}_{a,b} = \mathrm{FT}(\mathcal{F}_{a,b})$.

9.4.1. *Exponential sums II.* Let $d \in \mathbb{N}_{\geq 1}$ with d odd. For p large enough

$$\{\mathcal{L}_{e_p(bT+(a\overline{T})^d))}\}_{(a,b) \in \mathbb{F}_p^\times \times \mathbb{F}_p^\times}$$

is a 2-parameter family of Sp_{d+1}-type.

i) For any $(a,b) \in \mathbb{F}_p^\times \times \mathbb{F}_p^\times$, the Artin-Schreier sheaf $\mathcal{L}_{e_p(bT+(a\overline{T})^d)}$ is a Fourier, irreducible middle-extension ℓ-adic sheaf on $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$ pointwise pure of weight 0. Moreover its trace function is $t_{\mathcal{F}_{a,b}} : x \mapsto e_p(bx + (a\overline{x})^d)$.

ii) One has that $\mathrm{Sing}(\mathcal{L}_{e_p(bT+(a\overline{T})^d)}) = \{0, \infty\}$ for any $(a,b) \in \mathbb{F}_p^\times \times \mathbb{F}_p^\times$. Moreover, if $d < p$ one has

$$\mathrm{Swan}_0(\mathcal{L}_{e_p(bT+(a\overline{T})^d)}) = d, \quad \mathrm{Swan}_\infty(\mathcal{L}_{e_p(bT+(a\overline{T})^d)}) = 1.$$

Thus $c(\mathcal{L}_{e_p(bT+(a\overline{T})^d)}) = d + 4$ for any $(a,b) \in \mathbb{F}_p^\times \times \mathbb{F}_p^\times$.

iii) For any $a \in \mathbb{F}_p^\times$, the sheaf $\mathcal{K}_{a,1}$ is such that $G^{\mathrm{arith}}_{\mathcal{K}_{a,1}} = G^{\mathrm{geom}}_{\mathcal{K}_{a,1}}$ and

$$G^{\mathrm{geom}}_{\mathcal{K}_{a,1}} = \begin{cases} \mathrm{Sp}_{d+1}(\mathbb{C}) & \text{if } d \text{ is odd} \\ \mathrm{SL}_{d+1}(\mathbb{C}) & \text{if } d \text{ is even.} \end{cases}$$

This is done in [14, Paragraphs 7.12.3.1].

iv) Let $(a,b) \in \mathbb{F}_p^\times \times \mathbb{F}_p^\times$. By definition of the Fourier transform we have

$$\begin{aligned} t_{\mathcal{K}_{a,b}}(z) &= -\frac{1}{\sqrt{p}} \sum_{x \in \mathbb{F}_p^\times} e_p(bx + (a\overline{x})^d + xz) \\ &= -\frac{1}{\sqrt{p}} \sum_{x \in \mathbb{F}_p^\times} e_p((a\overline{x})^d + (b+z)x) \\ &= t_{\mathcal{K}_{a,1}}(b + z - 1) \end{aligned}$$

thus $t_{\mathcal{K}_{a,b}}(z) = t_{\mathcal{K}_{a,1}}(\gamma_z(b))$ for $\gamma_z := \begin{pmatrix} 1 & z-1 \\ 0 & 1 \end{pmatrix}$ as we wanted.

v) For any $a, z_1, z_2 \neq 0$, with $z_1 \neq z_2$, we need to prove that

$$[\gamma_{z_1}\gamma_{z_2}^{-1}]^* \mathcal{K}_{a,1} \neq \mathcal{K}_{a,1} \otimes \mathcal{L}, \tag{9.2}$$

for any ℓ-adic sheaf, $\mathcal{L}$, of rank 1, where $\gamma_{z_1}, \gamma_{z_2}$ are as in (iv). First of all observe that $\gamma_{z_1}\gamma_{z_2}^{-1} = \begin{pmatrix} 1 & z_1 - z_2 \\ 0 & 1 \end{pmatrix}$. Let us denote $\mathcal{K}_{a,0} := \mathrm{FT}(\mathcal{L}_{e_p((a\overline{T})^d)})$. Arguing as in (iv) one has that $\mathcal{K}_{a,1} = [\tau]^*\mathcal{K}_{a,0}$ where $\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Thus to check (9.2) it is enough to check that

$$[\gamma_z']^*\mathcal{K}_{a,0} \neq \mathcal{K}_{a,0} \otimes \mathcal{L},$$

for any ℓ-adic sheaf, $\mathcal{L}$, of rank 1, where $\gamma_z' = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ with $z \neq 0$. Since $\mathcal{L}_{e_p((a\overline{T})^d)}$ is lisse at $\{\infty\}$, then $\mathcal{K}_{a,0} = \mathrm{FT}(\mathcal{L}_{e_p((a\overline{T})^d)})$ is singular at $\{0, \infty\}$ ([13, Corollary 8.5.8]). Moreover using Laumon's Theory one gets that $\mathcal{K}_{a,0}$ has an unique slope at $d/(d+1)$ at ∞ and that it is tame at 0 ([14, Theorem 7.5.4]), thus

$$\mathrm{Swan}_0(\mathcal{K}_{a,0}) = 0, \qquad \mathrm{Swan}_\infty(\mathcal{K}_{a,0}) = d.$$

Then for any a, z one has $\mathrm{Sing}([\gamma_z']^*\mathcal{K}_{a,0}) = \{-z, \infty\}$. Moreover $[\gamma_z']^*\mathcal{K}_{a,0}$ has an unique slope at $d/(d+1)$ at ∞ and it is tame at $-z$. By contradiction, assume that there exists $\mathcal{L}$ lisse of rank 1 such that $[\gamma_z']^*\mathcal{K}_{a,0} = \mathcal{K}_{a,0} \otimes \mathcal{L}$. From the discussion above it would follow that $\{0, -z\} \subset \mathrm{Sing}(\mathcal{L}) \subset \{0, -z, \infty\}$ and $\mathcal{L}$ is tame everywhere. At this point it is useful to compute some data about $\mathrm{FT}(\mathcal{L})$:

a) $\mathrm{Sing}(\mathrm{FT}(\mathcal{L})) = \{0, \infty\}$, since $\mathcal{L}$ is tame at ∞ ([13, Corollary 8.5.8]),

b) $\mathrm{Rank}(\mathrm{FT}(\mathcal{L})) = \dim(H^1_c(\overline{\mathbb{A}}^1_{\mathbb{F}_p}, \mathcal{L} \otimes \mathcal{L}_{e_p(\alpha T)}))$ for any $\alpha \neq 0$. On the other hand, the Grothendieck-Ogg-Shafarevic formula ([15, Chapter 14]) implies

$$\begin{aligned}\dim(H^1_c(\overline{\mathbb{A}}^1_{\mathbb{F}_p}, \mathcal{L} \otimes \mathcal{L}_{e_p(\alpha T)})) &= -\mathrm{Rank}(\mathcal{L} \otimes \mathcal{L}_{e_p(\alpha T)}) + \sum_{x \in \overline{\mathbb{A}}^1_{\mathbb{F}_p}} \mathrm{Drop}_x(\mathcal{L} \otimes \mathcal{L}_{e_p(\alpha T)}) \\ &\quad + \sum_{x \in \overline{\mathbb{P}}^1_{\mathbb{F}_p}} \mathrm{Swan}_x(\mathcal{L} \otimes \mathcal{L}_{e_p(\alpha T)}) \\ &= -1 + 2 + 1 = 2.\end{aligned}$$

Hence, $\mathrm{Rank}(\mathrm{FT}(\mathcal{L})) = 2$.

c) Since $\mathcal{L}$ is not lisse on $\overline{\mathbb{A}}^1_{\mathbb{F}_p} \backslash \{0\}$, [13, Corollary 8.5.8] implies that $\mathrm{FT}(\mathcal{L})(\infty)$ has a break at 1. Thus $\mathrm{Swan}_\infty(\mathrm{FT}(\mathcal{L})) \geq 1$.

Now we have that

$$\begin{aligned} e_p(-zt - (a\bar{t})^d) &= \mathrm{FT}(t_{[\gamma_z']^*\mathcal{K}_{a,0}})(t) \\ &= \mathrm{FT}(t_{\mathcal{K}_{a,0}} \cdot t_{\mathcal{L}})(t) \\ &= -\frac{1}{\sqrt{p}} \sum_y \mathrm{FT}(t_{\mathcal{L}})(y)\, \mathrm{FT}(t_{\mathcal{K}_{a,0}})(t - y) \\ &= -\frac{1}{\sqrt{p}} \sum_y \mathrm{FT}(t_{\mathcal{L}})(y) e_p((a(\overline{y - t}))^d). \end{aligned} \tag{9.3}$$

Let us consider the ℓ-adic sheaf $\mathcal{G} := R^1 p_{1,!}(p_2^* \operatorname{FT}(\mathcal{L}) \otimes \mathcal{L}_{e_p((a(\overline{Y-T}))^d)})$. One has that

$$t_{\mathcal{G}} = -\frac{1}{\sqrt{p}} \sum_y \operatorname{FT}(t_{\mathcal{L}})(y) e_p((a(\overline{y-t}))^d).$$

Hence, $t_{\mathcal{G}}(t) = t_{\mathcal{L}_{e_p(-zT-(a\overline{T})^d)}}(t)$, for any $t \in \mathbb{F}_p$. Moreover the same computation as in (9.3) shows that for any $n \geq 1$ and any $t \in \mathbb{F}_{p^n}$, one has $t_{\mathcal{G},n}(t) = t_{\mathcal{L}_{e_p(-zT-(a\overline{T})^d)},n}(t)$. Thus, it follows that $\mathcal{L}_{e_p(-zT-(a\overline{T})^d)} = \mathcal{G}$ thanks to [6][Corollary 3.6] and [10][Theorem 2.3]. To get a contradiction it is enough to show that $\operatorname{Rank}(\mathcal{G}) \geq d+1$: in this case we would get that $1 = \operatorname{Rank}(\mathcal{L}_{e_p(-zT-(a\overline{T})^d)}) = \operatorname{Rank}\mathcal{G} \geq d+1 > 1$ which is absurd. We know that $\operatorname{Rank}(\mathcal{G})$ is equal to the dimension of the stalk $\mathcal{G}_t$ for any $t \in \mathbb{A}^1_{\mathbb{F}_p}$ where $\mathcal{G}$ is lisse. Using the Proper Base-Change Theorem one gets

$$\mathcal{G}_t = (R^1 p_{1,!}(p_2^* \operatorname{FT}(\mathcal{L}) \otimes \mathcal{L}_{e_p((a(\overline{Y-T}))^d)}))_t = H^1_c(\overline{\mathbb{A}}^1_{\mathbb{F}_p}, \operatorname{FT}(\mathcal{L}) \otimes \mathcal{L}_{e_p((a(\overline{Y-t}))^d)}).$$

Thus we need to compute

$$M := \dim(H^1_c(\overline{\mathbb{A}}^1_{\mathbb{F}_p}, \operatorname{FT}(\mathcal{L}) \otimes \mathcal{L}_{e_p((a(\overline{Y-t}))^d)})),$$

for some $t \notin \operatorname{Sing}(\mathcal{G})$. To simplify the notation let us denote $\mathcal{H} := \operatorname{FT}(\mathcal{L}) \otimes \mathcal{L}_{e_p((a(\overline{Y-t}))^d)}$ where $t \in \overline{\mathbb{A}}^1_{\mathbb{F}_p} \setminus \{0\}$. Observe that $\operatorname{Rank}(\mathcal{H}) = \operatorname{Rank}(\operatorname{FT}(\mathcal{L})) \cdot \operatorname{Rank}(\mathcal{L}_{e_p((a(\overline{Y-t}))^d)}) = 2$. Moreover, since $\operatorname{Sing}(\operatorname{FT}(\mathcal{L})) = \{0, \infty\}$ and $\operatorname{Sing}(\mathcal{L}_{e_p((a(\overline{Y-t}))^d)}) = \{t\}$ we have $\operatorname{Sing}(\mathcal{H}) = \{0, t, \infty\}$ and

$$\operatorname{Swan}_0(\mathcal{H}) = \operatorname{Swan}_0(\operatorname{FT}(\mathcal{L})) = 0, \quad \operatorname{Swan}_\infty(\mathcal{H}) = \operatorname{Swan}_\infty(\operatorname{FT}(\mathcal{L})) \geq 1,$$
$$\operatorname{Swan}_t(\mathcal{H}) = \operatorname{Swan}_t(\mathcal{L}_{e_p((a(\overline{Y-t}))^d)}) = d.$$

Thus, using the Grothendieck-Ogg-Shafarevic formula we get

$$\begin{aligned} M &= -\operatorname{Rank}(\mathcal{H}) + \operatorname{Drop}_0(\mathcal{H}) + \operatorname{Drop}_t(\mathcal{H}) + \operatorname{Swan}_\infty(\mathcal{H}) + \operatorname{Swan}_0(\mathcal{H}) + \operatorname{Swan}_t(\mathcal{H}), \\ &\geq -2 + 1 + 1 + 1 + d = d + 1 \end{aligned}$$

as we wanted.

vi) We start bounding

$$\begin{aligned} M_4 &:= \frac{1}{p^2} \sum_{a \in \mathbb{F}_p} \sum_{b \in \mathbb{F}_p} \Big| \frac{1}{\sqrt{p}} \sum_{n \in I} e_p(bn + a\overline{n}^d) \Big|^4 \\ &= \frac{1}{p^4} \sum_{a \in \mathbb{F}_p} \sum_{b \in \mathbb{F}_p} \sum_{n_1, n_2, m_1, m_2 \in I} e_p(b(n_1 + n_2 - m_1 - m_2) + a(\overline{n_1}^d + \overline{n_2}^d - \overline{m_1}^d - \overline{m_2}^d)). \end{aligned}$$

We use the same strategy as in [16, page 1505]: using the orthogonality of the additive characters one gets

$$M_4 = \frac{1}{p^3} \sum_{a \in \mathbb{F}_p} \sum_{\substack{n_1, n_2, m_1, m_2 \in I \\ n_1 + n_2 = m_1 + m_2}} e_p(a(\overline{n_1}^d + \overline{n_2}^d - \overline{m_1}^d - \overline{m_2}^d))$$

and then

$$M_4 = \frac{1}{p^2} \sum_{\substack{n_1,n_2,m_1,m_2 \in I \\ n_1+n_2=m_1+m_2 \\ \overline{n}_1^d+\overline{n}_2^d=\overline{m}_1^d+\overline{m}_2^d}} 1.$$

For $n_1 + n_2 \neq 0$, the system

$$\begin{cases} n_1 + n_2 = m_1 + m_2 \\ \overline{n}_1^d + \overline{n}_2^d = \overline{m}_1^d + \overline{m}_2^d \end{cases}$$

has at most $2d$ pairs of solutions (m_1, m_2). On the other hand, if $n_1 + n_2 = 0$ then one has $m_1 + m_2 = 0$. Thus we can bound M_4 as $M_4 \ll_d |I|^2 p^{-2}$. Now by positivity we get that

$$\frac{1}{p^2} \sum_{a \in \mathbb{F}_p^\times} \sum_{b \in \mathbb{F}_p^\times} \Big| \frac{1}{\sqrt{p}} \sum_{n \in I} e_p(bn + (a\overline{n})^d) \Big|^4 \leq (d, p-1) M_4 \ll_d |I|^2 p^{-2}.$$

Choosing $|I| \leq p^{1/2+1/6}$ we get the result.

9.4.2. *Hyper-Kloosterman sums.* For any $r \geq 2$, let $\mathcal{K}\ell_r$ denote the r-th Kloosterman sheaf. For any $r \geq 3$ odd the family

$$\{[x \mapsto \overline{ax}]^* \mathcal{K}\ell_r \otimes \mathcal{L}_{e_p(bT)}\}_{(a,b) \in \mathbb{F}_p^\times \times \mathbb{F}_p^\times}$$

is a 2-parameter family of Sp_{r+1}-type.

i) For any $(a, b) \in \mathbb{F}_p^\times \times \mathbb{F}_p^\times$, the sheaf $[\overline{ax}]^* \mathcal{K}\ell_r \otimes \mathcal{L}_{e_p(bT)}$ is a Fourier, irreducible middle-extension ℓ-adic sheaf on $\overline{\mathbb{P}}^1_{\mathbb{F}_p}$ pointwise pure of weight 0. Moreover, the trace function attached to $[x \mapsto \overline{ax}]^* \mathcal{K}\ell_r \otimes \mathcal{L}_{e_p(bT)}$ is given by

$$t_{\mathcal{F}_{a,b}} : x \mapsto \mathrm{Kl}_r(\overline{ax}; p) e_p(bx).$$

ii) Thanks to [8][Proposition 8.2] and [13][11.0.2], one has that $c(\mathcal{F}_{a,b}) \leq 5c([x \mapsto \overline{ax}]^* \mathcal{K}\ell_r)^2 c(\mathcal{L}_{e_p(bT)})^2 = 45c([x \mapsto \overline{ax}]^* \mathcal{K}\ell_r)^2 = 45(r+3)^2$.

iii) We start computing the Fourier transform of $t_{\mathcal{F}_{a,b}}$:

$$\begin{aligned} t_{\mathcal{K}_{a,b}}(z) &= -\frac{1}{\sqrt{p}} \sum_{x \in \mathbb{F}_p^\times} \mathrm{Kl}_r(\overline{ax}; p) e_p((b+z)x) \\ &= -\frac{1}{p^{r/2}} \sum_{x \in \mathbb{F}_p^\times} \Big(\sum_{x_1,\dots,x_{r-1} \in \mathbb{F}_p^\times} e_p(x_1 + \cdots + x_{r-1} + \overline{axx_1 \cdots x_{r-1}}) \Big) e_p((b+z)x) \\ &= -\frac{1}{p^{r/2}} \sum_{x,x_1,\dots,x_{r-1} \in \mathbb{F}_p^\times} e_p(x_1 + \cdots + x_{r-1} + (b+z)x + \overline{axx_1 \cdots x_{r-1}}). \end{aligned}$$

If $z \neq -b$, then we can use the change of variable $t = x(b+z)$ getting

$$\begin{aligned}t_{\mathcal{K}_{a,b}}(z) &= -\frac{1}{p^{r/2}} \sum_{t,x_1,\dots,x_{r-1}\in\mathbb{F}_p^\times} e_p(x_1+\cdots+x_{r-1}+t+(b+z)\overline{atx_1\cdots x_{r-1}})\\ &= \mathrm{Kl}_{r+1}(\overline{a}(b+z);p).\end{aligned}$$

Thus, we have that $t_{\mathcal{K}_{a,b}}(z) = \mathrm{Kl}_{r+1}(\overline{a}(b+z);p) = \mathrm{Kl}_{r+1}(\overline{a}(1+\gamma_z(b));p) = t_{[\gamma_z]^*\mathcal{K}_{a,1}}(b)$, where $\gamma_z = \begin{pmatrix}1 & z-1\\ 0 & 1\end{pmatrix}$.

iv) For any $a, b \in \mathbb{F}_p^\times$, the monodromy of $\mathcal{K}_{a,b} = [\gamma_{a,b}]^*\mathcal{K}\ell_{r+1}$ is the same as the one of $\mathcal{K}\ell_{r+1}$. Thus, $G^{\mathrm{arith}}_{\mathcal{K}_{a,b}} = G^{\mathrm{geom}}_{\mathcal{K}_{a,b}}$ and $G^{\mathrm{geom}}_{\mathcal{K}_{a,b}} = \mathrm{Sp}_{r+1}$.

v) We need to show that for any $a, z_1, z_2 \in \mathbb{F}_p^\times$ with $z_1 \neq z_2$, one has

$$[\gamma_{z_1}\gamma_{z_2}^{-1}]^*\mathcal{K}_{a,1} \neq \mathcal{K}_{a,1}\otimes\mathcal{L},$$

for any ℓ-adic sheaf, $\mathcal{L}$, of rank 1. This is just a consequence of ([9][Proposition 3.6]).

vi) We compute

$$\begin{aligned}M_4 &= \frac{1}{p^2}\sum_{a\in\mathbb{F}_p^\times}\sum_{b\in\mathbb{F}_p}\Big|\frac{1}{\sqrt{p}}\sum_{n\in I}\mathrm{Kl}_r(\overline{an};p)e_p(bn)\Big|^4\\ &= \frac{1}{p^4}\sum_{a\in\mathbb{F}_p^\times}\sum_{b\in\mathbb{F}_p}\sum_{n_1,n_2,m_1,m_2\in I}\mathrm{Kl}_r(\overline{an_1};p)\,\mathrm{Kl}_r(\overline{an_2};p)\,\mathrm{Kl}_r(-\overline{am_1};p)\,\mathrm{Kl}_r(-\overline{am_2};p)\times\\ &\times e_p(b(n_1+n_2-m_1-m_2)).\end{aligned}$$

By the orthogonality of the additive characters we get

$$M_4 = \frac{1}{p^3}\sum_{a\in\mathbb{F}_p^\times}\sum_{\substack{n_1,n_2,m_1,m_2\in I\\ n_1+n_2=m_1+m_2}}\mathrm{Kl}_r(\overline{an_1};p)\,\mathrm{Kl}_r(\overline{an_2};p)\,\mathrm{Kl}_r(-\overline{am_1};p)\,\mathrm{Kl}_r(-\overline{am_2};p).$$

On the other hand, the sum

$$\sum_{a\in\mathbb{F}_p^\times}\mathrm{Kl}_r(\overline{an_1};p)\,\mathrm{Kl}_r(\overline{an_2};p)\,\mathrm{Kl}_r(-\overline{am_1};p)\,\mathrm{Kl}_r(-\overline{am_2};p) \tag{9.4}$$

has size p if and only if either $n_1 = -n_2$ and $m_1 = -m_2$ or $n_1 = m_1$ and $n_2 = m_2$ or $n_1 = m_2$ and $n_2 = m_1$, and it has size $O_r(\sqrt{p})$ otherwise ([9][Corollary 3.3]). Let us choose $n_1, n_2 \in I$, we need to distinguish two cases

(1) $n_1 \neq -n_2$, thus we have at most two choices of m_1, m_2 such that the sum in (9.4) has size p,

(2) $n_1 = -n_2$, then the sum in (9.4) has size p for at most $|I|$ couples (m_1, m_2).

Thus we obtain

$$M_4 \ll_r |I|^2p^{-2} + |I|^3p^{-5/2}.$$

Choosing $|I| \leq p^{1/2+1/8}$ we get the result.

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I.M.B., Université de Bordeaux, 351, cours de la Libération, 33405 TALENCE, France
E-mail address: Pascal.Autissier@math.u-bordeaux.fr

ETH Zurich, Department of Mathematics, HG J 14.4, Rmistrasse 101, 8092 Zurich, Switzerland
E-mail address: dante.bonolis@math.ethz.ch

Institut Élie Cartan de Lorraine, Université de Lorraine, BP 70239, 54506 Vandoeuvre-lès-Nancy Cedex, France, and Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, ON, M3J1P3 Canada
E-mail address: youness.lamzouri@univ-lorraine.fr