BOUNDARY VALUES FOR THE CANONICAL SOLUTION TO $\bar\partial$-EQUATION AND $W^{1/2}$ ESTIMATES

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ABSTRACT. Let Ω be a bounded pseudo-convex domain in $\mathbb{C}^n$ for which there exist a smooth defining plurisubharmonic function. Then the $\bar\partial$-Neumann and the Bergman operators satisfiy $W^{1/2}$ estimates. The case of domains with Lipschitz boundary is considered, and we give applications of the method to L^p estimates with loss.

1. INTRODUCTION

Let Ω be a bounded pseudo-convex domain with smooth boundary, given by $\rho < 0$ where ρ is a $\mathscr{C}^\infty$ function such that $|\nabla\rho \neq 0|$ on $\partial\Omega$. For $\alpha = \sum \alpha_i d\bar z_i$ a $(0,1)$-form with smooth coefficients in Ω, let us define

$$\vartheta\alpha = -\sum \frac{\partial\alpha_i}{\partial z_i} \text{ and } <\alpha, \bar\partial\rho> = \sum \alpha_j \frac{\partial\rho}{\partial z_j}.$$

To obtain L^2estimates at the boundary for a solution to the $\bar\partial$-equation $\bar\partial u = \beta$strictly pseudo-convex domains, B. Berndtsson introduced in [1] the $(0,1)$-harmonics forms as the forms αsuch that $\bar\partial\alpha = \vartheta\alpha = 0$. We show in section2 that the fact that the operator giving the canonical solution to $\bar\partial$maps $L^2(\Omega)$ into $L^2(\Omega)$ is a immediate consequence of $(L^2(\Omega), W^{1/2}(\Omega)$ estimates and classical potential theory; but nevertheless the idea of B. Berndtsson to study the dual problem seemed to be powerfull, and we tried to develop it sistematically to get estimates when there is no compacity hypothesis as in [16] or [7].

We define pseudo-harmonic $(0,1)$-forms as $(0,1)$-forms such that $\bar\partial\alpha = \bar\partial\vartheta\alpha = 0$. We then prove that any smooth function on $\partial\Omega$ may be written as the boundary values (in a distribution sense) of $<\alpha, \bar\partial\rho>$, where α is pseudo-harmonic. Moreover this extension operator $<\alpha, \bar\partial\rho>_{|\partial\Omega} \mapsto \alpha$is the adjoint of the operator $\beta \mapsto \bar\partial^*\mathscr{N}\beta_{|\partial\Omega}$, where $\bar\partial^*\mathscr{N}$ gives the canonical solution to the $\bar\partial$-equation, while $<\alpha, \bar\partial\rho>_{|\partial\Omega} \mapsto -\vartheta\alpha$ is the adjoint to the operator $u \mapsto Bu_{|\partial\Omega}$, where B is the Bergman projection. This is proved in section 2. In section 3, we prove how Sobolev estimates for the operators $<\alpha, \bar\partial\rho>_{|\partial\Omega} \mapsto \alpha$ (and its analog for (p,q)-forms) imply Sobolev estimates for all operators related to $\bar\partial$-Neumann problem. Our results, here, are close to results of H. Boas and E. Straube [2]. In section 4 we develop identities for pseudo-harmonics forms and prove the following theorems:

Theorem 1.1. *Assume that ρ may be choosen plurisubharmonic in Ω. Then all operators related to the $\bar\partial$-Neumann map continuously $W^{1/2}(\Omega)$ into itself.*

Here $W^{1/2}(\Omega)$ means the Sobolev space, and a precise list of the operators which are considered is given in section 4. We had alredy proved this result for the Bergman projection, see [4]. Under a different hypothesis, which we do not know how to compare to ours, H. Boas and E. Straube give Sobolev estimates W^s for all s in [3]. Our method has the advantage not to ask for smoothness:

Theorem 1.2. *Let Ω be a bounded pseudo-convex domain with Lipschitz boundary for which there exist a Lipschitz defining function ρ which is plurisubharmonic inside Ω. Then the operators $\bar\partial^*\mathscr{N}$ and the Bergman projection B map $W^{1/2+\epsilon}(\Omega)$ into $W^{1/2}(\Omega)$, for all $\epsilon > 0$.*

In [10] J.E. Fornaess and N. Sibony have proved L^p estimates with loss for solutions to the $\bar\partial$-equation in all smooth pseudo-convex domains. In section 5 we show how these estimates can be deduced, via the notion of pseudo-harmonic forms extended to weighted $\bar\partial$-Neumann problem, from $W^{1/2}$ estimates, and somehow give more precise estimates than theirs. But of course we are far from critical results, which remain an open problem.

Some further developpements have been given in [5], [8].

NOTATIONS

Let Ω be a bounded domain of $\mathbb{C}^n$. A (p,q)-form on Ω may be written, as in [12], as

$$\alpha = {\sum_{I,J}}' \alpha_{IJ} dz^I \wedge d\bar z^J,$$

where the sum is taken over strictly increasing multi-indices. We shall write $\mathscr{C}^\infty_{pq}(\Omega)$, $\mathscr{C}^\infty_{pq}(\bar\Omega)$, $L^2_{pq}(\Omega)$, $W^s_{pq}(\Omega)$,... for the spaces of (p,q)-forms with coefficients in $\mathscr{C}^\infty(\Omega)$, $\mathscr{C}^\infty(\bar\Omega)$, $L^2(\Omega)$, $W^{1/2}(\Omega)$,...

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We set, for α and β two (p,q)-forms:

$$<\alpha,\beta>=\sum_{I,J}{}' \alpha_{IJ}\bar{\beta}_{IJ}\ ;\ |\alpha|=<\alpha,\alpha>^{1/2}.$$

ϑ is the formal adjoint of $\bar\partial$:

$$\vartheta\alpha=(-1)^p\sum_{|I|=p,|K|=q-1}{}'\sum_{i=1}^{n}\frac{\partial\alpha_{I,jK}}{\partial z_j}dz^I\wedge d\bar{z}^K.$$

If Ω has a $\mathscr{C}^1$ boundary and $\Omega=\{\rho<0\}$, $\rho\in\mathscr{C}^1(\bar\Omega)$ and $\nabla\rho\neq 0$ on Ω, let N be the $(0,1)$-normal vector field : $N=\sum\frac{\partial\rho}{\partial\bar z_j}\frac{\partial}{\partial z_j}$. The normal component of α is defined as:

$$\bar N\lrcorner\alpha=\sum_{|I|=p,|K|=q-1}{}'\sum_{j=1}^{n}\alpha_{I,jK}\frac{\partial\rho}{\partial z_j}dz^I\wedge d\bar z^K.$$

We shall also need (p,q)-forms on $\partial\Omega$: a (p,q)-form on $\partial\Omega$ is defined as a (p,q)-form such that $\bar N\lrcorner\alpha=0$ on $\partial\Omega$ (see [19]).

$\bar\partial^*$ will denote the adjoint of $\bar\partial$, $\Box=\bar\partial^*\bar\partial+\bar\partial\bar\partial^*$, $\mathscr{N}_{pq}$ the Neumann operator for (p,q)-form, B_{pq} the orthogonal projection in L^2_{pq} onto the subspace of $\bar\partial$-closed forms.

The definition of the Sobolev spaces $W^s(\Omega)$, for $s\in\mathbb{R}$, and $W^s(\partial\Omega)$, is given in [11] and [18] for instance. We shall use the following classical result (see [18]) :

Proposition 1.3. *Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$. If :*

(i) $f\in W^s(\Omega)$ and $\triangle f\in W^{-1}(\Omega)$, then f has a trace at the boundary in $W^{s-1/2}(\partial\Omega)$. Moreover f is the sum of the Poisson integral of its trace and the solution in $W^1_0(\Omega)$ of the equation $\triangle g=\triangle f$, and :

$$\|f\|_{W^{s-1/2}(\partial\Omega)}\leq C\left\{\|\triangle f\|_{W^{-1}(\Omega)}+\|f\|_{W^s(\Omega)}\right\}.$$

(ii) f has a trace in $W^{s-1/2}(\partial\Omega)$ and $\triangle f\in W^{s-2}(\Omega)\cap W^{-1}(\Omega)$, then f belongs to $W^s(\Omega)$ and :

$$\|f\|_{W^s(\Omega)}\leq C\left\{\max\left[\|\triangle f\|_{W^{-1}(\Omega)},\|\triangle f\|_{W^{s-2}(\Omega)}\right]+\|f\|_{W^{s-1/2}(\partial\Omega)}\right\}.$$

We shall need the fact that Proposition 1.3 generalises to elliptic systems (see again [18]).

2. Boundary Values for the Canonical Solution to $\bar\partial$-Equation and Related Operators; Pseudo-harmonic Forms and T_{pq} Operators

It is well known that holomorphic functions f in $L^2(\Omega)$ have trace in $W^{-1/2}(\partial\Omega)$. This generalises easily by Proposition 1.3 to functions $f\in L^2(\Omega)$ for which $\bar\partial f\in L^2_{01}(\Omega)$. More generally, one has the following Proposition :

Proposition 2.1. *Let Ω be a bounded domain with smooth boundary. For all $f\in L^2_{pq}(\Omega)$, such that $\bar\partial f\in L^2_{pq+1}(\Omega)$, there exists a unique $g\in W^{-1/2}_{pq}(\Omega)$ so that, for $\varphi\in\mathscr{C}^\infty_{pq}(\bar\Omega)$ with $\bar N\lrcorner\varphi=0$ on $\partial\Omega$:*

$$\int_{\partial\Omega}<g,\varphi>d\sigma=\int_\Omega<\bar\partial f,\varphi\wedge\bar\partial\rho>dV-\int_\Omega<f,\vartheta(\varphi\wedge\bar\partial\rho>dV. \tag{2.1}$$

(We assume here that $\Omega=\{\rho<0\}$, with $\rho\in\mathscr{C}^s(\mathbb{C}^n)$, s large enough, $|\nabla\rho|=1$ on $\partial\Omega$.

When $f\in\mathscr{C}^1_{pq}(\bar\Omega)$, (2.1) is valid with g given by the boundary values of f : this is a direct consequence of Stokes formula. It justifies the following definition :

Definition 2.2. g is called the trace of f, and denoted by f^b.

Proof of proposition 2.1. For $\psi\in\mathscr{C}^1_{pq}(\partial\Omega)$, let ψ' be the harmonic extention of ψ, coefficient by coefficient. As

$$\|\psi'\|_{W^1_{pq}(\Omega)}\leq C\|\psi\|_{W^{1/2}(\partial\Omega)},$$

$$\psi\mapsto\int_\Omega<\bar\partial f,\psi'\wedge\bar\partial\rho>dV-\int_\Omega<f,\vartheta(\psi'\wedge\bar\partial\sigma>dV,$$

is a continuous linear form on $W^{1/2}_{pq}(\partial\Omega)$: let g be this current in $W^{-1/2}_{pq}(\partial\Omega)$. We have proved unicity. To prove (2.1), let ψ be $\varphi_{|\partial\Omega}$: we have to prove that

$$\begin{aligned}\int_\Omega<\bar\partial f,\varphi\wedge\bar\partial\rho>dV &- \int_\Omega<f,\vartheta(\varphi\wedge\bar\partial\rho)>dV\\ &= \int_\Omega<\bar\partial f,\psi'\wedge\bar\partial\rho>dV-\int_\Omega<f,\vartheta(\psi'\wedge\bar\partial\rho)>dV.\end{aligned}$$

It is true for $f \in \mathscr{C}^1_{pq}(\bar\Omega)$ as both sides are equal to $\displaystyle\int_{\partial\Omega} < f_{|\partial\Omega}, \psi > d\sigma$. For general f there exists, by Friedrich's regularisation as in [12], a sequence (f_n) in $\mathscr{C}^1_{pq}(\bar\Omega)$ such that $f_n \to f$ in $L^2_{pq}(\Omega)$ and $\bar\partial f_n \to \bar\partial f$ in $L^2_{pq+1}(\Omega)$. As equality is valid for each f_n it also holds for the limit f. □

Remark 2.3. If f belongs to the domain of $\bar\partial^*$, then $(N\lrcorner f)^b = 0$. In this case all the coefficients of f at the boundary are known as soon as one knows f^b (f is "tangential" at the boundary). In particular, if one knows f^b, $\bar\partial f$ and $\bar\partial^* f$, one can write f as the sum of the Poisson integral of f^b (coefficient by coefficient) and the solution of a Dirichlet problem (see Proposition 1.3). Conversely, if $(N\lrcorner f)^b = 0$ and $\vartheta f \in L^2_{pq-1}(\Omega)$, then f is in $\mathrm{Dom}(\bar\partial^*)$.

Proposition 2.1 has immediate corollary :

Proposition 2.4. *Let Ω be pseudoconvex and as in proposition 2.1. Then, for $\beta \in L^2_{pq}(\Omega)$, $\mathscr{N}\beta$, $\bar\partial^*\mathscr{N}_{pq}\beta$, $B_{pq}\beta$, $(\bar\partial^*\mathscr{N}_{pq})^*\beta$ have boundary values in $W^{-1/2}_{pq}(\partial\Omega)$ (resp. $W^{-1/2}_{pq-1}(\partial\Omega)$, $W^{-12}_{pq}(\partial\Omega)$, $W^{-1/2}_{pq+1}(\partial\Omega)$).*

Notation 2.5. We will denote by $\mathscr{N}^b_{pq}$, $(\bar\partial^*\mathscr{N}_{pq})^b$, B^b_{pq} and $(\bar\partial^*\mathscr{N}_{pq})^{*b}$ the associated operators.

Remark 2.6. $\mathscr{N}\beta$, $\bar\partial^*\mathscr{N}_{pq}\beta$, $(\bar\partial^*\mathscr{N}_{pq})^*\beta$ and $B_{p0}\beta$ are in $\mathrm{Dom}(\bar\partial^*)$, so Remark 2.3 applies.

Let us prove it for $(\bar\partial^*\mathscr{N}_{pq})^b\beta$: by definition, if β and ψ are in $L^2_{pq-1}(\Omega)$ and $\bar\partial\psi \in L^2_{pq}(\Omega)$:

$$\int_\Omega < (\bar\partial^*\mathscr{N}_{pq})^*\beta, \bar\partial\psi > dV = \int_\Omega < \beta, \psi - B_{pq-1}\psi > dV.$$

This means that $(\bar\partial^*\mathscr{N}_{pq})^*\beta$ is in $\mathrm{Dom}(\bar\partial^*)$, and :

$$\bar\partial^*(\bar\partial^*\mathscr{N}^*_{pq}\beta = \beta - B_{pq-1}\beta.$$

Proposition 2.4 allows us to define the adjoint operators of $\mathscr{N}^b_{pq}$, $(\bar\partial^*\mathscr{N}_{pq})^b$, B^b_{pq} and $(\bar\partial^*\mathscr{N}_{pq})^{*b}$. In particular we shall call :

$$T_{pq} \,:\, W^{1/2}_{pq-1}(\partial\Omega) \to L^2_{pq}(\Omega),\ 1 \le q \le n,$$

the adjoint operator of $(\bar\partial^*\mathscr{N}_{pq})^b$, defined by :

$$\int_{\partial\Omega} < (\bar\partial^*\mathscr{N}^b_{pq}\beta, f > d\sigma = \int_\Omega < \beta, T_{pq}f > dV, \tag{2.2}$$

for $f \in W^{1/2}_{pq-1}(\partial\Omega)$ and $\beta \in L^2_{pq}(\Omega)$; we shall call :

$$S_{pq} \,:\, W^{1/2}_{pq}(\partial\Omega) \to L^2_{pq}(\Omega),\ 0 \le q \le n-1,$$

adjoint operator of B^b_{pq}, defined by :

$$\int_{\partial\Omega} < B^b_{pq}\beta, f > d\sigma = \int_\Omega < \beta, S_{pq}f > dV, \tag{2.3}$$

for $f \in W^{1/2}_{pq-1}(\partial\Omega)$ and $\beta \in L^2_{pq}(\Omega)$.

When Ω is a domain of $\mathbb{C}$, T_{pq} is easily deduced from harmonic extension. Our aim now is to find the corresponding property in $\mathbb{C}^n$. Let us define :

Definition 2.7. Let $f \in \mathscr{C}^\infty_{pq}(\Omega)$. We say that f is pseudo-harmonic (resp. harmonic) if $\bar\partial f = \bar\partial\vartheta f = 0$ (resp. $\bar\partial f = \vartheta f = 0$).

Alternatively, f is pseudo-harmonic if and only if $f \in L^2_{pq}(\Omega)$, is $\bar\partial$-closed and has harmonic coefficients in the canonical basis.

The following Proposition characterizes $T_{pq}f$ in terms of pseudo-harmonic form :

Proposition 2.8. *Let Ω be pseudoconvex as in Proposition 2.1. Let T_{pq}, $1 \le q \le n$ and S_{pq}, $0 \le q \le n-1$ be defined by (2.2) and (2.3). Then :*

(i) for all $f \in W^{1/2}_{pq-1}(\partial\Omega)$, $T_{pq}f$ is a pseudo-harmonic form in $L^2_{pq}(\Omega)$ and $\vartheta T_{pq}f$ belongs to L^2pq-1 (Ω) .

(ii) for all $f \in W^{1/2}_{pq}(\partial\Omega)$, $S_{pq}f$ is a harmonic form in $L^2_{pq}(\Omega)$ and $S_{pq}f = -\vartheta T_{pq+1}f$;

(iii) for all $f \in W^{1/2}_{pq-1}(\partial\Omega)$, $T_{pq}f$ is the unique pseudo-harmonic form $\alpha \in \mathscr{C}^\infty_{pq+1}(\Omega)$ such that :

(a) $\alpha \in L^2_{pq}(\Omega)$ and $\vartheta\alpha \in L^2_{pq-1}(\Omega)$;

(b) f is the boundary values of $\bar N\lrcorner\alpha$ on $\partial\Omega$. Moreover, $T_{pq}f$ is harmonic if and only if :

$$\int_{\partial\Omega} < \varphi^b, f > d\sigma = 0$$

for all $\varphi \in L^2_{pq-1}(\Omega)$ such that $\bar\partial\varphi = 0$.

proof of proposition 2.8. Let us prove that $T_{pq}f$ is pseudo-harmonic : we have to prove that :

$$\int_\Omega < \varphi, \bar\partial T_{pq}f > dV = 0 \, \varphi \in \mathscr{D}_{pq+1}(\Omega), \tag{2.4}$$

and

$$\int_\Omega < \varphi, \bar\partial\vartheta T_{pq}f > dV = 0, \, \varphi \in \mathscr{D}_{pq}(\Omega). \tag{2.5}$$

To prove (2.4), we write (2.2) with $\beta = \vartheta\varphi = \bar\partial^*\varphi$. Then β is orthogonal to $\bar\partial$-closed forms, so $B_{pq}\beta = 0$, and $\bar\partial^*\mathscr{N}_{pq}\beta = 0$. To prove (2.5), we write 2.2 with $\beta = \bar\partial\vartheta\varphi = \bar\partial\bar\partial^*\varphi$: $\bar\partial^*\mathscr{N}_{pq}\beta = \bar\partial^*\varphi$ is compactly supported in Ω, so $(\bar\partial^*\mathscr{N}_{pq})^b\beta = 0$.

Now to finish the proof of (i) and (ii), as we already know that $T_{pq}f$ and $S_{pq}f$ are in $L^2_{pq}(\Omega)$, we have only to prove that $S_{pq-1}f = -\vartheta T_{pq}f$: the fact that $S_{pq-1}f$ is a harmonic form is an immediate consequence. So let us show that, for $\varphi \in \mathscr{D}_{pq-1}(\Omega)$:

$$\int_\Omega < \varphi, S_{pq-1}f > dV = -\int_\Omega < \bar\partial\varphi, T_{pq}f > dV.$$

By definition the left hand side is :

$$\int_{\partial\Omega} < B^b_{pq-1}\varphi, f > d\sigma,$$

while the right hand side is :

$$\int_{\partial\Omega} < \bar\partial^*\mathscr{N}_{pq})^b\bar\partial\varphi, f > d\sigma.$$

But $\bar\partial^*\mathscr{N}_{pq}\bar\partial\varphi = \varphi - B_{pq-1}\varphi$, and, as $\varphi^b = 0$,

$$(\bar\partial\mathscr{N}_{pq})^b(\bar\partial\varphi) = -B^b_{pq-1}\varphi.$$

Now let us prove that $f = (\bar N \lrcorner T_{pq}f)^b$. Let f' be the harmonic extension of f, coefficient by coefficient. Then $\bar\partial(f'\rho)$ belongs to $W^1_{pq}(\Omega)$, and $(\bar N \lrcorner \bar\partial(f'\rho))^b = f$. By Remark 2.3, to prove that $\bar N \lrcorner T_{pq}f$ and $\bar N \lrcorner \bar\partial(f'\rho)$ have the same boundary values it is sufficient to prove that :

$$T_{pq}f - \bar\partial(f'\rho) \in \mathrm{Dom}(\bar\partial^*),$$

or, which is equivalent :

$$\int_\Omega < \bar\partial\varphi, T_{pq}f - \bar\partial(f'\rho) > dV = -\int_\Omega < \varphi, \vartheta T_{pq}f - \vartheta\bar\partial(f'\rho) > dV. \tag{2.6}$$

By Stokes formula, for $\psi \in L^2_{pq-1}(\Omega)$ with $\bar\partial\psi \in L^2_{pq}(\Omega)$:

$$\int_{\partial\Omega} < \psi^b, f > d\sigma = \int_\Omega < \bar\partial\psi, \bar\partial(f'\rho) dV - \int_\Omega < \psi, \vartheta\bar\partial(f'\rho) dV. \tag{2.7}$$

In particular if $\psi = \bar\partial^*\mathscr{N}_{pq}\beta\backslash$:

$$\begin{aligned}
\int_\Omega < \beta, T_{pq}f > dV &= \int_{\partial\Omega} < \psi^b, f > d\sigma \\
&= \int_\Omega < \bar\partial\bar\partial^*\mathscr{N}_{pq}\beta, \bar\partial(f'\rho) > dV - \int_\Omega < \bar\partial^*\mathscr{N}_{pq}\beta, \vartheta\bar\partial(f'\rho) > dV \\
&= \int_\Omega < \beta, \bar\partial(f'\rho) > dV - \int_\Omega < \bar\partial^*\mathscr{N}_{pq}\beta, \vartheta\bar\partial(f'\rho) > dV,
\end{aligned}$$

as $\bar\partial\bar\partial^*\mathscr{N}_{pq}\beta = \beta - \bar\partial^*\bar\partial\mathscr{N}_{pq}\beta$, and the last term is orthogonal to $\bar\partial$-closed forms. Finally, if $\beta = \bar\partial\varphi$ we find that the left hand side of (2.6) is equal to

$$-\int_\Omega < \varphi - B_{pq-1}\varphi, \vartheta\bar\partial(f'\rho) dV.$$

Now, by (2.7) used with $\psi = B_{pq-1}\varphi$:

$$\begin{aligned}
\int_\Omega < B_{pq-1}\varphi, \vartheta\bar\partial(f'\rho) > dV &= \int_{\partial\Omega} < B^b_{pq-1}\varphi, f > d\sigma \\
&= \int_\Omega < \varphi, \vartheta T_{pq}f > dV.
\end{aligned}$$

It remains to prove unicity of α satisfying (a) and (b). But if α_1 and $\alpha_2$are two such forms, then $\alpha = \alpha_1 - \alpha_2$ belongs to $\mathrm{Dom}(\bar\partial^*)$ by (a) and Remark 2.3. Moreover $\Box\alpha = 0$, so $\alpha = 0$.

Finally, if $T_{pq}f$ is harmonic, $S_{pq-1}f = 0$, so

$$\int_{\partial\Omega} < B_{pq}\beta, f > d\sigma = 0, \, \beta \in L^2_{pq}(\Omega).$$

In particular $\int_{\partial\Omega} < \varphi^b f > d\sigma = 0$ if $\bar\partial\varphi = 0$. Conversely if this condition is satisfied for all φ's, then $S_{pq-1}f = 0$ and $T_{pq}f$ is harmonic by (ii). □

Remark 2.9. In the proof we have obtained the following formula :

$$\int_\Omega < \beta, T_{pq}f > dV = \int_\Omega < \beta, \bar\partial(f'\rho) > dV - \int_\Omega < \bar\partial^* \mathcal{N}_{pq}\beta, \vartheta\bar\partial(f'\rho) > dV,$$

which gives immediately :

$$T_{pq}f = \bar\partial(f'\rho) - \mathcal{N}_{pq}\bar\partial\vartheta\bar\partial(f'\rho). \tag{2.8}$$

Remark 2.10. Let us define T^*_{pq} as the adjoint of $(\bar\partial^*\mathcal{N}_{pq})^{*b}$:

$$T^*_{pq} : W^{1/2}_{pq}(\partial\Omega) \to L^2_{pq-1}(\Omega),\ 1 \le q \le n,$$

is given by :

$$\int_{\partial\Omega} < (\bar\partial^*\mathcal{N}_{pq})^{*b}\beta > d\sigma = \int_\Omega < \beta, T^*_{pq}f > dV, \tag{2.9}$$

for $f \in W^{1/2}_{pq}(\partial\Omega)$ and $\beta \in L^2_{pq-1}(\Omega)$.

Using (2.7) with $\psi = (\bar\partial^*\mathcal{N}_{pq})^*\beta$, it follows that

$$\begin{aligned}\int_\Omega < \beta, T^*_{pq}f > dV &= -\int_\Omega < (\bar\partial\mathcal{N}_{pq})^*\beta, \vartheta\bar\partial(f'\rho) > dV \\ &= -\int_\Omega < \beta, \bar\partial^*\mathcal{N}_{pq}\vartheta\bar\partial(f'\rho) > dV,\end{aligned}$$

so

$$T_{pq}f = -\bar\partial^*\mathcal{N}_{pq}\vartheta\bar\partial(f'\rho). \tag{2.10}$$

From (2.10) it follows that $T^*_{pq}f \in \mathrm{Dom}(\bar\partial^*)$, $\bar\partial^* T^*_{pq}f = 0$, and $\vartheta\bar\partial T^*_{pq}f = 0$. In particular, the coefficients of $T^*_{pq}f$ are harmonic functions.

Remark 2.11. Let us define R_{pq} as the adjoint of $\mathcal{N}^b_{pq}$:

$$R_{pq} : W^{1/2}_{pq}(\partial\Omega) \to L^2_{pq-1}(\Omega),\ 1 \le q \le n,$$

is given by :

$$\int_{\partial\Omega} < \mathcal{N}^b_{pq}\beta, f > d\sigma = \int_\Omega < \beta, R_{pq}f > dV,$$

for $f \in W^{1/2}_{pq}(\partial\Omega)$ and $\beta \in L^2_{pq}(\Omega)$.

From the well known formula :

$$\mathcal{N}_{pq} = (\bar\partial^*\mathcal{N}_{pq})^*(\bar\partial^*\mathcal{N}_{pq}) + (\bar\partial^*\mathcal{N}_{pq+1})(\bar\partial^*\mathcal{N}_{pq+1})^*,$$

it follows that:

$$R_{pq} = (\bar\partial^*\mathcal{N}_{pq})^* T^*_{pq} + (\bar\partial^*\mathcal{N}_{pq+1})T_{pq+1}. \tag{2.11}$$

Now $(\bar\partial\vartheta + \vartheta\bar\partial)R_{pq}f = -\bar\partial T^*_{pq}f + \vartheta T_{pq}f$. Using formulas (2.8) and (2.10) and the fact that $\mathcal{N}_{pq+1}\bar\partial = \bar\partial\mathcal{N}_{pq}$, it follows that $(\bar\partial\vartheta + \vartheta\bar\partial)R_{pq} = 0$: R_{pq} has harmonic coefficients.

3. Relations betwen Sobolev Estimates for T_{pq} and for the Neumann Operator $\mathcal{N}$

Our first Proposition shows equivalence betwen Sobolev estimates for T_{pq} and $\bar\partial^*\mathcal{N}_{pq}$ and relates them with Sobolev estimates for B_{pq} projection :

Proposition 3.1. *Let Ω be a pseudoconvex bounded domain with smooth boundary; s_1 and s_2 are two positive numbers, $s_1 \le s_2 + 1$. Then the following are equivalent :*

(i) T_{pq} maps continuously $W^{-s_1+1/2}_{pq-1}(\partial\Omega)$ into $W^{-s_2}_{pq}(\Omega)$;

(ii) $\bar\partial^*\mathcal{N}_{pq}$ maps continuously $W^{s_2}_{pq}(\Omega)$ into $W^{s_1}_{pq-1}(\Omega)$;

(iii) T^*_{pq} maps continuously $W^{s_2+1/2}_{pq}(\partial\Omega)$ into $W^{s_1}_{pq-1}(\Omega)$.

Moreover, when these conditions are satisfied, then B_{pq} and B_{pq-1} map continuously $W^{s_2}_{pq}(\Omega)$ into $W^s_{pq}(\Omega)$ (resp. $W^{s_2}_{pq}(\Omega)$ into $W^s_{pq-1}(\Omega)$), with $s = \min(s_1, s_2)$.

Proof.

(ii)⇒(i) : we have to prove that

$$\left(\bar\partial^*\mathcal{N}_{pq}\right)^b\ :\ W^{s_1}_{pq}(\Omega)\to W^{s_1-1/2}_{pq-1}(\partial\Omega).$$

As $(\bar\partial\vartheta+\vartheta\bar\partial)\bar\partial^*\mathcal{N}_{pq}=B_{pq}$, the coefficients of $\bar\partial^*\mathcal{N}_{pq}f$, for $f\in W^{s_2}_{pq}(\Omega)$, have Laplacians in $W^{-1}(\Omega)$. They have a trace in $W^{s_1-1/2}(\partial\Omega)$ by Proposition 1.3.

(i)⇒(ii) : Let us prove first that for all s_1 (i) implies :

$$\left(\bar\partial^*\mathcal{N}_{pq}\right)^b\ :\ W^{s_2}_{pq}(\Omega)\to W^{s_1-1/2}_{pq-1}(\partial\Omega).$$

It is a consequence of the fact that, if $g\in\mathscr{C}(\bar\Omega)$ and $h\in L^2(\Omega)$ is harmonic, then :

$$\left|\int_\Omega g\bar h dV\right|\leq C\left\|g\right\|_{W^s(\Omega)}\left\|h\right\|_{W^{-s}(\Omega)}, \tag{3.1}$$

see [20]. We know that T_{pq} maps into forms with harmonic coefficients. Now, by proposition 1.3, as $\bar\partial^*\mathcal{N}_{pq}f$, for $f\in\mathscr{C}^\infty_{pq}(\bar\Omega)$, has boundary values in $W^{s_1-1/2}_{pq-1}(\partial\Omega)$, it is in $W^{s_1}_{pq-1}(\Omega)$ if the Laplacians of its coefficients are in $W^{\max(-1,s_1-2)}(\Omega)$. They are given by $\vartheta B_{pq}f$, so it is the case if $s_1\leq 1$, and we have proved (ii) in this case. Let us prove it in general by induction on m, where the integer m is such that $m<s_{1\leq m+1}$. We can assume $m\geq 1$, and by induction hypothesis we alredy know that $\bar\partial^*\mathcal{N}_{pq}$ maps $W^{s_2}_{pq}(\Omega)$ into $W^m_{pq-1}(\Omega)$. So, using the proof below, B_{pq} maps $W^{s_2}_{pq}(\Omega)$ into $W^{\min(s_2,m)}_{pq}(\Omega)$, and ϑB_{pq} maps $W^{s_2}(\Omega)$ into $W^{s_1-2}(\Omega)$ as $s_1\leq s_2+1$. We conclude from this.

(ii)⇒continuity for B_{pq} and B_{pq-1} : it is a consequence of the following formulas given by H. Boas and E. Straube [2] :

$$B_{pq-1}=w_tB^t_{pq-1}w_{-t}-\bar\partial^*\mathcal{N}_{pq}\left(\bar\partial w_t\wedge B^t_{pq-1}w_{-t}\right), \tag{3.2}$$

$$B_{pq}=B^t_{pq}+\left(\bar\partial^*\mathcal{N}_{pq}\right)^*\left(\bar\partial^*-\bar\partial^*_t\right)\left(\mathrm{id}-B^t_{pq}\right), \tag{3.3}$$

where w_t is the weight $w_t(z)=\exp\left(-t\left|z\right|^2\right)$, t big enough and B^t_{pq} the associated projection which is known to be bounded in $W^s(\Omega)$ by [16]. Continuity of B_{pq-1} follows at once from (3.2). For B_{pq} take the adjoints in (3.3), and use the fact that $\bar\partial^*-\bar\partial^*_t$ is a zero order operator.

(ii)⇒(iii) : Follows from (2.10).

(iii)⇒(i) : Using formula (3.1) again, it follows from (iii) that :

$$\left(\bar\partial^*\mathcal{N}_{pq}\right)^{*b}\ :\ W^{-s_1}_{pq-1}(\Omega)\cap\mathscr{H}_{pq}\to W^{-s_2-1/2}_{pq}(\partial\Omega),$$

where $\mathscr{H}_{pq}$ is the space of forms whose coefficients are finite linear combinations of harmonic functions multiplied by given smooth functions. In particular :

$$f\mapsto\left(\bar\partial^*\mathcal{N}_{pq}\right)^{*b}\vartheta\bar\partial\left(f'\rho\right)$$

maps $W^{-s_1+1/2}_{pq-1}(\partial\Omega)$ into $W^{-s_2-1/2}_{pq}(\partial\Omega)$. By formula (2.8), the coefficients of $T_{pq}f$ have boundary values in $W^{-s_2-1/2}_{pq}(\partial\Omega)$ (as the coefficients of $\bar\partial\left(f'\rho\right)$ have boundary values in $W^{-s_1+1/2}(\partial\Omega)$ and $s_1\leq s_2+1$). As they are harmonic, they are in $W^{-s_2}_{pq}(\partial\Omega)$.

This concludes the proof of Proposition 3.1. □

The similar Proposition, with T_{pq} replaced by T^*_{pq} and conversely, can be proved in the same way. We will only write :

Proposition 3.2. *Proposition 3.1 still holds with T_{pq} replaced by T^*_{pq}, T^*_{pq} by T_{pq}, and $\bar\partial^*\mathcal{N}_{pq}$ by $\left(\bar\partial^*\mathcal{N}_{pq}\right)^*$.*

Conversely, continuity of B_{pq} and B_{pq-1} imply continuity for $\bar\partial^*\mathcal{N}_{pq}$ and $\left(\bar\partial^*\mathcal{N}_{pq}\right)^*$; the following Lemma is implicit in [2] :

Lemma 3.3. *If B_{pq+i}, with $i=-1,0$, maps continuously $W^s_{pq+i}(\Omega)$ into $W^r_{pq+i}(\Omega)$, with $r\leq s$, then, if $r_1=\dfrac{r^2}{s}$,*

(i) $\bar\partial^*\mathcal{N}_{pq}$ maps continuously $W^s_{pq}(\Omega)$ into $W^{r_1}_{pq-1}(\Omega)$;

(ii) $\left(\bar\partial^*\mathcal{N}_{pq}\right)^*$ maps continuously $W^s_{pq}(\Omega)$ into $W^{r_1}_{pq-1}(\Omega)$;

(iii) $\mathcal{N}_{pq}$, restricted to the kernel of $\bar\partial$, maps continuously $W^s_{pq}(\Omega)$ into $W^{r_2}_{pq}(\Omega)$, with $r_2=\dfrac{r_1^3}{s^2}$; $\mathcal{N}_{pq-1}$, restricted to the orthogonal of $\ker(\bar\partial)$, maps $W^s_{pq-1}(\Omega)$ into $W^{r_2}_{pq-1}(\Omega)$.

(i) follows from the classical formula :

$$\bar\partial^*\mathcal{N}_{pq}=\left(\mathrm{Id}-B_{pq-1}\right)\left(\bar\partial^*_t\mathcal{N}^t_{pq}\right)B_{pq},$$

and the fact that, by interpolation, B_{pq-1} maps $W^r_{pq-1}(\Omega)$ into $W^{r_1}_{pq-1}(\Omega)$. Take the adjoints for $\left(\bar\partial^*\mathcal{N}_{pq}\right)^*$. Finally,

$$\mathcal{N}_{pq}=\left(\bar\partial^*\mathcal{N}_{pq}\right)^*\bar\partial^*\mathcal{N}_{pq}+\left(\bar\partial^*\mathcal{N}_{pq+1}\right)\left(\bar\partial^*\mathcal{N}_{pq+1}\right)^*,$$

reduces to the first term on the kernel of $\bar\partial$, to the second on its orthogonal.

From Proposition 3.1, 3.2 and Lemma 3.3, we get :

Proposition 3.4. *The following are equivalent :*

(i) T_{pq} *maps continuously* $W^{-s+1/2}_{pq-1}(\partial\Omega)$ *into* $W^{-s}_{pq}(\Omega)$ *;*

(ii) For $i=-1,0$, B_{pq+i} *maps continuously* $W^s_{pq}(\Omega)$ *into itself ;*

(iii) T_{pq} *maps continuously* $W^{s+1/2}_{pq-1}(\partial\Omega)$ *into* $W^s_{pq}(\Omega)$.

Proposition 3.5. *Under one of the following conditions :*

(i) T_{pq+i} *maps continuously* $W^{-s+1/2}_{pq+i}(\partial\Omega)$ *into* $W^{-s_2}_{pq+i}(\Omega)$, $i=0,1$ *;*

(ii) T_{pq+i} *maps continuously* $W^{s_2+1/2}_{pq+i}(\partial\Omega)$ *into* $W^{s_1}pq+i\ (\Omega)\setminus$, $i=0,1$ *;*

then $\mathcal{N}_{pq}$ *maps continuously* $W^{s_2}_{pq}(\Omega)$ *into* $W^s_{pq}(\Omega)$ *with* $s=s_2$ *if* $s_{2\leq s_1}$, $s=\dfrac{s_1^3}{s_2^2}$ *if* $s_1\leq s_2$.

Remark 3.6. We have just written Proposition 3.4 for the case $s_1=s_2=s$. In general one gets a weaker result with no equivalence.

Remark 3.7. Conversely, if $\mathcal{N}_{pq}$ is hypoelliptic and maps $W^s_{pq}(\Omega)$ into $W^s_{pq}(\Omega)$, H. Boas and E. Straube have proved in [2] that B_{pq+i} maps continuously $W^s_{pq+i}(\Omega)$ into $W^s_{pq+i}(\Omega)$ for $i=-1,0,1$. Proposition 3.4, T_{pq+i} maps, for $i=0,1$, $W^{s+1/2}_{pq+i-1}(\partial\Omega)$ into $W^s_{pq+i}(\Omega)$ and $W^{-s+1/2}_{pq+i-1}(\partial\Omega)$ into $W^{-s}_{pq+i}(\Omega)$.

4. $W^{1/2}$ Estimates for the $\bar\partial$-Neumann Operator when Ω has a Plurisubharmonic Defining Function

We shall first prove the $W^{1/2}$ estimates for all operators under a smoothness assumption on the boundary. We shall then prove that 1/2 is not critical, one can as well get a $W^{1/2+\epsilon}$ estimate. Finally, we shall prove estimates for $\bar\partial\mathcal{N}_{pq}$ and B_{00} under the assumption that the boundary of Ω is Lipschitz.

Theorem 4.1. *Let* Ω *be a bounded pseudo-convex domain with smooth boundary. Assume that there exists a* $\mathscr{C}^2(\bar\Omega)$ *defining function* ρ *which is plurisubharmonic in* Ω*. Then the operators* $\bar\partial^*\bar N_{pq}$, *(resp.* $(\bar\partial^*\mathcal{N}_{pq})^*$, B_{pq} *and* $\mathcal{N}_{pq}$*) maps continuously* $W^{1/2}_{pq}(\Omega)$ *into* $W^{1/2}_{pq-1}(\Omega)$ *(resp.* $W^{1/2}_{pq-1}(\Omega)$ *into* $W^{1/2}_{pq}(\Omega)$, $W^{1/2}_{pq}(\Omega)$ *into itself).*

Remark 4.2. C. Kiselman has shown in [15] that the worm domain introduced by K. Diederich and E. Fornaess gives the example of a $\mathscr{C}^\infty$ pseudo-convex domain which has no pluirsubharmonic defining function.

Remark 4.3. In order to proove Theorem 4.1, by section 3, it is sufficient to prove that T_{pq} maps $L^2(\partial\Omega)$ into $W^{-1/2}_{pq}(\Omega)$. As $T_{pq}f$ has harmonic coefficients and $|\rho|$ is equivalent to the distance to the boundary, it is even sufficient to prove that T_{pq} maps $L^2_{pq-1}(\partial\Omega)$ into $L^2_{pq}(\Omega;(-\rho)dV)$. One may ask which smoothness on $\partial\Omega$ and ρ is really necessary to be able to use the techniques of section 3 and conclude from estimates on T_{pq} (and S_{pq}). As far as $\bar\partial^*\mathcal{N}_{pq}$ and B_{00} are concerned, it is easy to see that it is sufficient to assume $\rho\in\mathscr{C}^1(\bar\Omega)$ and that the two key points of potential theory that we used are the following :

Fact 4.4. *a) Harmonics functions which have boundary values in* $L^2(\partial\Omega)$ *are in* $W^{1/2}(\Omega)$.

b) Harmonics functions which are in $L^2(\delta dV)$, δ *beeing the distance to the boundary, are in the dual of* $W^{1/2}(\Omega)$.

The first assertion is true as soon as $\partial\Omega$ is Lipschitz by B. Dahlberg's theorem (see [9] for instance). We shall use it below. We did not find a discussion of the second assertion in the literature. Going through the proof of Lions-Magenes, it seems to be the case for $\mathscr{C}^3$ boundaries, but it is probably valid for $\mathscr{C}^{1+\epsilon}$ boundaries. The smoothness that we ask for B_{pq}, $q\geq 1$, is at least the smoothness which is asked by the proof of J.J. Kohn [16] for B^t_{pq}.

Theorem 4.1 is a corollary of the following Proposition :

Proposition 4.5. *Let* Ω *be a bounded pseudo-convex domain of* $\mathbb{C}^n$ *with smooth boundary which is given by* $\Omega=\{\rho<0\}$, *with* $\rho\in\mathscr{C}^2(\bar\Omega)$, $|\nabla\rho|\neq 0$ *on* $\partial\Omega$ *and plurisubharmonic in* Ω*. Then, for* $f\in\mathscr{C}^1_{pq-1}(\partial\Omega)$ *:*

$$\int_\Omega\left\{(\mathrm{Diam}(\Omega))^{-2}\left|T_{pq}f\right|^2+\left|S_{pq}f\right|^2\right\}(-\rho)dV\leq C\int_{\partial\Omega}|f|^2\,|\nabla\rho|\,d\sigma,$$

with C *independant of* Ω*;* $\mathrm{Diam}(\Omega)$ *is the diameter of* Ω.

Proof. The following lemma gives a Hörmander-Morrey type identity for all forms in $\mathscr{C}^1(\bar\Omega)$. A similar identity has been obtained by Bo Berndtsson in [1].

Lemma 4.6. *Let Ω be a bounded domain with $\mathscr{C}^2(\bar\Omega)$ defining function ρ and $\alpha \in \mathscr{C}^1_{pq}(\bar\Omega)$. Then :*

$$\begin{aligned}\int_\Omega (-\rho)\left|\bar\partial\alpha\right|^2 dV \quad &+ \quad 2(-1)^p \Re \int_\Omega (-\rho\partial\vartheta\alpha, \alpha > dV + \int_{\partial\Omega} \left|\bar N \lrcorner \alpha\right|^2 \frac{d\sigma}{|\nabla\rho|} \\ &= \quad {\sum_{I,J}}' \int_\Omega \sum_{j,k=1}^n \alpha_{I,jK}\bar\alpha_{I,jK} \frac{\partial^2\rho}{\partial z_j \partial\bar z_k} dV \\ &\quad + \int_\Omega (-\rho)\left|\vartheta\alpha\right|^2 dV + {\sum_{I,J}}' \int_\Omega (-\rho) \sum_{j=1}^n \left|\frac{\partial\alpha_{I,J}}{\partial\bar z_j}\right|^2 dV.\end{aligned}$$

Proof of Lemma 4.6. As in [13], p. 83, let us start from :

$$\left|\bar\partial\alpha\right|^2 = {\sum_{I,J}}' \sum_j \left|\frac{\partial\alpha_{IJ}}{\partial\bar z_j}\right|^2 - {\sum_{I,K}}' \sum_{j,k} \frac{\partial\alpha_{I,jK}}{\partial\bar z_k} \overline{\frac{\partial\alpha_{I,kK}}{\partial\bar z_j}},$$

multiply by $(-\rho)$, integrate on Ω i and integrate by parts the last term. Writing

$$(-1)^p < \bar\partial\vartheta\alpha, \alpha >,$$

instead of :

$${\sum_{I,K}}' \sum_{j,k} \frac{\partial^2\alpha_{I,jK}}{\partial z_j \bar z_k} \bar\alpha_{I,kK},$$

we get :

$$\int_\Omega (-\rho)\left|\bar\partial\alpha\right|^2 dV \quad + \quad (-1)^p \int_\Omega (-\rho) < \bar\partial\vartheta\alpha, \alpha > dV$$

$$- \quad {\sum_{I,J}}' \int_\Omega (-\rho) \sum_j \left|\frac{\partial\alpha_{IJ}}{\partial\bar z_j}\right|^2 dV \tag{4.1}$$

$$= \quad \int_\Omega {\sum_{I,K}}' \sum_{i,k} \frac{\partial\rho}{\partial z_j} \frac{\partial\alpha_{I,jK}}{\partial\bar z_k} \bar\alpha_{I,jK} dV. \tag{4.2}$$

Another integration by parts gives :

$$(-1)^p \int_\Omega < \alpha, \bar\partial\vartheta\alpha > dV \quad + \quad \int_\Omega (-\rho)\left|\vartheta\alpha\right|^2 dV \tag{4.3}$$

$$= \quad \int_\Omega {\sum_{I,K}}' \sum_{j,k} \frac{\partial\rho}{\partial z_j} \alpha_{I,jK} \frac{\partial\bar\alpha_{I,kK}}{\partial\bar z_k} dV. \tag{4.4}$$

To conclude, we use the divergence formula to prove that :

$$\begin{aligned}\int_{\partial\Omega} \left|\sum \beta_j \frac{\partial\rho}{\partial z_j}\right|^2 \frac{d\sigma}{|\nabla\rho|} \quad &= \quad \int_\Omega \sum \frac{\partial\rho}{\partial z_j} \frac{\partial}{\partial\bar z_k}(\beta_j\bar\beta_k) dV \\ &\quad + \int_\Omega \sum_{j,k} \frac{\partial^2\rho}{\partial z_j \partial\bar z_k} \beta_j \bar\beta_k dV,\end{aligned}$$

use it for each term I and K being fixed, and add them. □

To be able to prove Proposition 4.5, we shall use Lemma 4.6 to obtain weighted inequalities : let us write Lemma 4.6 for $\beta = e^\lambda \alpha$, where λ is a plurisubharmonic function in $\mathscr{C}^2(\bar\Omega)$: then $\bar N \lrcorner \beta = e^\lambda \left(\bar N \lrcorner \alpha\right)$, $\bar\partial\beta = e^\lambda \bar\partial\alpha + e^\lambda \left(\bar\partial\lambda \wedge \alpha\right)$,

$$\begin{aligned}e^{-2\lambda} < \bar\partial\vartheta\beta, \beta > \quad &= \quad < \bar\partial\theta\alpha, \alpha > + < \bar\partial\lambda \wedge \vartheta\alpha, \alpha > - {\sum_{I,K}}' \sum_j \left|\alpha_{I,jK} \frac{\partial\lambda}{\partial z_j}\right|^2 \\ &\quad - \sum \frac{\partial^2\lambda}{\partial z_j \partial\bar z_k} \alpha_{I,jK} \bar\alpha_{I,jK} - \sum \frac{\partial\lambda}{\partial z_j} \frac{\partial\alpha_{I,jJ}}{\partial\bar z_k} \bar\alpha_{I,kK}.\end{aligned}$$

So neglecting positive terms on the left hand side and using Schwarz inequality, we can write :

$$\begin{aligned}&\int_\Omega (-\sigma) e^{2\lambda} \sum \frac{\partial^2\lambda}{\partial z_j \partial\bar z_k} \alpha_{I,jK} \bar\alpha_{I,kK} dV \\ &\quad = \int_{\partial\Omega} \left|\bar N \lrcorner \alpha\right|^2 e^{2\lambda} \frac{d\sigma}{|\nabla\rho|} + \int_\Omega (-\rho) e^{2\lambda} \left\{\left|\bar\partial\alpha\right|^2 + |\vartheta\alpha|^2 + \sum \left|\frac{\partial\alpha_{IK}}{\partial\bar z_k}\right|^2\right\} dV \\ &\quad\quad + 2\left|\int_\Omega (-\rho) < \bar\partial\vartheta\alpha, \alpha > e^{2\lambda} dV\right| + \sup_\Omega |\nabla\rho|^2 \int_\Omega (-\rho) |\alpha|^2 e^{2\lambda} dV.\end{aligned}$$

Let us take $\lambda(z) = t\,|z|^2$, with $t < 1$. Then the last term is absorbed by the left hand side when $\dfrac{1}{t} \geq C(\mathrm{Diam}(\Omega))^2$. With this choice, using Lemma 4.6, one gets :

$$\begin{aligned} t\int_\Omega (-\rho)\,|\alpha|^2\,dV + \int_\Omega (-\rho)\,|\vartheta\alpha|^2\,dV \quad &\leq \quad C\int_{\partial\Omega} \left|\bar N\lrcorner\alpha\right|^2 \frac{d\sigma}{|\nabla\rho|} + C\int_\Omega (-\rho)\left|\bar\partial\alpha\right|^2 dV \\ &\quad +2\left|\int_\Omega (-\rho) < \bar\partial\vartheta\alpha, \alpha > e^{2\lambda} dV\right| \\ &\quad +2\left|\int_\Omega < \bar\partial\vartheta\alpha, \alpha > dV\right|. \end{aligned}$$

We want to use this inequality for $\alpha = T_{pq}f$, with $f \in \mathscr{C}^2_{pq-1}(\partial\Omega)$. Let $\alpha^{(n)} \in \mathscr{C}^2_{pq}(\bar\Omega)$ so that $\alpha^{(n)} \to \alpha$ in L^2 norm, $\vartheta\alpha^{(n)} \to \vartheta\alpha$ in L^2, $\bar\partial\alpha^{(n)} \to 0$, and $(\bar N\lrcorner\alpha^{(n)})_{|\partial\Omega} = (\bar N\lrcorner\alpha)_{|\partial\Omega} = |\nabla\rho|\,f$. Writing the inequality for $\alpha^{(n)}$ and taking the limit, we find that :

$$(\mathrm{Diam}(\Omega))^{-2}\int_\Omega (-\rho)\,|\alpha|^2\,dV + \int_\Omega (-\rho)\,|\vartheta\alpha|^2\,dV \leq C\int_{\partial\Omega} |f|^2\,|\nabla\rho|\,d\sigma,$$

where C a universal constant as soon as we have proven that :

$$\int_\Omega (-\rho) < \bar\partial\vartheta\alpha^{(n)}, \alpha^{(n)} > dV \to 0$$

as well as the other term. But $(-\rho)\alpha^{(n)}$ belongs to $\mathrm{Dom}(\bar\partial^*)$, so

$$\int_\Omega (-\rho) < \bar\partial\vartheta\alpha^{(n)}, \alpha^{(n)} > dV = \int_\Omega < \vartheta\alpha^{(n)}, \vartheta\left[(-\rho)\alpha^{(n)}\right] > dV,$$

which tends to $\displaystyle\int_\Omega < \vartheta\alpha, \vartheta(-\rho\alpha) > dV$. Again $-\rho\alpha$ belongs to $\mathrm{Dom}(\bar\partial^*)$, so this last integral is equal to

$$\int_\Omega < (-\rho) < \bar\partial\vartheta\alpha, \alpha > dV = 0.$$

The same for the other term. □

Remark 4.7. For $q > 1$, the hypothesis ρ plurisubharmonic can be weakned in a standart way, see [5].

Following a suggestion given by J.J. Kohn, one can obtain as well a $W^{1/2+\epsilon} \to W^{1/2+\epsilon}$ result :

Theorem 4.8. *Let \|it Ω be as in Theorem 4.1. Then there exists $\epsilon > 0$ such that the operators $\bar\partial^{*\mathscr{N}_{pq}}$, $\left(\bar\partial^*\mathscr{N}_{pq}\right)^*$, B_{pq} and $\mathscr{N}_{pq}$ map continuously $W^{1/2+\epsilon}_{pq-1}(\Omega)$ into $W^{1/2+\epsilon}_{pq-1}(\Omega)$ (resp. $W^{1/2+\epsilon}_{pq}(\Omega)$ into $W^{1/2+\epsilon}_{pq+1}(\Omega)$, $W^{1/2+\epsilon}_{pq}(\Omega)$ into itself).*

Skech of the proof. Let us remark first that while we proved Theorem 4.1, we have proven that if $\beta \in L^2_{pq}(\Omega)$, $\bar\partial\beta$ and $\vartheta\beta$ belong to $L^2_{pq\pm1}(\Omega)$ and $\bar N\lrcorner\beta_{|\partial\Omega} \in L^2(\partial\Omega)$ (just remark that regularisation works also for the $\dfrac{\partial\beta_{IJ}}{\partial\bar z_j}$),

$$\begin{aligned} \int_\Omega (-\rho)\,|\beta|^2\,dV \ + \ &\int_\Omega (-\rho)\sum\left|\frac{\partial\beta_{I,J}}{\partial\bar z_j}\right|^2 dV + \int_\Omega (-\rho)\,|\vartheta\beta|^3\,dV \\ &\leq \left\{\int_{\partial\Omega} \left|\bar N\lrcorner\beta\right|^3 d\rho + \int_\Omega (-\rho)\left[\left|\bar\partial\beta\right|^2 + |\vartheta\beta|^2 + \left|< \bar\partial\vartheta\beta, \beta >\right|\right] dV\right\}. \end{aligned}$$

Let $f \in \mathscr{C}^1_{pq-1}(\partial\Omega)$, $\alpha = T_{pq}f$: we want to prove the a priori estimate :

$$\|\alpha\|_{W^{-1/2-\epsilon}(\Omega)} \leq C\,\|f\|_{W^{-\epsilon}(\partial\Omega)}\,,$$

with C independent of f. Let us first remark that, as α has harmonic coefficients,

$$\begin{aligned} \|\alpha\|_{W^{-1/2-\epsilon}(\Omega)} \quad &\simeq \quad \int_\Omega (-\rho)^{1+\frac{\epsilon}{2}}\,\|\alpha\|^2\,dV \\ &\simeq \quad \int_\Omega (-\rho)\,|\Lambda_\epsilon\alpha|^2\,dV, \end{aligned}$$

where Λ_ϵ i is defined in the following way : if ϵ_0 is such that $\{-\epsilon_0 < \rho < 0\}$ is diffeomorphic to $\partial\Omega\times]0,\epsilon_0[$ and $g \in \mathscr{C}^2(\Omega)$, then :

$$\Lambda_\epsilon g = \varphi g + \Lambda_\epsilon[(1-\varphi)g]$$

where φ is $\mathscr{C}^2$ with compact support in $\left\{\rho < -\dfrac{\epsilon_0}{2}\right\}$ and is 1 for $\{\rho \leq -\epsilon_0\}$; for g supported in $\{-\epsilon_0 < \rho < 0\}$, Λ_ϵ acts as $\Lambda'_\epsilon \otimes \psi(\eta) + 1 - \psi(\eta))\mathrm{Id}$, where Λ'_ϵ is a pseudo-differential operator of order $-\epsilon$ on $\partial\Omega$, ψ is smooth, compactly supported, and 1 near 0 ; moreover, Λ'_ϵ is choosen in such a way that

$$\|g\|_{W^{-\epsilon}(\partial\Omega)} \sim \|\Lambda'_\epsilon g\|_{L^2(\partial\Omega)}$$

and, for D a derivative and a a smooth function, $[D,\Lambda_\epsilon]$ and $[a,\Lambda_\epsilon]$ are tangential pseudo-differentials operators of order $-\epsilon$ and $-1-\epsilon$ with norm bounded by $C\epsilon$.

Let us write (3.7) with $\beta=\Lambda_\epsilon\alpha$, where $\alpha=T_{pq}f$. Then

$$\bar\partial\beta=\Lambda_\epsilon(\bar\partial\alpha)+[\bar\partial,\Lambda_\epsilon]\alpha,$$

and

$$\begin{aligned}\bar\partial\vartheta\beta &= \bar\partial\Lambda_\epsilon\vartheta\alpha+\bar\partial[\Lambda_\epsilon,\vartheta]\alpha\\ &= \Lambda_\epsilon\bar\partial\vartheta\alpha+[\bar\partial,\Lambda_\epsilon]\vartheta\alpha\\ &\quad +\sum(\text{Operators of order }-\epsilon)\left(\frac{\partial\alpha_{IJ}}{\partial\bar z_k}\right)+(\text{Operator of order }-\epsilon)\,(\alpha),\end{aligned}$$

so :

$$\begin{aligned}\int_\Omega(-\rho)\left|\Lambda_\epsilon\alpha\right|^2dV &+ \int_\Omega(-\rho)\sum\left|\Lambda_\epsilon\frac{\partial\alpha_{IJ}}{\partial\bar z_j}\right|^2dV+\int_\Omega(-\rho)\left|\Lambda_\epsilon\vartheta\alpha\right|^2dV\\ &\le C\left\|f\right\|_{W^{-\epsilon}(\partial\Omega)}+C\epsilon^2\left\|\alpha\right\|_{W^{-1-\epsilon}(\partial\Omega)}\\ &\quad +C\epsilon^2\int_\Omega(-\rho)\left[\left|\Lambda_\epsilon\alpha\right|^2+\sum\left|\Lambda_\epsilon\frac{\partial\alpha_{IJ}}{\partial\bar z_j}\right|^2+\left|\Lambda_\epsilon\vartheta\alpha\right|^2\right]dV.\end{aligned}$$

Choosing ϵ small enough and using the fact that for harmonic functions it is equivalent to belong to $W^{-1-\epsilon}(\partial\Omega)$ or $W^{-1/2-\epsilon}(\Omega)$, we ge the announced result. □

We shall now deals with Lipschitz domains :

Theorem 4.9. *Let Ω be a bounded pseudo-convex domain with Lipschitz boundary for which there exists a Lipschitz defining function ρ which is plurisubharmonic inside Ω. Then the operators $\bar\partial^*\mathcal{N}_{pq}$ Bergman projection B_{00} map continuously $W^{1/2+\epsilon}_{pq}(\Omega)$ into $W^{1/2}_{pq-1}(\Omega)$ (resp. $W^{1/2+\epsilon}(\Omega)$ into $W^{1/2}(\Omega)$) for any $\epsilon>0$.*

Let us remind that ρ i is a Lipschitz defining function for Ω if and only if, after a $\mathscr{C}^1$ change of variable, near a point at the boundary, it may be written locally as $x_{2n}-f(x_1,\dots,x_{2n-1})$, for some Lipschitz function f. Under this hypothesis, $\nabla\rho$ is defined a.e. near $\partial\Omega$ with $C_1<|\nabla\rho|<C_2$, and $-\rho$ is equivalent to the distance to the boundary. We shall use an exhaustion of the domain Ω by domains Ω_{ϵ_1} where

$$\Omega_{\epsilon_1}=\{z\in\Omega\,;\,\rho*\varphi_{\epsilon_1}<-\epsilon_2(\epsilon_1)\}\,,$$

with $\varphi\ge0$, supported in the unit ball, $\varphi\in\mathscr{C}^\infty(\mathbb{C}^n)$ and $\int_{\mathbb{C}^n}\varphi=1$, $\varphi_{\epsilon_1}(z)=\frac{1}{\epsilon_1^{2n}}\varphi\left(\frac{z}{\epsilon_1}\right)$. It is possible to choose $\epsilon_2=\epsilon_2(\epsilon_1)=k\epsilon_1$ so that $\rho*\varphi_\epsilon$ is plurisubharmonic in Ω_{ϵ_1}. All conditions for Theorem 4.1 are satisfied by Ω_{ϵ_1} and, for $f\in W^{1/2+\epsilon}_{pq}(\Omega)$, $\alpha_{\epsilon_1}=\bar\partial^*\mathcal{N}^{\epsilon_1}_{pq}f$ belongs to $W^{1/2}_{pq}(\Omega_{\epsilon_1})$. Let us verify that its norm in $W^{1/2}_{pq-1}(\Omega_{\epsilon_1})$ is bounded by a constant which is independent of ϵ_1 ; then $\alpha_{\epsilon_{1_k}}$ converges weakly to $\alpha\in W^{1/2}_{pq-1}(\Omega)$ ($\alpha_{\epsilon_{1_k}}$ may be considered as an element of $W^{1/2}(\Omega)$ as functions in $W^{1/2}(\Omega_{\epsilon_{1_k}})$ extends to functions of $W^{1/2}(\mathbb{C}^n)$ (see [11] for instance)), and it is orthogonal to $\bar\partial$-closed forms $\beta\in L^2-pq-1\backslash\,(\Omega)$ as :

$$\int_\Omega<\alpha,\beta>dV=\lim_{k\to\infty}\int_\Omega<\alpha_{\epsilon_{1_k}},\beta>dV=0,$$

and $\bar\partial\alpha=B_{pq}f$ as, for $\beta\in\mathscr{C}^\infty_{pq}(\Omega)$ compactly supported :

$$\begin{aligned}\int_\Omega<\bar\partial\alpha,\beta>dV &= \int_\Omega<\alpha,\vartheta\beta>dV=\lim_{k\to\infty}\int_{\Omega_{\epsilon_{1_k}}}<\alpha_{\epsilon_{1_k}},\vartheta\beta>dV\\ &= \lim_{k\to\infty}\int_{\Omega_{\epsilon_{1_k}}}<B^{\epsilon_{1_k}}_{pq}f,\beta>dV.\end{aligned}$$

But $B^{\epsilon_{1_k}}_{pq}f$ is bounded in L^2 uniformly, so again there is a subsequence weakly convergent. The limit is easily shown to be $B_{pq}f$.

So the proof relies on the fact that α_{ϵ_1} belongs to $W^{1/2}_{pq}(\Omega_{\epsilon_1})$ uniformly. We use the fact 4.4, and the facts that $\mathrm{Diam}(\Omega_{\epsilon_1})\le\mathrm{Diam}(\Omega)$, $\sup_{\partial\Omega_{\epsilon_1}}|\nabla[\rho*\varphi_{\epsilon_1}]|$ is bounded by $C\sup|\nabla\rho|$ on a neighborhood of $\partial\Omega$ iand the constants which give the equivalence between $-\rho*\varphi_{\epsilon_1}-\epsilon_2$ and $\mathrm{dist}(.,\partial\Omega_{\epsilon_1})$ may be choosen independent of ϵ_1. It remains to show that the fact 4.4, with $W^{1/2}(\Omega)$ replaced by $W^{1/2+\epsilon}(\Omega)$ in the second assertion, is satisfied for Ω_{ϵ_1} with constants which do not depend on ϵ_1. The first fact follows from B. Dahlberg's theorem (see, for instance [14] to see that the constants depends only on the Lipschitz constants). The second one follows also from B. Dahlberg's theorem and the fact that functions in $W^{1/2+\epsilon}(\Omega)$ have traces at the boundary. As we did not find any reference, we sketch the proof in the appendix.

For the bergman projection the proof follows the same lines.

Remark 4.10. Examples in $\mathbb{C}$ prove that $W^{1/2}$ is the best possible result for domains with Lipschitz boundary.

5. Further Results

The aim of this paragraph is to show how one can deduce L^p results with loss from W^s results for the canonical solution to the $\bar\partial$-equation. For simplicity, we will consider only domains with smooth boundary; but using technics similar to those developed in the last paragraph, one can prove the same results with a $\mathscr{C}^2$ boundary. The key point is contained in the following Proposition :

Proposition 5.1. *Let Ω be bounded pseudo-convex domain with smooth boundary; δ denotes the distance to the boundary. Let $0<s<1/2$. Then the following are equivalent :*

(a) $\bar\partial^*\mathscr{N}_{pq}$ maps continuously $W^s_{pq}(\Omega)$ into $W^s_{pq-1}(\Omega)$;

(b) $\bar\partial^*\mathscr{N}_{pq}$ maps continuously $L^2_{pq}\left(\frac{dV}{\delta^{2s}}\right)$ into $L^2_{pq-1}\left(\frac{dV}{\delta^{2s}}\right)$.

Moreover the analogous equivalences are valid for $\left(\bar\partial^*\mathscr{N}_{pq}\right)^*$, $\mathscr{N}_{pq}$ and B_{p0}, as well as for $\bar\partial^*_t\mathscr{N}^t_{pq}$, $\left(\bar\partial^*_t\mathscr{N}^t_{pq}\right)^*$, $\mathscr{N}^t_{pq}$ and B^t_{p0}, where $\mathscr{N}^t_{pq}$ ... denotes the $\bar\partial$-Neumann operator for the weight $e^{-t|z|^2}$ which was considered by J.J. Kohn in [16].

Let us prove that (a)$\Rightarrow$ (b). As $W^s_{pq-1}(\Omega)\hookrightarrow L^2_{pq-1}\left(\frac{dV}{\delta^{2s}}\right)$, we only have to prove that $\bar\partial^*\mathscr{N}_{pq}$ maps $L^2_{pq}\left(\frac{dV}{\delta^{2s}}\right)$ into $W^s_{pq-1}(\Omega)$. But by ( ), it is sufficient to prove that $\left(\bar\partial^*\mathscr{N}_{pq}\right)^b$ maps $L^2_{pq}\left(\frac{dV}{\delta^{2s}}\right)$ into $W^{s-1/2}_{pq-1}(\partial\Omega)$, or, by duality, that T_{pq} maps $W^{-s+1/2}_{pq-1}(\partial\Omega)$ into $W^{-s}_{pq}(\Omega)$. By hypothesis and Proposition 3.1, T_{pq} maps $W^{-s+1/2}_{pq-1}(\partial\Omega)$ into $W^{-s}_{pq}(\Omega)$. But $T_{pq}f$ has harmonic coefficients, so it is equivalent for $T_{pq}f$ to be in $W^{-s}_{pq}(\Omega)$ $L^2_{pq}(\delta^{2s}dV)$.

Let us prove that (b)$\Rightarrow$ (a). We have now to prove that $\bar\partial^*\mathscr{N}_{pq}$ maps continuously $L^2_{pq}\left(\frac{dV}{\delta^{2s}}\right)$ into $W^s_{pq-1}(\Omega)$. But, if $\bar\partial^*\mathscr{N}_{pq}f=u$, $u=v+h$, where $v\in W^1_{0,pq-1}(\Omega)$ is the solution to the Dirichlet problem, coefficient by coefficient, and h has harmonic coefficients. Now, by hypothesis, h belongs to $L^2_{pq-1}\left(\frac{dV}{\delta^{2s}}\right)$ so h belongs to $W^s_{pq-1}(\Omega)$.

The same proof is valid for $\left(\bar\partial^*_t\mathscr{N}^t_{pq}\right)^*$, $\mathscr{N}_{pq}$ and the Bergman projection B_{p0}. As the property that we used for harmonic functions are also valid for any elliptic operator, the Lebesgue measure can be replaced by the measure $e^{-t|z|^2}dV(z)$ to obtain the same result for the weighted $\bar\partial$-Neumann operator.

We shall need the following Lemma :

Lemma 5.2. *([10] for $q=0$, [6] for general q) Let $u\in L^2_{pq}\left(\frac{dV}{\delta^{2s}}\right)$, with $\bar\partial u\in L^2_{pq+1}(dV)$ and $\vartheta u\in L^2_{pq-1}(dV)$. Then u is in $L^r_{pq}(dV)$ with $\frac{1}{r}=\frac{1}{2}-\frac{s}{n+1}$.*

Corollary 5.3. *a) Let us assume that $\bar\partial^*_t\mathscr{N}^t_{pq}$ maps $W^{1/2}_{pq}(\Omega)$ into $W^{1/2}_{pq-1}(\Omega)$. Then $\bar\partial^*_t\mathscr{N}^t_{pq}$ maps $L^r_{pq}(\Omega)$ into $L^{r_1}_{pq-1}(\Omega)$ for $r>2$ and $r_1<2+\frac{4}{nr+2}$.*

b) Let us assume that $\left(\bar\partial^*_t\mathscr{N}^t_{pq}\right)^*$ maps $W^{1/2}_{pq}(\Omega)$ into $W^{1/2}_{pq-1}(\Omega)$; then $\bar\partial^*_t\mathscr{N}^t_{pq}$ maps $L^r_{pq}(\Omega)$ into $L^{r_1}_{pq-1}(\Omega)$ for $r_1<2$and $r>2-\frac{4-2r_1}{nr_1+2}$.

Proof. a) follows immediately from Proposition 5.1 and Lemma 5.2, using the fact that $L^r(\Omega)\hookrightarrow L^2\left(\frac{dV}{\delta^{2s}}\right)$when $s<\frac{1}{2}-\frac{1}{r}$and W^sestimates for $\bar\partial^*_t\mathscr{N}^t$.b) follows from the same proof with $\left(\bar\partial^*_t\mathscr{N}^t_{pq}\right)^*$ instead of $\bar\partial^*_t\mathscr{N}^t_{pq}$. We conclude by duality. □

Remark 5.4. The hypothesis of a) and b) are valid in two cases : t big enough by [16] ; for all $t\geq 0$ by theorem 4.1 when Ω has a plurisubharmonic defining function.

This allows to give a new proof of a theorem of J.E. Fornaess and N. Sibony [10] with best indices and for known operators solving the $\bar\partial$.

Remark 5.5. A similar Corollary can be given for $\mathscr{N}^t_{pq}$, $\left(\bar\partial^*_t\mathscr{N}^t_{pq}\right)^*$, and B^t_{p0}.

Modifying Lemma 5.2, we can obtain L^r estimates with weights.

Proposition 5.6. *Assume that $\bar\partial^*_t\mathscr{N}^t_{pq}$ maps $W^{1/2}_{pq}(\Omega)$into $W^{1/2}_{pq-1}(\Omega)$. Then :*

a) For $r>2$, and $2+\frac{2r-4}{nr+2}\leq r_1<\infty$,$\bar\partial^*_t\mathscr{N}^t_{pq}$ maps the $\bar\partial$-closed forms of $L^r_{pq}(\Omega)$ into $L^{r_1}_{pq-1}(\delta^{\tau r_1}dV)$if the following conditions are satisfied :

$$\frac{1}{r_1}\geq\frac{1}{r}-\frac{1}{2n},\ \tau>(n+1)\left(\frac{1}{2}-\frac{1}{r_1}\right)-\left(\frac{1}{2}-\frac{1}{r}\right).$$

b) For $r > 2n$, $\delta^r \bar\partial_t^* \mathcal{N}_{pq}^t$maps the $\bar\partial$-closed forms of $L^r_{pq}(\Omega)$into $L^\partial_{pq}(\Omega)$if $\tau > \frac{n}{2} - \frac{1}{r}$.

To prove Proposition 5.6, it is sufficient to prove the following Lemma, which is in the same spirit as Lemma 5.2. For simplicity, we shall consider only $(0,1)$-forms at first.

Lemma 5.7. *Let $u \in L^2\left(\frac{dV}{\delta^{2s}}\right)$. Then :*

(i) for $r_1 \geq \frac{2(n+1)}{n+1-s}$, u belongs to $L^{r_1}(\delta^{\tau r_1} dV)$if

$$\frac{1}{r_1} \geq \frac{1}{r} - \frac{1}{2n} \text{ and } \tau = (n+1)\left(\frac{1}{2} - \frac{1}{r_1}\right) - s\,;$$

(ii) $\delta^r u$belongs to L^∞if $r > 2n$and $\tau = \frac{n+1}{} - s$.

Proof. The key inequality is the following :

$$\left(\int_{|z|<1} |f|^{r_1} dV\right)^{\frac{1}{r_1}} \leq C\left(\int_{|z|<2} |f|^2 dV\right)^{\frac{1}{2}} + C\left(\int_{|z|<2} \left|\bar\partial f\right|^r dV\right)^{\frac{1}{r}}, \tag{5.1}$$

where $\frac{1}{r_1} \geq \frac{1}{r} - \frac{1}{2n}$if t $r_1 < \infty$, $r > 2n$ for $r_1 = \infty$. To prove it, just write $f = g + h$, where $\triangle g = \triangle f$ in the ball of radius 2 with $g = 0$ at the boundary, and h is harmonic : g is in the space $W^{1,r}$, which is contained in L^{r_1} for $\frac{1}{r_1} \geq \frac{1}{r} - \frac{1}{2n}$. Then, by homogeneity, if E_ϵ is the ellipsoïd :

$$E_\epsilon = \left\{\frac{|z_1|^2}{\epsilon^2} + \sum_2^n \frac{|z_j|^2}{\epsilon} < 1\right\},$$

$$\left(\int_{E_\epsilon} |f|^{r_1} dV\right)^{\frac{1}{r_1}} \leq C\epsilon^{-(n+1)\left(\frac{1}{2} - \frac{1}{r_1}\right)}\left(\int_{E_\epsilon^*} |f|^2 dV\right)^{\frac{1}{2}} + \sqrt{\epsilon}\epsilon^{-(n+1)\left(\frac{1}{r} - \frac{1}{r_1}\right)}\left(\int_{E_\epsilon^*} \left|\bar\partial f\right|^r dV\right)^{\frac{1}{r}},$$

where E_ϵ^* is the double of E_ϵ. Now, to conclude, one follows the proof of [10], using a covering of Ω by ellipsoïds E_ϵ with ϵ equivalent to the distance to the boundary. □

For (p,q)-forms, Lemma 5.7 is still valid with the additional assumption

$$\vartheta_t u \in L^r_{pq-1}(\Omega),$$

which is satisfied by $\bar\partial_t^* \mathcal{N}_{pq}^t \beta$: for a (p,q)-form f, (5.1) is still true when

$$C\left(\int_{|z|<2} |\vartheta_t f|^r dV\right)^{\frac{1}{r}}$$

is added on the right hand side, by elliptic theory.

6. APPENDIX

We want to sketch the proof of the following result :

Lemma 6.1. *if Ω is a bounded domain which Lipschitz boundary, then any harmonic function which is in $L^2(\delta dV)$is in the dual of $W^{1/2+\epsilon}(\Omega)$.*

Let u be harmonic in Ω, $u \in L^2(\delta dV)$. Then u may be written as Xv where v is harmonic and X is a $\mathscr{C}^1$ vector field so that

$$\inf_{\partial\Omega} < X, N >> 0,$$

where N is the exterior unit normal to $\partial\Omega$ (if Ω was $\{x_{2n} > f(x_1, \ldots x_{2n-1}\}$, where f is Lipschitz, take $X = \frac{\partial}{\partial x_{2n}}$; oterwise take a partition of unity with supports in neighborhoods of boundary points of this form). Now, by Rellich's lemma (see [14] for instance),

$$\int_\Omega |\nabla v|^2 \delta dV \leq C \int_\Omega |Xv|^2 \delta dV.$$

It follows that v belongs to$W^{1/2}(\Omega)$, and, by B. Dahlberg's theorem, that v has boundary values in $L^2(\partial\Omega)$. Let w be in $W^{1/2+\epsilon}(\Omega)$. Then w has boundary values in $^2(\partial\Omega)$ (see [11] p.37) and its derivatives belong to $W^{-1/2+\epsilon}(\Omega)$. As

$$\int_\Omega (Xv)\, w dV = -\int_\Omega v\,(Xw)\, dV + \int_\Omega avw dV + \int_{\partial\Omega} bvw d\sigma,$$

where a b are bounded functions,

$$\left|\int_\Omega (Xv)\, wdV\right| \leq C\left(\int_\Omega |u|^2\,\delta dV^{1/2}\,\|w\|_{W^{1/2+\epsilon}(\Omega)}\,.\right)$$

It may be seen that, applying this to the domains Ω_{ϵ_1}, the constants C may be taken independent of ϵ_1.

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