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Another Proof for Non-Supercyclicity in Finite Dimensional Complex Banach Spaces

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Abstract. We give an elementary proof, based on linear algebra and on a simple and well-known technique from the theory of dynamical systems, for the non-existence of supercyclic linear operators defined on a finite dimensional complex Banach space with dimension greater than or equal to two.

1. INTRODUCTION. Consider X to be a real or complex Banach space X, and denote by $\mathcal{L}(X)$ the set consisting of all bounded linear operators going from X into X. An operator $T \in \mathcal{L}(X)$ is said to be *supercyclic* if there is some $x \in X$ such that the set $\{\lambda T^n x : \lambda \in \mathbb{K}, n = 0, 1, \ldots\}$ is dense in X. In this case, x is called a supercyclic vector for T. Supercyclicity is an intermediate property between cyclicity and hypercyclicity [**1**].

Recall that any linear operator defined on a finite dimensional Banach space is bounded. So, if $\dim X = 1$, we have that any nonzero linear operator $T : X \to X$ is supercyclic. When X is a real Banach space with $\dim X = 2$, then X also has supercyclic operators. For example, in $X := \mathbb{R}^2$ a rotation (with respect to the origin) by an irrational number defines a supercyclic operator. G. Herzog established in 1992 that separable infinite dimensional Banach spaces always have supercyclic operators and, on the other side, that finite dimensional complex Banach spaces with dimension greater than 2 do not have supercyclic operators [**2**]. He also proved that in the real case, this holds when $\dim X \geq 3$. The proof we present for the complex case will follow readily from two simple results. The first is based on linear algebra and the second uses a simple and well-known technique from the theory of dynamical systems [**3**, p. 6–8].)

2. THE PROOF.

Lemma 1. *Let X be a two-dimensional complex Banach space. If $T : X \to X$ is a linear operator, then T is not supercyclic.*

http://dx.doi.org/10.4169/amer.math.monthly.120.05.466
MSC: Primary 47A16, Secondary 47A15

Proof. We will first consider the situation where T has two linearly independent eigenvectors, say v and w, corresponding to eigenvalues α and β. We will also assume that $\mid \alpha \mid \leq \mid \beta \mid$ and, since the zero operator is not supercyclic, $0 < |\beta|$. Consider a sequence $\{x_n\} \subset X,\ x_n := a_n v + b_n w, a_n, b_n \in \mathbb{C}$. Notice that $x_n \to 0$ in X, if and only if $a_n \to 0$ and $b_n \to 0$.

If $x \in X$ with $x := av + bw$ for $a, b \in \mathbb{C}$, then

$$T^n x = a\alpha^n v + b\beta^n w, \quad \text{for all } n \in \mathbb{N}_0. \tag{1}$$

If $a = 0$ or $b = 0$, then clearly x is not a supercyclic vector for T. So we now suppose that $a \neq 0$ and $b \neq 0$. Assume there are sequences $\{\lambda_k\} \subset \mathbb{C}$ and $\{n(k)\} \subset \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ such that $\lambda_k T^{n(k)} x \to cv + w$, for some $c \in \mathbb{C}$. Then from (1) we obtain that

$$a\lambda_k \alpha^{n(k)} \to c,\, b\lambda_k \beta^{n(k)} \to 1.$$

Since $\frac{|\alpha|}{|\beta|} \leq 1$, this implies that

$$\mid c \mid = \left|\frac{a}{b}\right| \lim_{k\to\infty} \left|\frac{\alpha^{n(k)}}{\beta^{n(k)}}\right| \leq \left|\frac{a}{b}\right|.$$

It follows that x is not a supercyclic vector for T.

Let us now consider the case $\sigma(T) = \{\alpha\}$. If $\dim n(T - \alpha I) = 2$, then we are in the above situation and so T is not a supercyclic operator. So we now assume that $\dim n(T - \alpha I) = 1$. We can then find a basis $\{v, w\}$ for X such that the associated matrix for T with respect to this basis is

$$A := \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}.$$

Hence

$$A^n = \begin{pmatrix} \alpha^n & n\alpha^{n-1} \\ 0 & \alpha^n \end{pmatrix}, \quad \text{for all } n \in \mathbb{N}.$$

If we now take $x \in X$ with $x := av + bw$, then

$$T^n x = (a\alpha^n + bn\alpha^{n-1})v + b\alpha^n w,\ \text{for all } n \in \mathbb{N}. \tag{2}$$

If $b = 0$, then clearly x is not a supercyclic vector for T. So we suppose that $b \neq 0$. Take $c \neq 0$, and let us assume there is a sequence $\{\lambda_k\} \subset \mathbb{C}$ and a sequence $\{n(k)\} \subset \mathbb{N}_0$ such that

$$\lambda_k T^{n(k)} x \to (a + c)v + bw. \tag{3}$$

Since x and $(a + c)v + bw$ are linearly independent, we must have $n(k) \in \mathbb{N}$ for k large enough. Hence, by (2) and (3) we have

$$\lambda_k (a\alpha^{n(k)} + bn(k)\alpha^{n(k)-1}) \to a + c,\ \text{and } \lambda_k b\alpha^{n(k)} \to b. \tag{4}$$

It follows that $\lambda_k \alpha^{n(k)} \to 1$ and $\lambda_k n(k)\alpha^{n(k)-1} \to \frac{c}{b}$. Hence we have $\alpha \neq 0$ and

$$\frac{n(k)}{\alpha} = \lambda_k n(k)\alpha^{n(k)-1} \frac{1}{\lambda_k \alpha^{n(k)}} \to \frac{c}{b}.$$

This allows us to conclude that there is $K \in \mathbb{N}$ such that $n(k) = m \in \mathbb{N},$ for all $k \geq K$ and $c = \frac{mb}{\alpha}$. Thus x is not a supercyclic vector for T. ■

Lemma 2. *Let X be a finite dimensional complex Banach space and $T \in \mathcal{L}(X)$. If $\dim X \geq 3$ and T is supercyclic, then there exists a Banach space Y with $\dim Y = \dim X - 1$ and a supercyclic operator $S \in \mathcal{L}(Y)$.*

Proof. Let α be an eigenvalue of T. Take v to be a corresponding eigenvector and $M := \{\lambda v : \lambda \in \mathbb{C}\}$. Then M is a closed subspace that is invariant under T. Hence, we can consider the quotient space X/M, and obtain a bounded linear operator $\tilde{T} : X/M \to X/M$ by defining $\tilde{T}[x] := [Tx]$. In terms of the canonical map $\pi : X \to X/M$, this means that

$$\tilde{T}\pi = \pi T.$$

It follows that $\tilde{T}^n \pi = \pi T^n,$ for all $n \in \mathbb{N}$. Since π preserves dense sets, this implies that if $x \in X$ is a supercyclic vector for T, then $[x]$ is a supercyclic vector for $\tilde{T}$. ■

Theorem 1. *Let X be a complex Banach space. If $2 \leq \dim X < \infty$ and $T : X \to X$ is a linear operator, then T is not supercyclic.*

Proof. Let A consist of those $n \in \mathbb{N}$ such that, for any n-dimensional complex Banach space X, the set $\mathcal{L}(X)$ does not not have a supercyclic operator. Lemma 1 indicates that $2 \in A$. Using Lemma 2, it now follows that $A = \{2, 3, \ldots\}$, and so the proof is complete. ■

ACKNOWLEDGMENTS. The author acknowledges the referees for their suggestions, which greatly improved the exposition.

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